Propositional Equivalences
Discrete Mathematics — CSE 131

Definitions: Tautology, Contradiction and Contingency

Definition

A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a **tautology**.

Definition

A compound proposition that is always false, no matter what the truth values of the propositions that occur in it, is called a **contradiction**.

Definition

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example of a Tautology

The compound proposition $p \vee \neg p$ is a tautology because it is always true.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Example of a Contradiction

The compound proposition $p \wedge \neg p$ is a contradiction because it is always false.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Definition: Logical Equivalence

Definition

The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Note: The notation $p \Leftrightarrow q$ is also commonly used.

Example of a Logical Equivalence

The following truth table shows that the biconditional statement $(\neg p \vee q) \leftrightarrow (p \rightarrow q)$ is always true no matter what the truth values of the propositions p and q.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$(\neg p \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Therefore $(\neg p \vee q) \equiv (p \rightarrow q)$. This equivalence is called the **disjunctive normal form of the implication** (DNFI).

Augustus De Morgan

Born on June 27, 1806 in Madras, India.
Died on Mars 18, 1871 in London, England.

www-groups.dcs.st-and.ac.uk/
~history/Mathematicians/
De_Morgan.html

De Morgan's Law 1

The compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Therefore $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$.

De Morgan's Law 2

The compound propositions $\neg(p \land q)$ and $\neg p \lor \neg q$ are logically equivalent.

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$	$\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Therefore $\neg(p \land q) \equiv (\neg p \lor \neg q)$.

Logical Equivalences

Equivalence	Name
$p \wedge \mathrm{T} \equiv p$ $p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}$ $p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee \neg p \equiv \mathrm{T}$ $p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Logical Equivalences (continued)

Equivalence	Name
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv (\neg p \lor \neg q)$ $\neg(p \lor q) \equiv (\neg p \land \neg q)$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws

Logical Equivalences Involving Conditional Statements

$$p \rightarrow q \equiv \neg p \vee q \quad \text{(DNFI)}$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p \quad \text{(contrapositive)}$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Logical Equivalences Involving Biconditionals

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Disjunctive Normal Form

Definition

A compound proposition is said to be in **disjunctive normal form** if it is a disjunction of conjunctions of the variables or their negations.

For example: $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$.

Let a compound proposition that uses n propositional variables. This compound proposition is logically equivalent to a disjunctive normal form. Indeed, it is sufficient to write a conjunction for each combination of truth values for which the compound proposition is true.

Disjunctive Normal Form

The truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

is given by

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

From the first and the third row, this compound proposition is logically equivalent to the disjunctive normal form:

$$(p \wedge q) \vee (\neg p \wedge q).$$

Functionally Completeness

Definition

A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

For example, because any compound proposition is equivalent to a disjunctive normal form, then the collection of logical operators $\{\vee, \wedge, \neg\}$ is functionally complete.