Forward Kinematics

Israt Jahan
Lecturer
Department of Computer Science and Engineering
Daffodil International University

Forward Kinematics

Given: The values of the joint variables

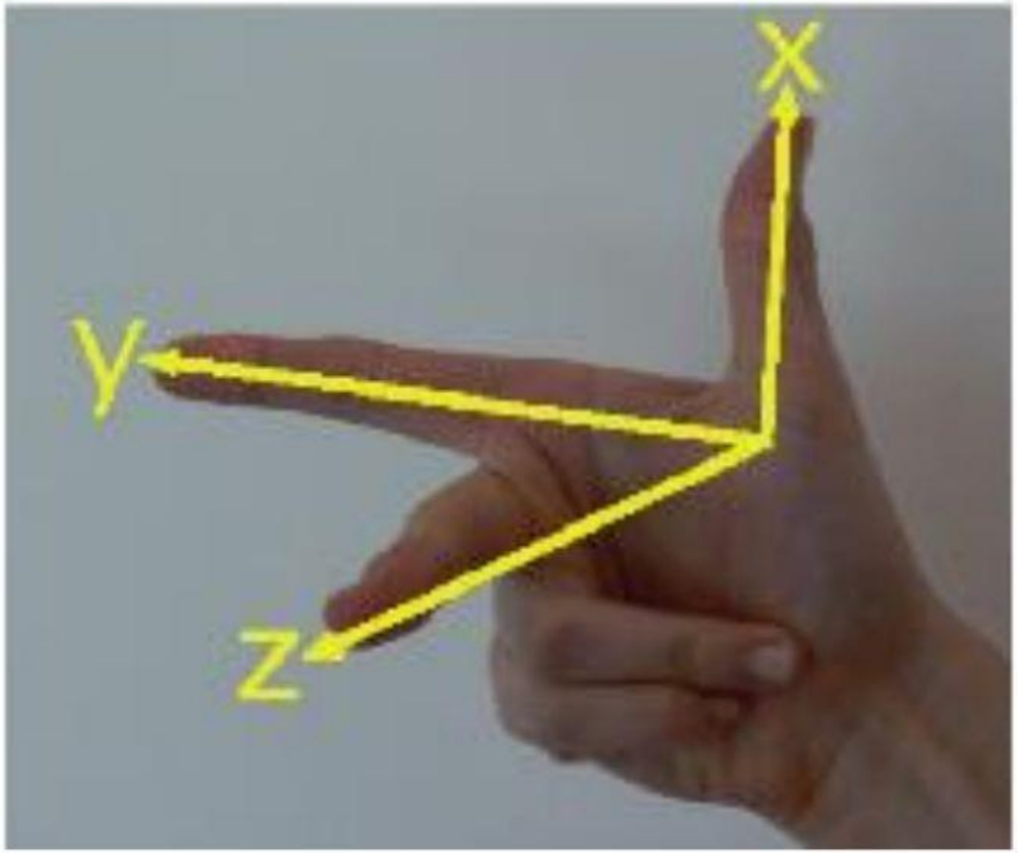
Identified: The position and orientation of the end effector

Forward Kinematics

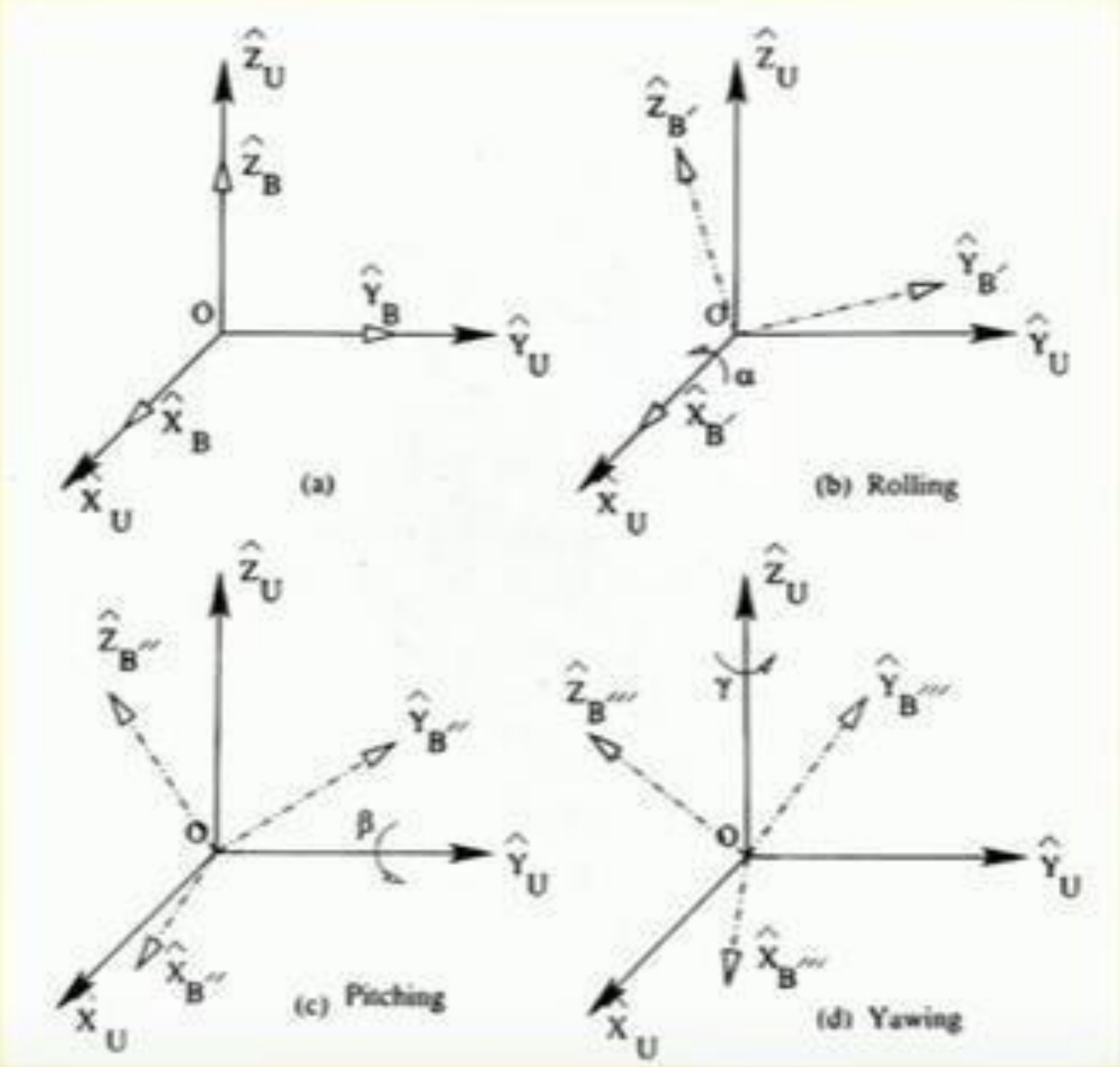
Forward kinematics refers to the use of the kinematic equations of a robot to compute the position of the end-effector from specified values for the joint parameters..

The kinematics equations of the robot are used in robotics, computer games, and animation.

Right hand Rule



Roll, Pitch and Yaw Angles



We know, Z-axis rotation, X-axis rotation, Y-axis rotation

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$ {}^{U}_{B}R_{composite,\,rpy} = ROT(\hat{Z}_U,\gamma)ROT(\hat{Y}_U,\beta)ROT(\hat{X}_U,\alpha) $$

$$ = \begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + s\alpha s\beta c\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha c\beta \end{bmatrix} $$

We compare with

$$ {}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

We get

$$ \alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) $$

$$ \beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right) $$

$$ \gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) $$

A Numerical Example

The concept of roll, pitch and yaw angles has been used to represent the rotation of a frame {B} with respect to the reference frame {U}, that is ${}^{U}_{B}R$. Let us suppose that the above rotation can also be expressed by a 3X3 rotation matrix as given below.

$$\alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

$${}^{U}_{B}R = \begin{bmatrix} -0.250 & 0.433 & -0.866 \\ 0.433 & -0.750 & -0.500 \\ -0.866 & -0.500 & 0.000 \end{bmatrix}$$

Determine the angles of rolling, pitching and yawing.

Solution:

Angle of rolling $\alpha = \tan^{-1}\frac{r_{32}}{r_{33}} = \tan^{-1}\frac{-0.500}{0.000} = 90°$

Angle of pitching $\beta = \tan^{-1}\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}$

$$= \tan^{-1}\frac{0.866}{\sqrt{(-0.250)^2 + (0.433)^2}}$$

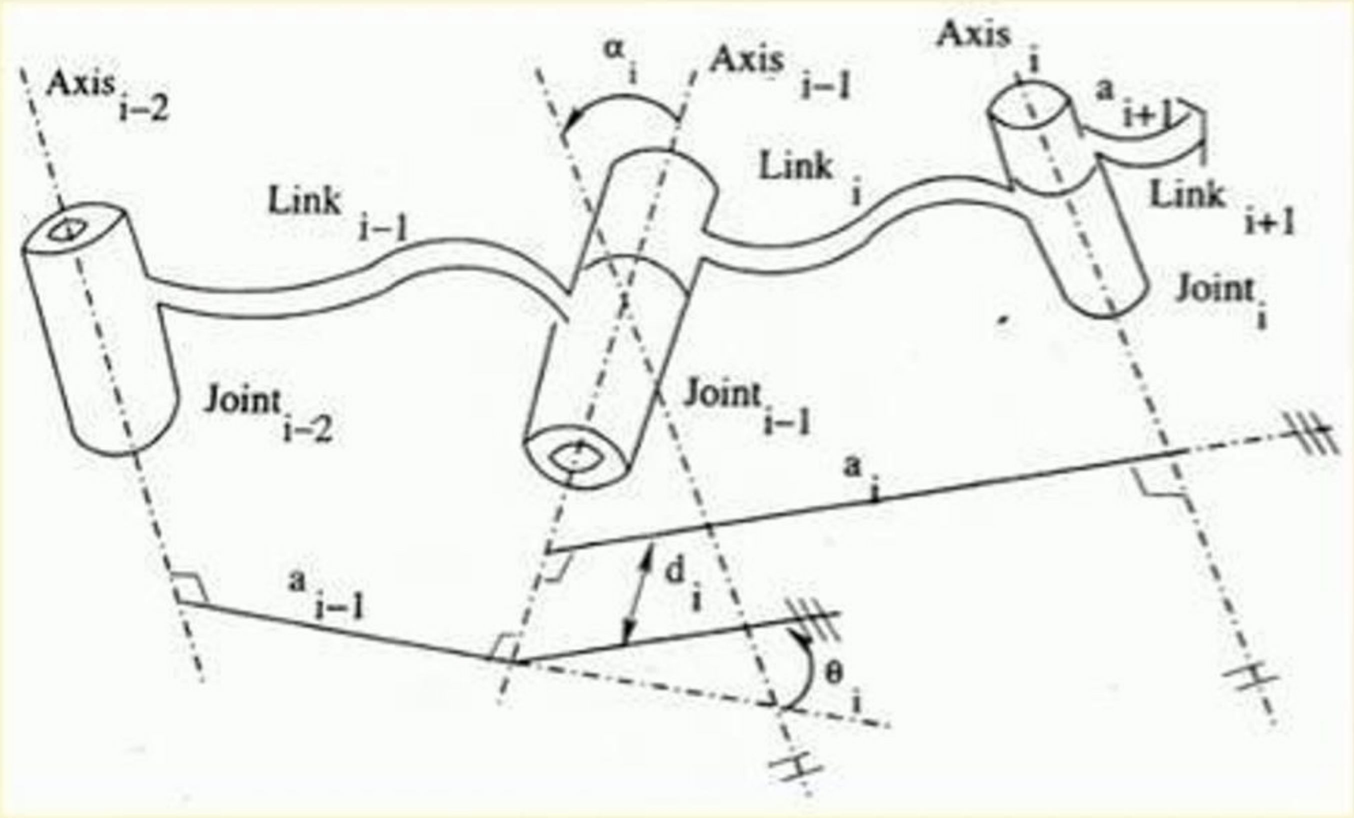
Angle of yawing $\gamma = \tan^{-1}\frac{r_{21}}{r_{11}} = \tan^{-1}\frac{0.433}{-0.250}$

$$= -59.99 \approx -60°$$

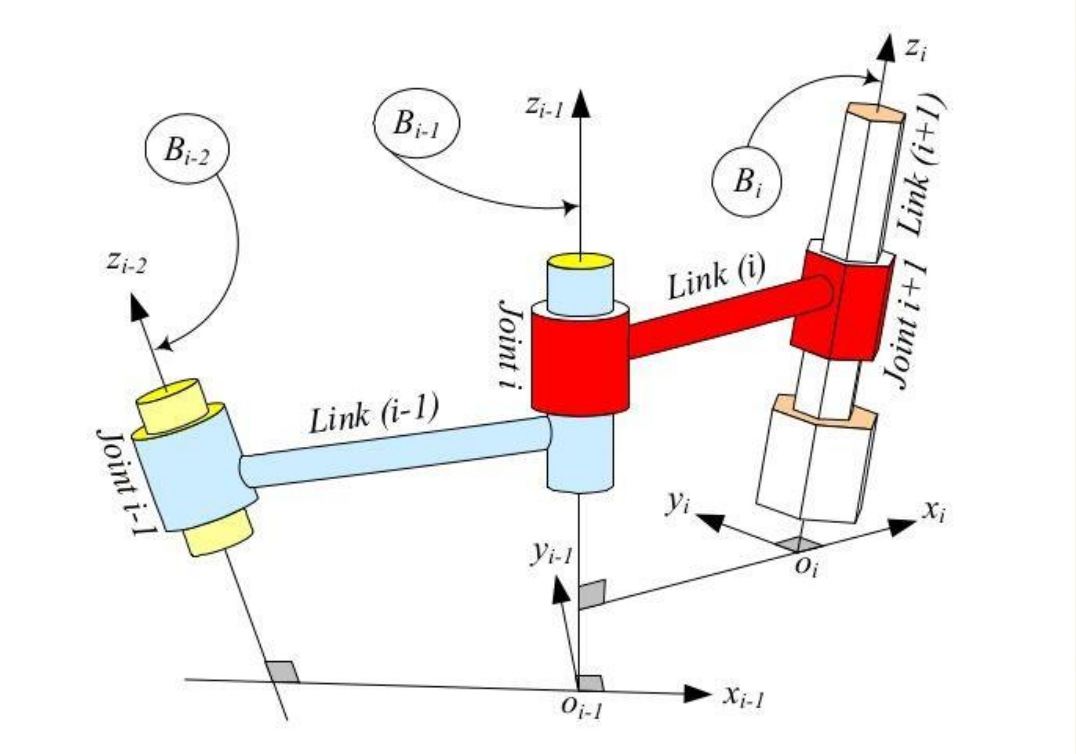
Denavit-Hartenberg Notations

[redacted]

Link and Joint Parameters



- **Length of link$_i$ (a_i):** It is the mutual perpendicular distance between Axis$_{i-1}$ and Axis$_i$
- **Angle of twist of link$_i$ (α_i):** It is defined as the angle between Axis$_{i-1}$ and Axis$_i$



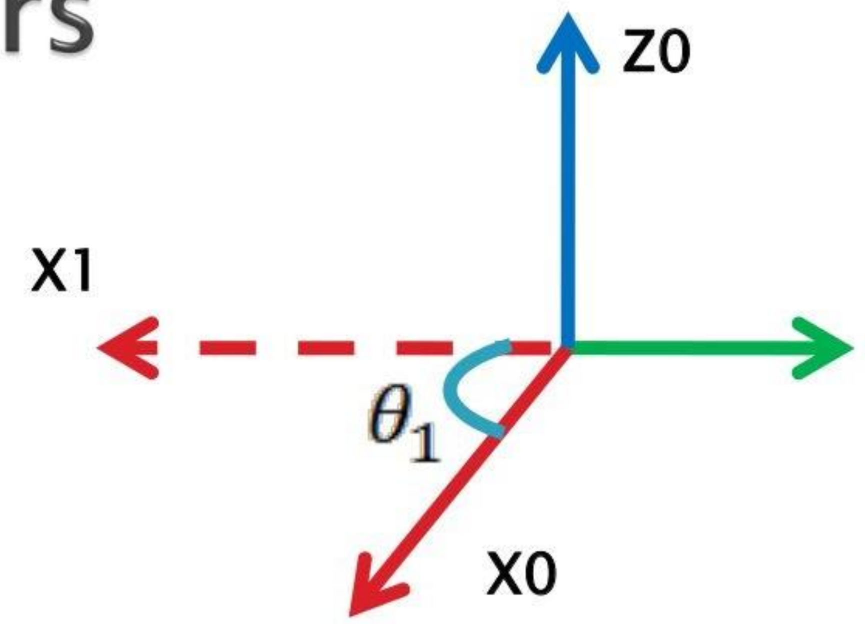
Notes:

- **Revolute joint:** θ_i is variable
- **Prismatic joint:** d_i is variable

- **Offset of link$_i$ (d_i):** It is the distance measured from a point where a_{i-1} intersects the Axis$_{i-1}$ to the point where a_i intersects the Axis$_{i-1}$ measured along the said axis

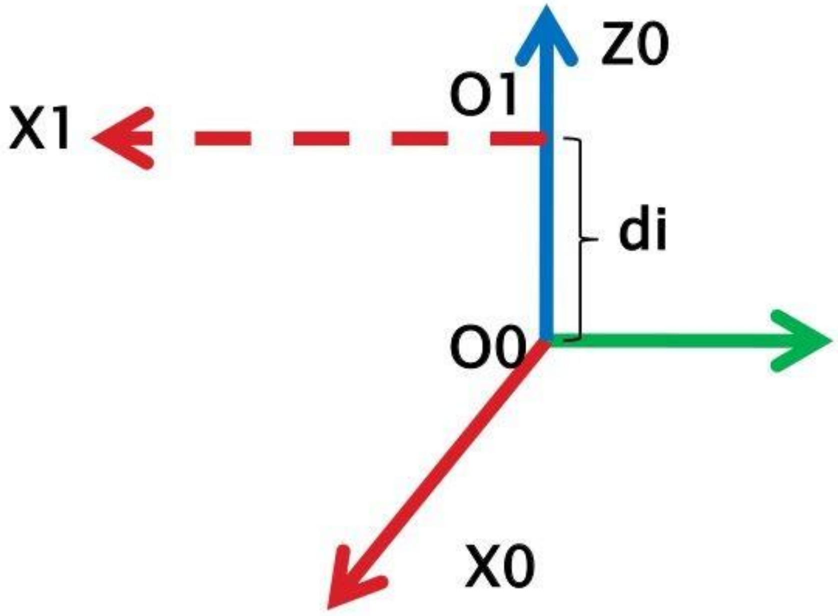
- **Joint Angle (θ_i):** It is defined as the angle between the extension of a_{i-1} and a_i measured about the Axis$_{i-1}$

DH prameters



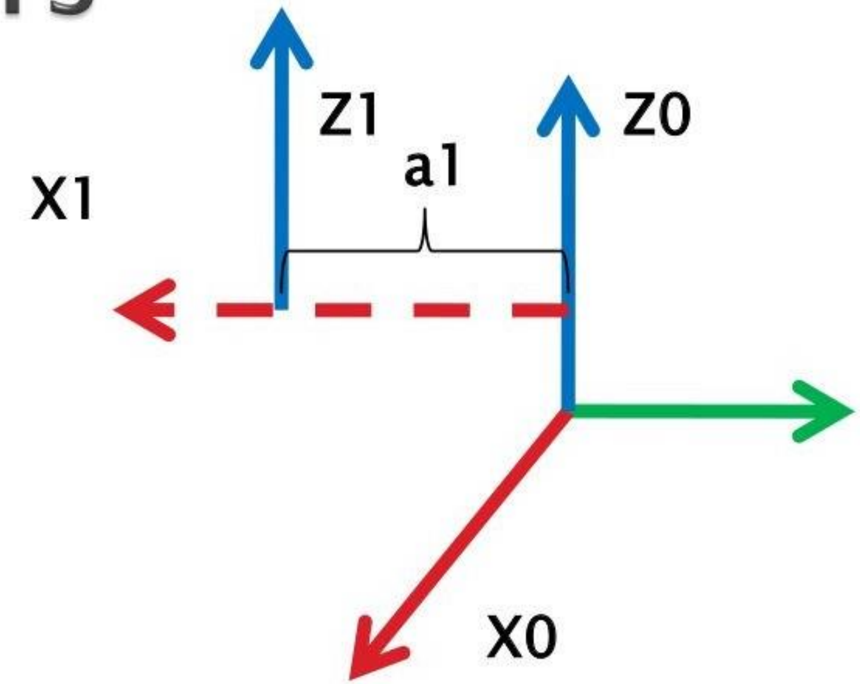
- (joint angle) θ_1 is angle from x0 to x1 measured about Z0

DH prameters



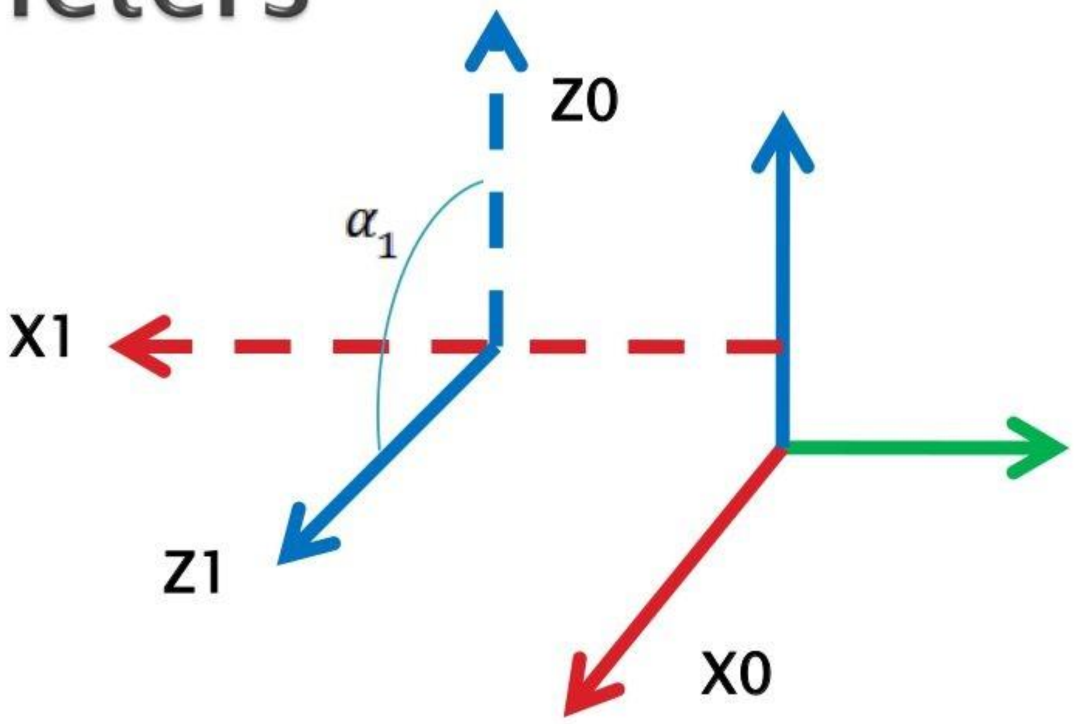
- (Link Offset) d1 distance from O0 to O1 measured along z0

DH prameters



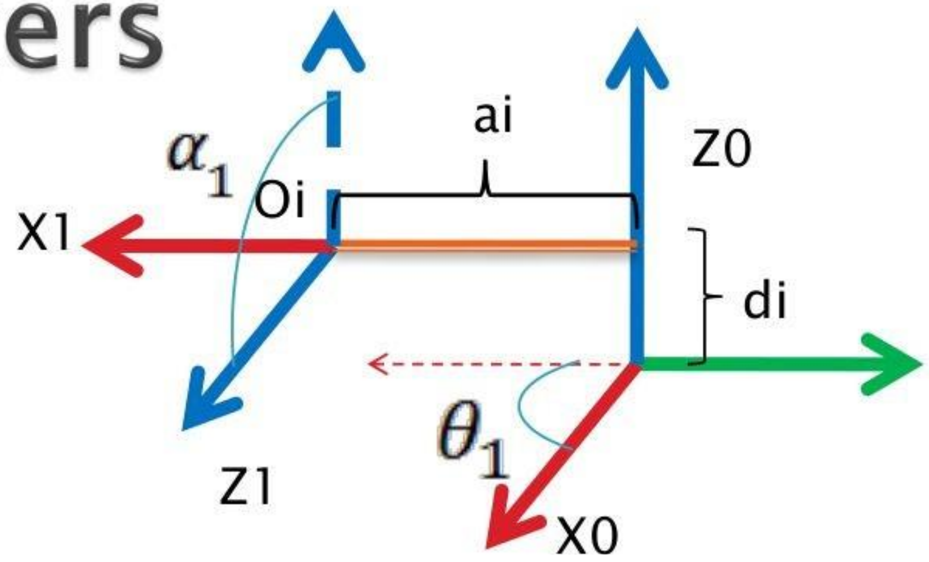
- (Link Length) a1 distance from z0 to z1 measured along x1

DH prameters



- (Link twist) α_1 is angle from z0 to z1 measured about x1

DH prameters



1. *Link length* a_i is the distance between z_{i-1} and z_i axes along the x_i-axis. a_i is the *kinematic length* of link (i).

2. *Link twist* α_i is the required rotation of the z_{i-1}-axis about the x_i-axis to become parallel to the z_i-axis.

3. *Joint distance* d_i is the distance between x_{i-1} and x_i axes along the z_{i-1}-axis. Joint distance is also called *link offset.*

4. *Joint angle* θ_i is the required rotation of x_{i-1}-axis about the z_{i-1}-axis to become parallel to the x_i-axis.

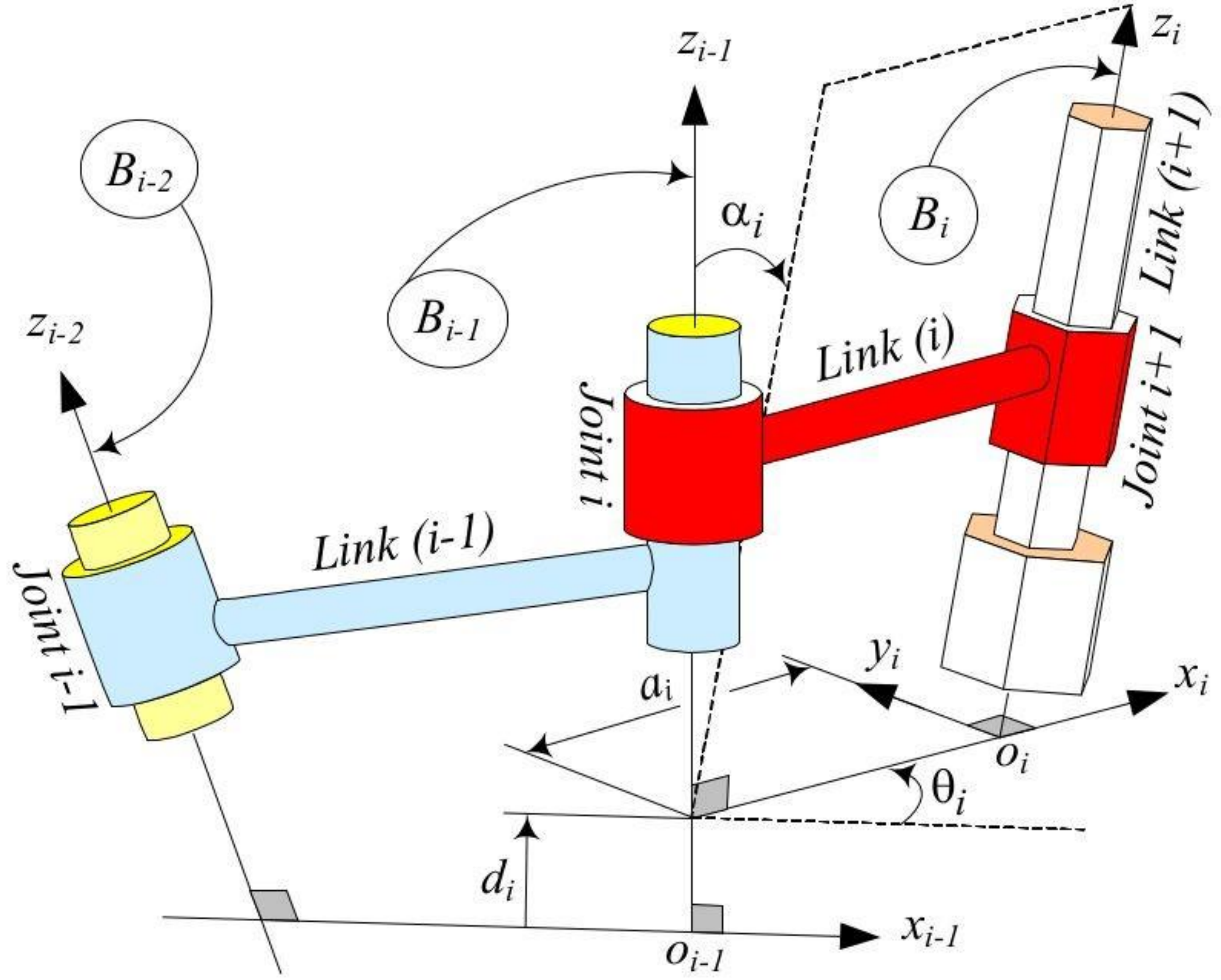


FIGURE 5.3. DH parameters $a_i, \alpha_i, d_i, \theta_i$ defined for joint i and link (i).

DH Techniques

- Matrix A_i representing the four movements is found by: four movements

1. Rotation of θ about current Z axis
2. Translation of d along current Z axis
3. Translation of a along current X axis
4. Rotation of α about current X axis

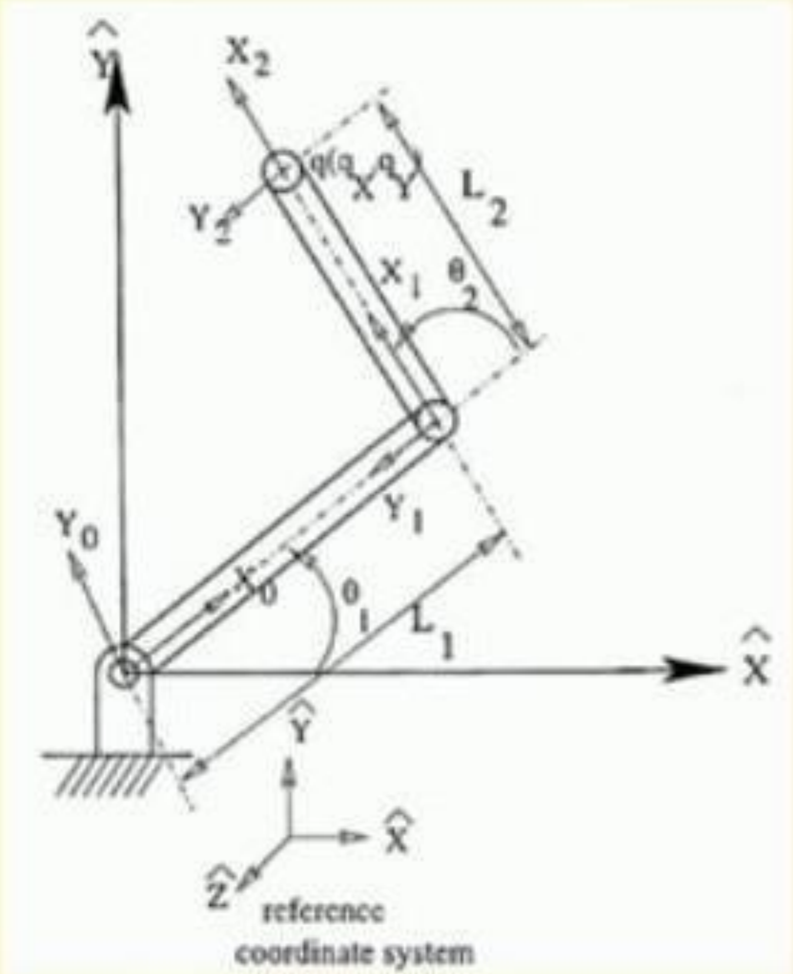
$$A_i = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$A_i = R_{z,\theta_i} \mathrm{Trans}_{z,d_i} \mathrm{Trans}_{x,a_i} R_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1

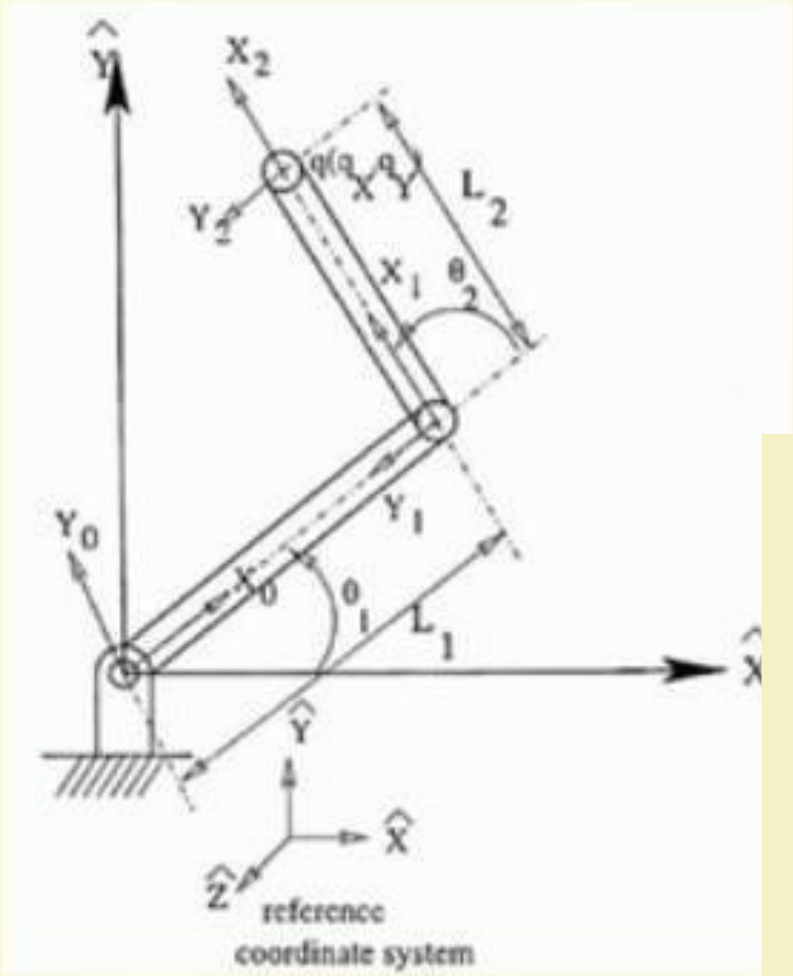


Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$^{Base}_{2}T = {}^{Base}_{1}T\,^{1}_{2}T$$

$$\begin{aligned} ^{Base}_{1}T &= ROT(\hat{Z},\theta_1)TRANS(\hat{X},L_1) \\ &= \begin{bmatrix} c_1 & -s_1 & 0 & L_1c_1 \\ s_1 & c_1 & 0 & L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

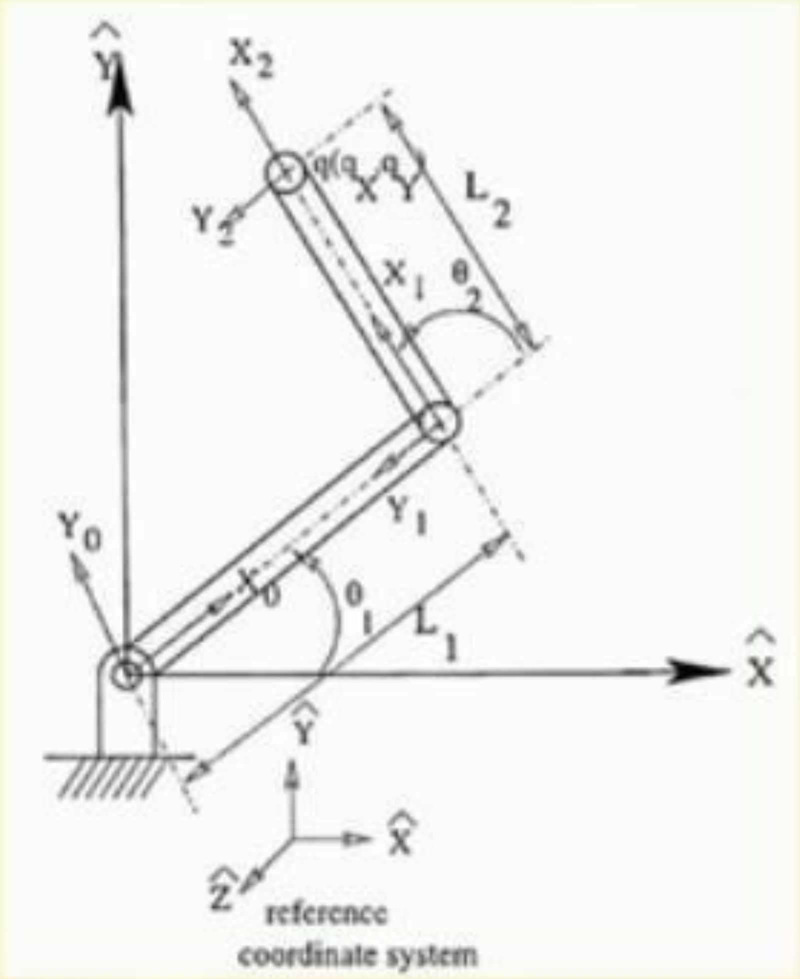
Example 1



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$
{}^{1}_{2}T = ROT(\hat{Z},\theta_2)TRANS(\hat{X},L_2) = \begin{bmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

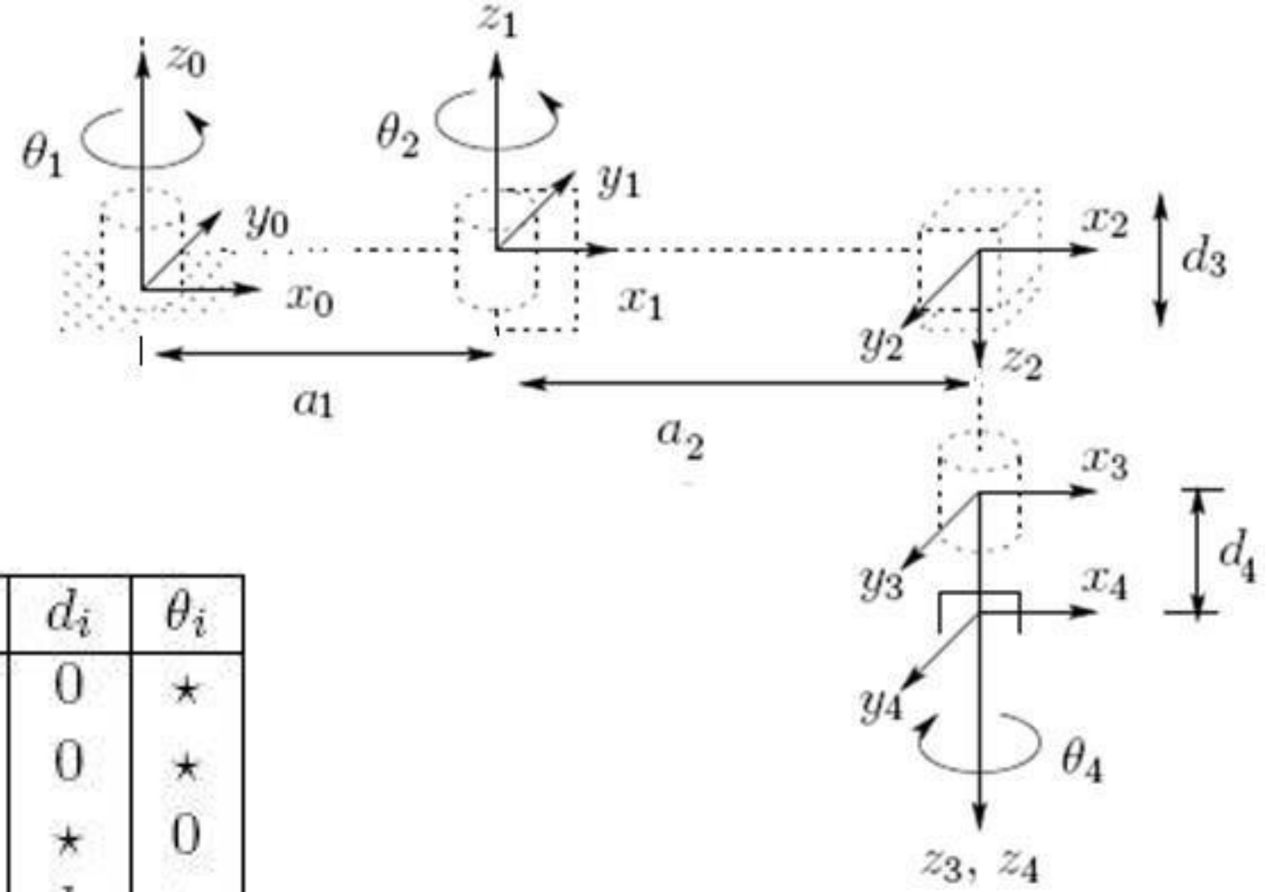
Example 1



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$
\begin{aligned}
{}^{Base}_{2}T &= {}^{Base}_{1}T\,{}^{1}_{2}T \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & L_1c_1 + L_2c_{12} \\ s_{12} & c_{12} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example 2



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

Example 2

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

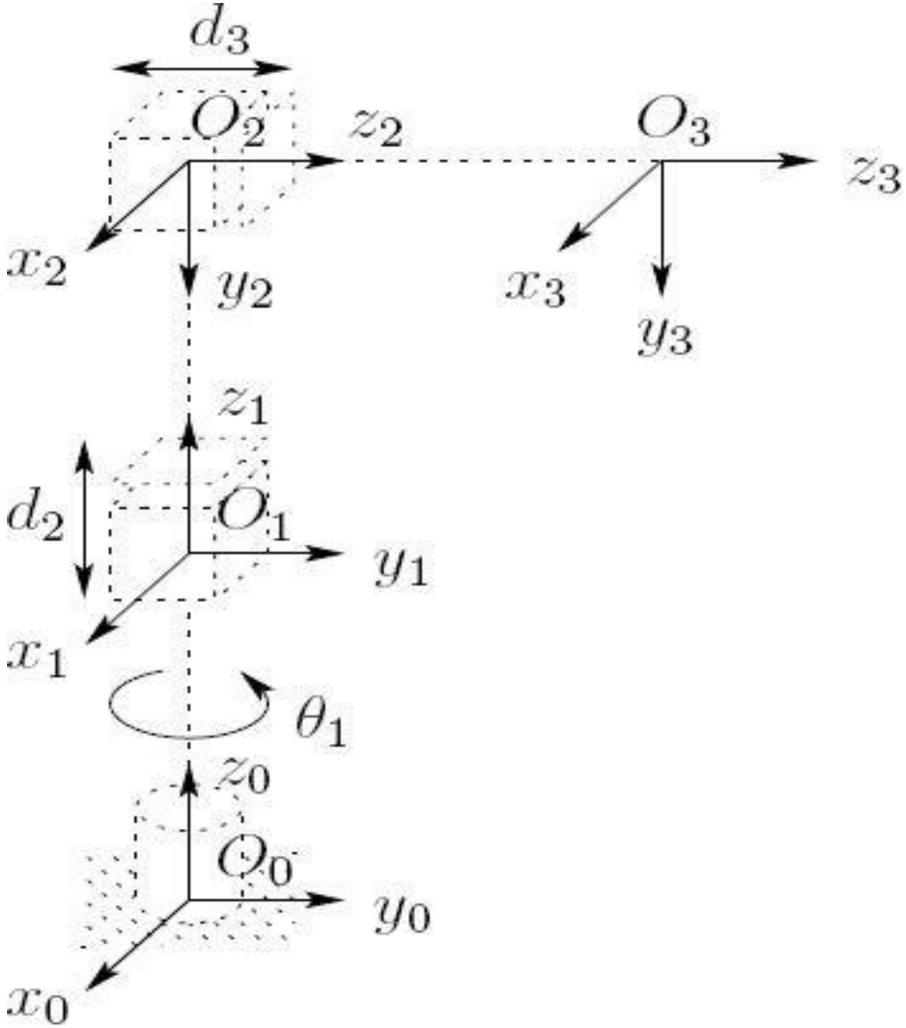
$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3
The three links cylindrical

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable



Example 3
The three links cylindrical

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

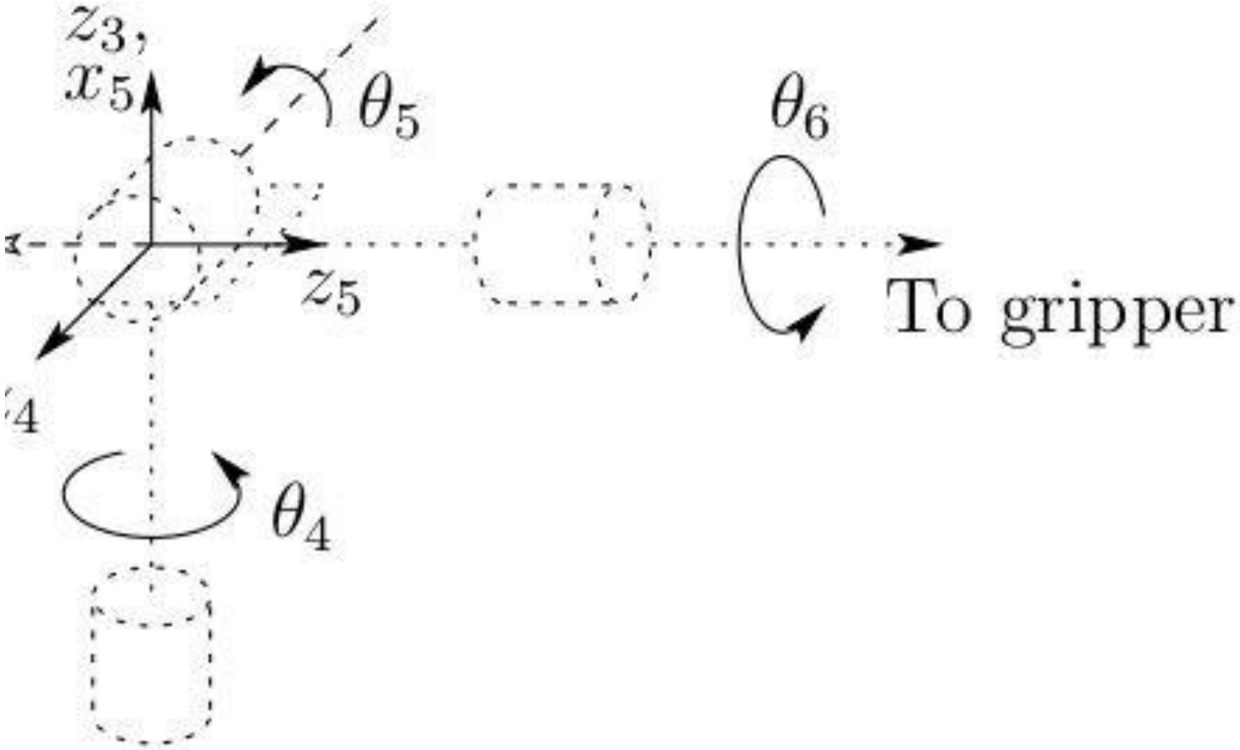
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4
Spherical wrist

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable



Example 4
Spherical wrist

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

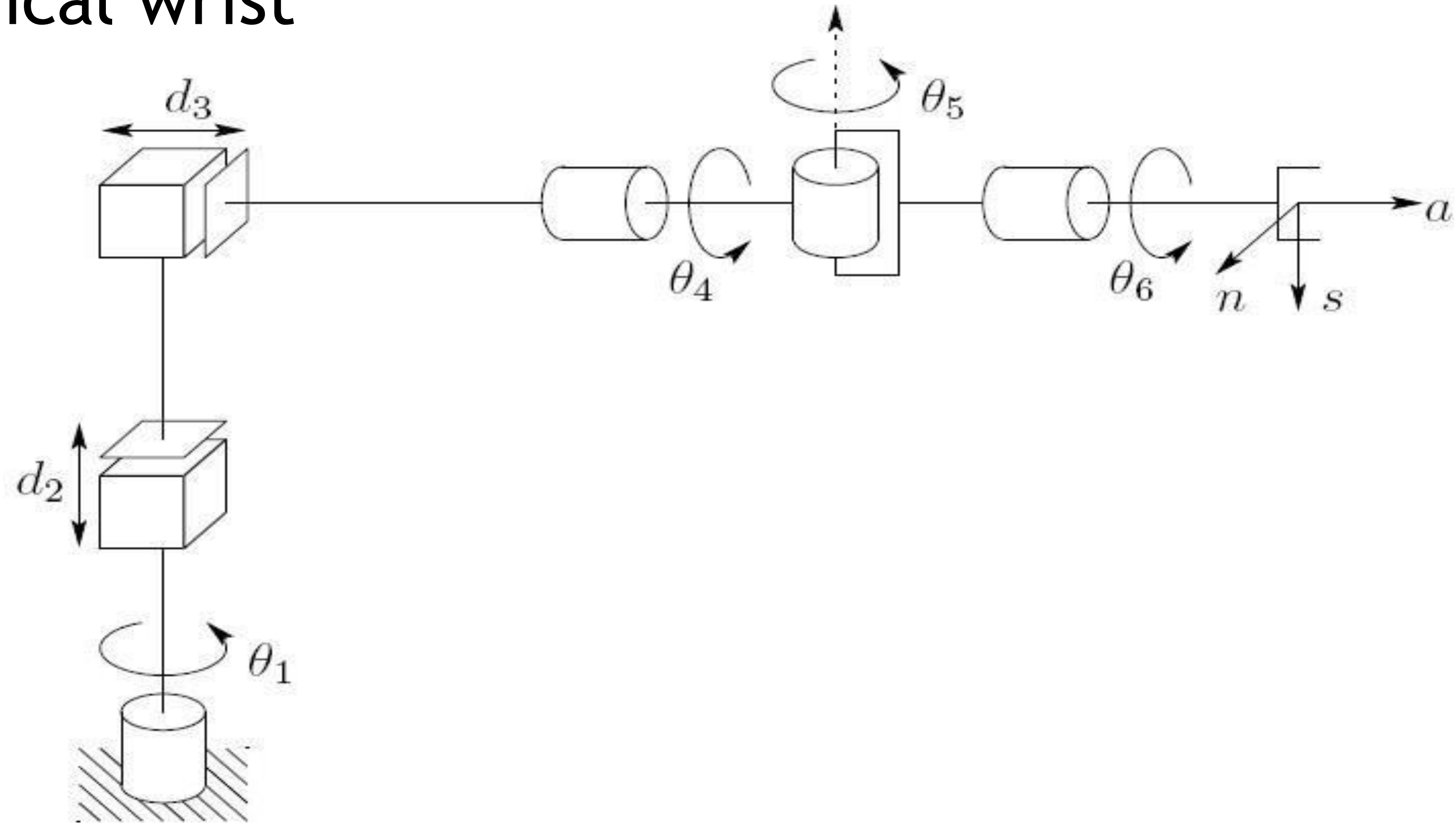
$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} R_6^3 & O_6^3 \\ 0 & 1 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 5

The three links cylindrical with

Spherical wrist



Example 5

The three links cylindrical with

Spherical wrist

$$T_6^0 = T_3^0 T_6^3$$

- given by example 3 given by example 4.

T_3^0

T_3^6

Example 5

The three links cylindrical with Spherical wrist

$$T_6^0 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_1 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2.
\end{aligned}$$

Frame No.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3

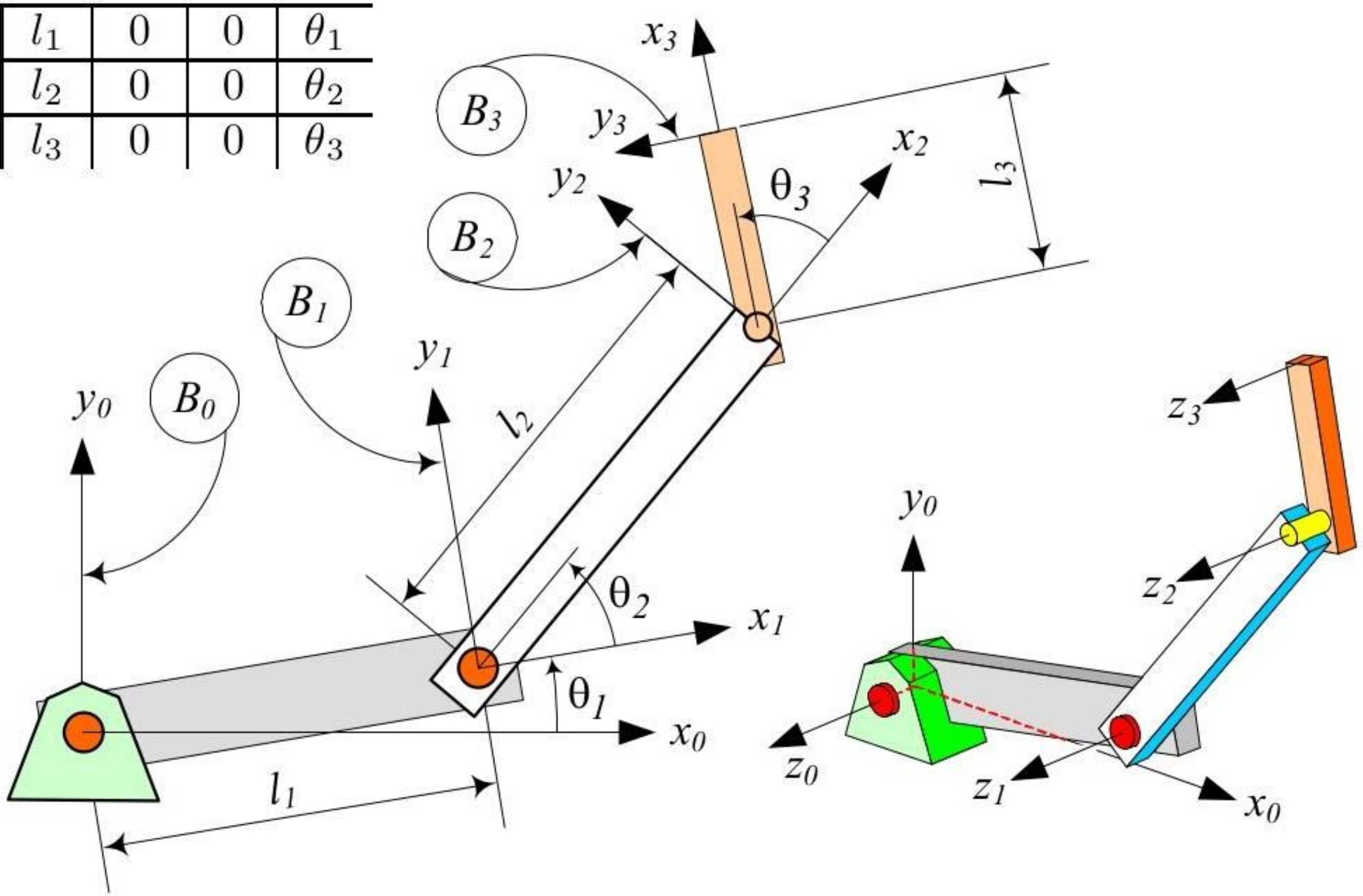
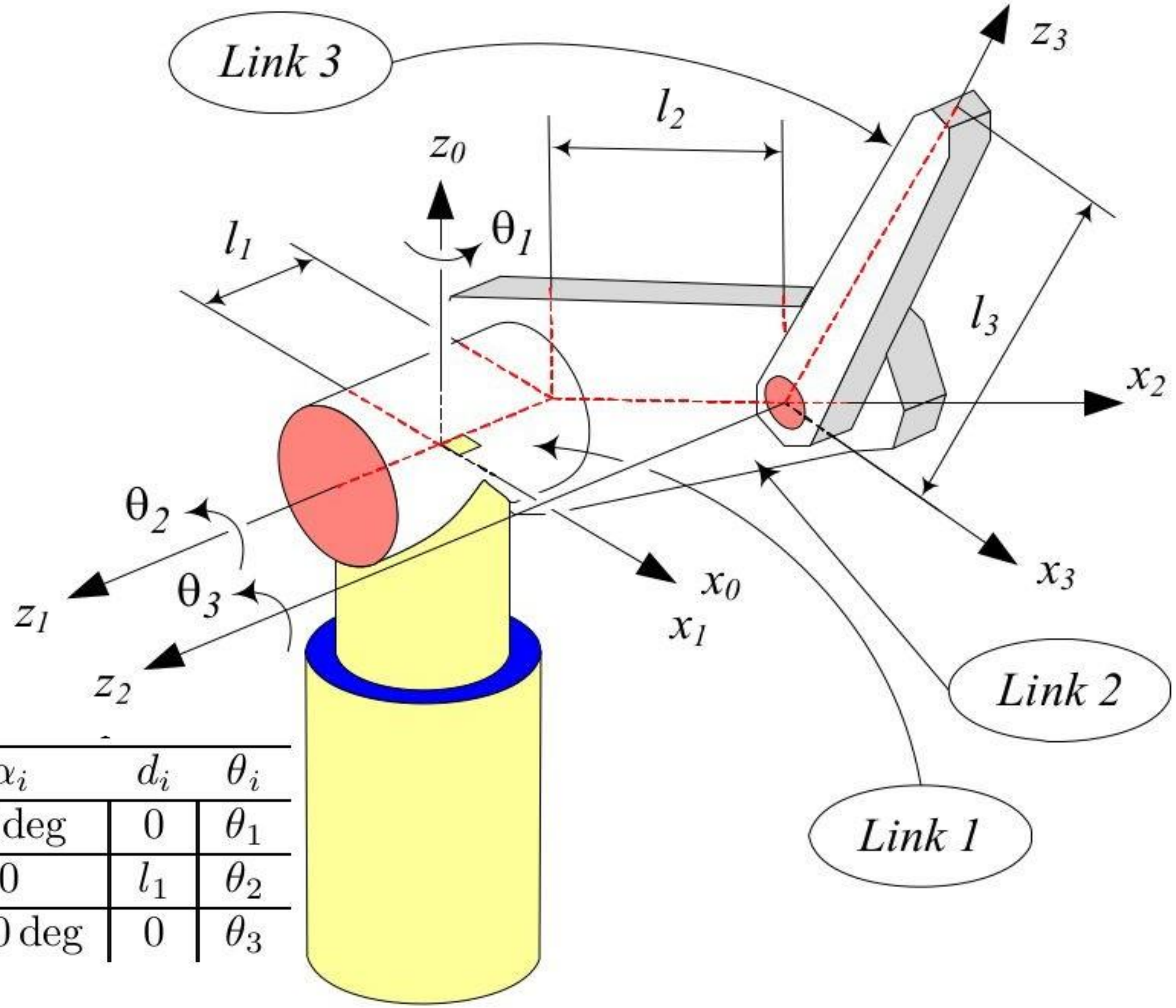


FIGURE 5.4. Illustration of a 3R planar manipulator robot and DH frames of each link.



Frame No.	a_i	α_i	d_i	θ_i
1	0	90 deg	0	θ_1
2	l_2	0	l_1	θ_2
3	0	−90 deg	0	θ_3

FIGURE 5.5. 3*R* *PUMA* manipulator and links coordinate frame.

References

- Lecture on Kinematics-Fall2019 by Honorable Prof. Dr. Syed Akhter Hossain Sir

- Lectures by honourable Prof D K Pratihar of NPTEL
- https://youtu.be/6Wb0rmIvllI
- https://youtu.be/AbRhzpReb2Q
- https://youtu.be/h4_2xAPj3y0