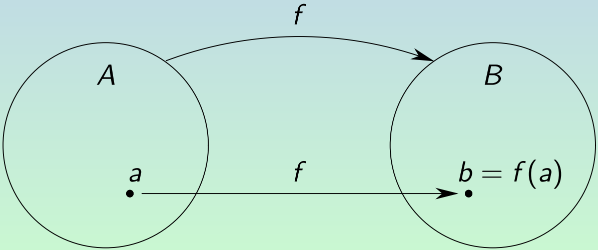
Functions
Discrete Mathematics

Definition: Function

Definition

Let A and B be non empty sets. A **function** f from A to B is an assignment of *exactly one* element of B to each element of A. We write $f(a) = b$ if b is the *unique* element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \rightarrow B$.



Definitions: Domain, Codomain, Image, Preimage and Range

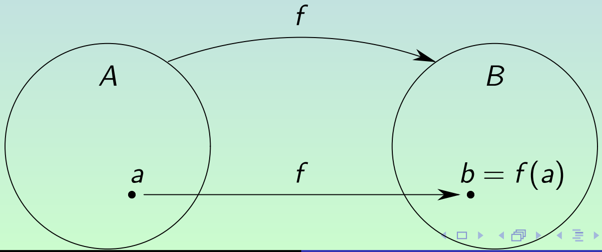
Definition

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.

If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b.

The **range** of f is the set of all images of elements of A.

Also, If f is a function from A to B, we say that f **maps** A to B.

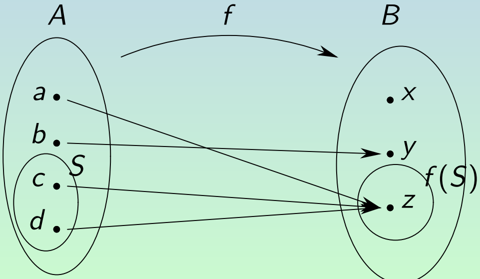


Definition

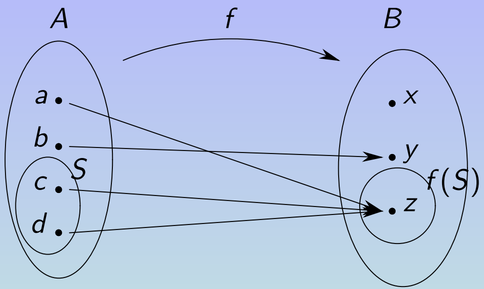
Let f be a function from the set A to the set B and let S be a subset of A. The **image** of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \in B \mid \exists s \in S \text{ with } (t = f(s))\}.$$

We also use the shorthand $f(S) = \{f(s) \mid s \in S\}$ to denote this set.



Example



- The domain of f is $A = \{a, b, c, d\}$.
- The codomain of f is $B = \{x, y, z\}$.
- $f(a) = z$.
- The image of a is z.
- The preimages of z are a, c and d.
- The range of f is $f(A) = \{y, z\} \subseteq B$.
- The image of the subset $S = \{c, d\} \subseteq A$ is $f(S) = \{z\} \subseteq B$.

Definition: One-To-One (Injective) Function

Definition

A function f from A to B is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain A. A function is said to be an **injection** if it is injective.

By taking the contrapositive of the implication in this definition, a function is injective if and only if $a \neq b$ implies $f(a) \neq f(b)$.

Another way to understand it, a function is injective means that if an element of the codomain has a preimage, then it is a unique preimage.

Definition: Onto (Surjective) Function

Definition

A function f from A to B is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a **surjection** if it is surjective.

Another way to understand it, a function is surjective means that each element of the codomain has at least one preimage.

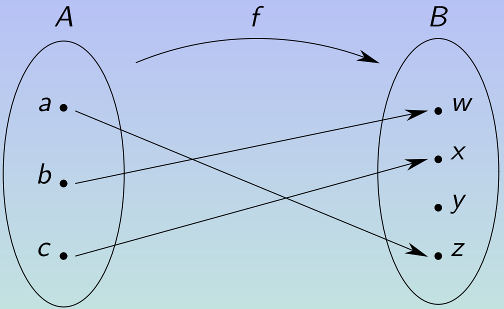
Definition: One-To-One Correspondence (Bijective) Function

Definition

The function f is a **one-to-one correspondence** if it is both one-to-one and onto.

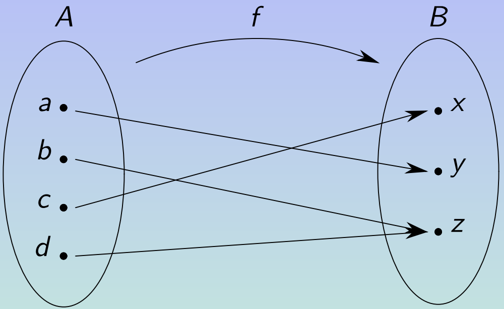
The function f is is said to be **bijective** if it is both injective and surjective. A function is said to be a **bijection** if it is bijective.

Example 1



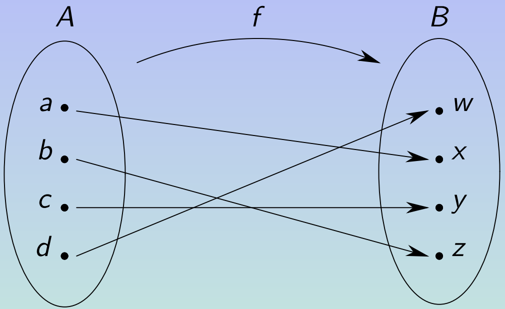
Is f injective?
Is f surjective?
Is f bijective?

Example 2

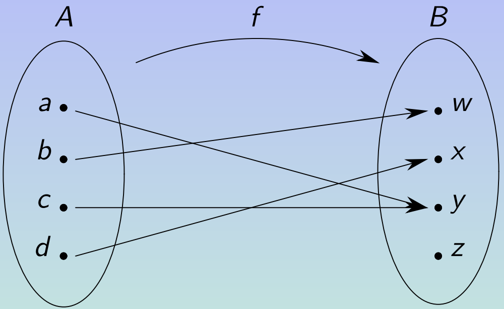


Is f injective?
Is f surjective?
Is f bijective?

Example 3

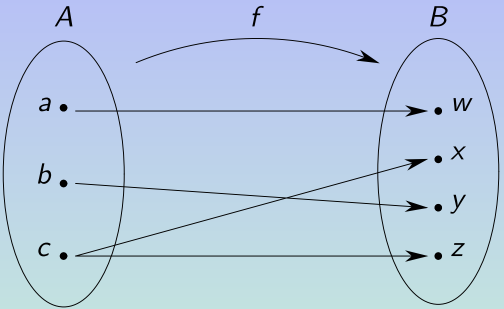


Is f injective?
Is f surjective?
Is f bijective?



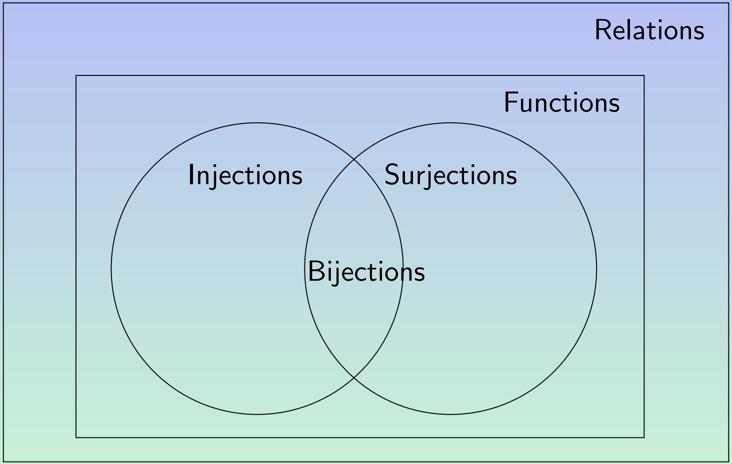
Is f injective?
Is f surjective?
Is f bijective?

Example 5



Is f injective?
Is f surjective?
Is f bijective?

Venn Diagram of Function Classification



Addition and Product of Functions

Definition

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

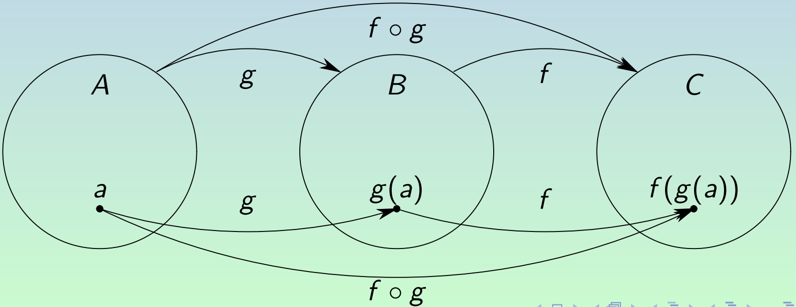
$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Definition: Composition of Functions

Definition

Let g be a function from the set A to the set B, and let f be a function from the set B to the set C. The **composition of the functions** f **and** g, denoted by $f \circ g$, is defined by

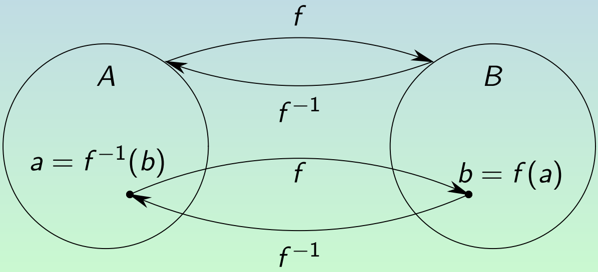
$$(f \circ g)(a) = f(g(a)).$$



Definition: Inverse Function

Definition

Let f be a bijection from the set A to the set B. The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$. The inverse function is also a bijection.



Identity Function

Definition

Identity function (also called identity mapping): The identity mapping $\mathbb{1}_X : X \to X$ is the function with domain and codomain X defined by

$$\mathbb{1}_X(x) = x, \quad \forall x \in X.$$

Left and Right Inverse

Definition

Let $f : X \to Y$ be a fonction with domain X and codomain Y , and $g : Y \to X$ be a fonction with domain Y and codomain X.

The function g is a **left inverse** of f if $g \circ f = \mathbb{1}_X$.

The function g is a **right inverse** of f if $f \circ g = \mathbb{1}_Y$.

The function g is an **inverse** of f if g is both a left and right inverse of f. When f has an inverse, it is often written f^{-1}.

Left and Right Inverse

Theorem

A function is **injective** *if and only if it has a* **left inverse***.*

A function is **surjective** *if and only if it has a* **right inverse***.*

A function is **bijective** *if and only if it has an* **inverse***.*

If a function has an inverse, then this **inverse is unique***.*

Note: The left and right inverses are not necessarily unique.