Sets
Discrete Mathematics

Historical Note: Georg Ferdinand Ludwig Philipp Cantor

Born March 3, 1845 in St. Petersburg, Russia.
Died January 6, 1918 in Halle, Germany.

www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Cantor.html

Definitions: Set and Object

Definition

A **set** is an unordered collection of **objects**, finite or infinite, that all possess the same property: membership of the set.

Definition

The objects in a set are called the **elements**, or **members**, of the set. The elements of a set are said to **belong** to that set. A set is said to **contain** its elements.

Notation: Membership

We write $a \in A$ to denote that a **is an element of** (or **belongs to**) the set A.

We write $a \notin A$ to denote that a **is not an element of** (or **does not belong to**) the set A.

Describing Sets

One way to describe a set is to list all the elements between braces.

The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

Describing Sets

Sometimes the brace notation is used to describe a set without listing all its members. Some elements of the set are listed, and then **ellipses** (...) are used when the general pattern of the elements is obvious.

- The set C of all the positive integers less than 100 can be written $C = \{1, 2, 3, ..., 99\}$.
- The set $\mathbb{N}$ of natural numbers can be written $\mathbb{N} = \{0, 1, 2, 3, ...\}$.
- The set $\mathbb{Z}$ of integers can be written $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.

Describing Sets

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members.

- The set O of all odd positive integers less than 10 can be written as
  $O = \{x \,|\, (x \in \mathbb{N}) \wedge (x < 10) \wedge (x \text{ is odd})\}$.
- The set $\mathbb{Q}$ of rational numbers can be written as
  $\mathbb{Q} = \{a/b \,|\, (a \in \mathbb{Z}) \wedge (b \in \mathbb{Z}) \wedge (b \neq 0)\}$.

Important Sets

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**.
- $\mathbb{Q} = \{p/q \mid (p \in \mathbb{Z}) \wedge (q \in \mathbb{Z}) \wedge (q \neq 0)\}$, the set of **rational numbers**.
- $\mathbb{R}$, the set of **real numbers**.
- $\mathbb{C}$, the set of **complex numbers**.

Definition: Equality of Sets

Definition

Two sets are **equal** if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

We write $A = B$ if A and B are equal sets.
Example: The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal.

Historical Note: John Venn

Born August 4, 1834 in Hull, England.
Died April 4, 1923 in Cambridge, England.

www-groups.dcs.st-and.ac.uk/
~history/Mathematicians/Venn.html

Definition: Venn Diagram

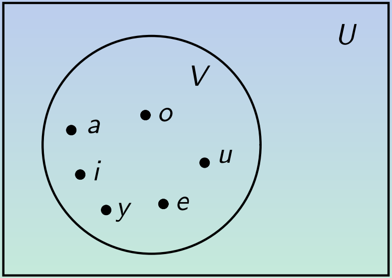
In **Venn diagrams**, the **universal set** U, which contains all the objects under consideration, is represented by a rectangle. Note that the universal set varies depending on which objects are of interest.

Inside this rectangle, circles or other geometrical figures are used to represent sets.

Sometimes points are used to represent the particular elements of the set.

Example of a Venn Diagram

Here is the Venn diagram of the set V of vowels in the English alphabet:



Note: Certain authors place the symbols V and U outside of the circle and rectangle respectively.

Definition: Empty Set and Singleton

Definition

There is a special set that has no elements. This set is called the **empty set**. It is denoted by $\emptyset$ or by $\{\,\}$.

Definition

A set that contains exactly one element is called a **singleton**.

Definition: Subset

Definition

The set A is said to be a **subset** of the set B, denoted $A \subseteq B$, if and only if every element of A is also an element of B.

- We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

  is true.
- When we wish to emphasize that a set A is a subset of the set B, but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. Some authors also use the notation $A \subsetneq B$.

Two Subsets of a Non-Empty Set

Theorem

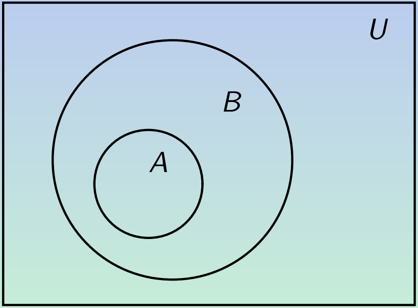
The empty set is a subset of all the sets. That is, $\emptyset \subseteq S$*, for any set* S.

Theorem

Every set is a subset of itself. That is, $S \subseteq S$*, for any set* S.

Subset

Here is a Venn diagram showing that A is a subset of B.



Definition: Cardinality

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a non negative integer, we say that S is a **finite set** and that n is the **cardinality** of S.

The cardinality of S is denoted by $|S|$.

A set is said to be **infinite** if it is not finite.

The cardinality of the empty set is 0. That is, $|\emptyset| = 0$.

Definition: Power Set

Definition

Given a set S, the **power set** of S is the set of all the subsets of S. The power set of S is denoted by $P(S)$.

Let $S = \{0, 1, 2\}$. The power set of S is given by:
$P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

If $|S| = n$ then $|P(S)| = 2^n$.

Definition: Ordered n-tuple

Definition

The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Two n-tuples are equal if and only if each corresponding pair of their elements is equal.

We call 2-tuples **couples** or **ordered pairs**.

Definition: Cartesian Product

Definition

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Note: The Cartesian products $A \times B$ and $B \times A$ are, in general, not equal.

Definition: Cartesian Product

Definition

The **Cartesian product** of n sets A_1, A_2, ..., A_n, denoted by $A_1 \times A_2 \times ... \times A_n$ is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$, where a_i belongs to A_i, for $i = 1, 2, ..., n$. In other words,

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) \,|\, a_i \in A_i, \text{ for } i = 1, 2, ..., n\}.$$

Some authors use the notation

$$A_1 \times A_2 \times ... \times A_n = \prod_{i=1}^{n} A_i.$$