Graphs and Graph Models

Definition: Pair

Definition

Let V be a set and u and v be two elements belonging to this set. A **pair** in V is a set of two elements of V. In other words, a pair in V is a set $\{u, v\}$ such as $u, v \in V$.

Remarks:

- The two pairs $\{u, v\}$ and $\{v, u\}$ are identical; the order of the elements in the set has no significance.
- The pair $\{u, v\}$ implies that $u \neq v$.
- The pair $\{u, u\}$ is written like a singleton $\{u\}$.

Definition: Ordered Pair

Definition

The **ordered pair** (u, v) is the **ordered** collection that has u as its first element and v as its second element.

Let V be a set. The **Cartesian product** in V, written $V \times V$, is

$$V \times V = \{(u, v) \,|\, u \in V \wedge v \in V\}.$$

Remarks:

- Two ordered pairs are equal only if the corresponding elements are equal.
- There is an order: $(u, v) \neq (v, u)$.
- The ordered pair (u, u) is an element of the set $V \times V$.

Definition: Graph

Definition (p. 589)

A **graph** $G = (V, E)$ consists of V, a non empty set of **vertices** (or **nodes**) and E, a set of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.

Remarks

Remark: The set of vertices can not be empty. The set of vertices V of a graph G may be infinite. A graph with an infinite Vertex set is called an **infinite graph**, and in comparison, a graph with a finite vertex set is called a **finite graph**. $\diamond$

Remark: The set of edges may be empty. $\diamond$

Definition: Edges

Vertices are linked two by two by a possibly empty set E of edges.

- An edge between the vertices u and v can be **bidirectional** (or **undirected**) and is defined by the **pair** $\{u, v\}$.
- An edge between the vertices u and v can be **directional** (or **directed**). It is defined by the **ordered pair** (u, v).
- We can allow (or not) many edges between two vertices.
- We can allow (or not) edges to be between a vertex and itself (a **loop**).
- According to the type of edges allowed, graphs are classified as simple graphs, multigraphs, pseudographs, directional graphs and directional multigraphs.

Definition: Simple Graph

Definition

A **simple graph** $G = (V, E)$ consists of a non empty set V of vertices and a set E of edges made of **pairs** of distinct elements of V.

Remarks:

- $E = \{\{u, v\} \mid u, v \in V \wedge u \neq v\}$.
- Edges are not directed.
- A simple graph is also a **undirected graph**.
- There can not be more than one edge between two vertices.
- There can not be a loop.

Definition: Multigraph

Definition

A **multigraph** $G = (V, E)$ consists of a non empty set V of vertices, a set E of edges made of **pairs** of distinct elements of V, and a function f from E to $\{\{u, v\} \mid u, v \in V \wedge u \neq v\}$. Edges e_1 and e_2 are called **multiple edges** (or **parallel edges**) if $f(e_1) = f(e_2)$.

Remarks:

- Edges are not directed.
- There can be more than one edge between two vertices. When there are m different edges associated to the same pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m.
- There can not be a loop.

Definition: Pseudograph

Definition

A **pseudograph** $G = (V, E)$ consists of a non empty set V of vertices, a set E of edges made of **pairs** of elements of V, and a function f from E to $\{\{u, v\} \mid u, v \in V \wedge u \neq v\}$. An edge is a **loop** if $f(e) = \{u, u\} = \{u\}$ for a $u \in V$.

Remarks:

- Edges are not directed.
- There can be more than one edge between two vertices.
- There can be a loop.

Definition: Simple Directed Graph

Definition

A **simple directed graph** (or **digraph**) $G = (V, E)$ consists of a non empty set of vertices V and a set of **directed edges** (or **arcs**) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to **start** at u and **end** at v.

Remarks:

- $E = \{(u, v) \mid u, v \in V\} \subseteq V \times V$.
- The edges are directed.
- There can not be multiple edges in the same direction between two vertices.
- There can not be loops.

Definition: Directed Multigraph

Definition

A **directed multigraph** $G = (V, E)$ consists of a non empty set of vertices V and a set of **directed edges** (or **arcs**) E, and a function f from E to $\{(u, v) \mid u, v \in V\}$. Edges e_1 and e_2 are called **multiple directed edges** if $f(e_1) = f(e_2)$.

Remarks:

- The edges are directed.
- There can be multiple edges in the same direction between two vertices.
- There can be loops.

Graph Terminology

Type	Edges	Multiple Edges?	Loops?
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Both	Yes	Yes

Note: The generic term **graph** describes graphs with or without directed edges, loops and multiple edges.