# Graph Terminology and Special Types of Graphs
# Discrete Mathematics

# Definition: Adjacent Vertices

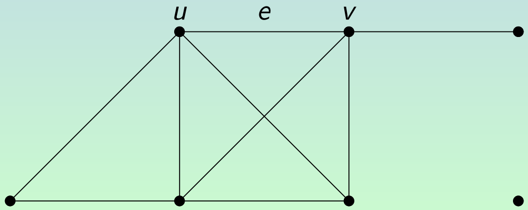
**Definition**

Two vertices $u$ and $v$ in an undirected graph $G$ are called **adjacent** (or **neighbors**) in $G$ if $u$ and $v$ are endpoints of an edge of $G$.

If $e$ is associated with $\{u, v\}$, the edge $e$ is called **incident with** the vertices $u$ and $v$.

The edge $e$ is also said to **connect** $u$ and $v$.

The vertices $u$ and $v$ are called **endpoints** of an edge associated with $\{u, v\}$.

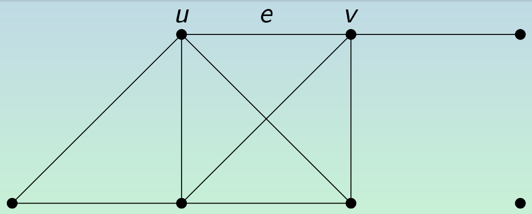


# Definition: The Degree of a Vertex

## Definition

The **degree of a vertex in an undirected graph** is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of a vertex.

The degree of the vertex $v$ is denoted by $\deg(v)$.

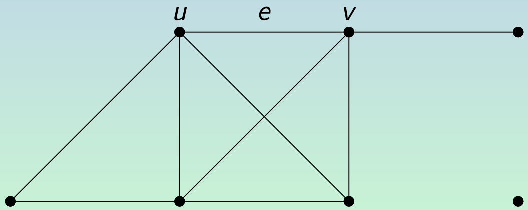


# Definition: Isolated and Pendant Vertices

## Definition

A vertex of **degree zero** is called **isolated**. It follows that an isolated vertex is not adjacent to any vertex.

A vertex is **pendant** if and only if it has a **degree one**. Consequently, a pendant vertex is adjacent to exactly one other vertex.



# The Handshaking Theorem

**Theorem**

*Let* $G = (V, E)$ *be an undirected graph with e edges. Then*

$$2e = \sum_{v \in V} \deg(v).$$

Note that this applies even if multiple edges and loops are present.

# Vertices of Odd Degree

**Theorem**

*An undirected graph has an even number of vertices of odd degree.*

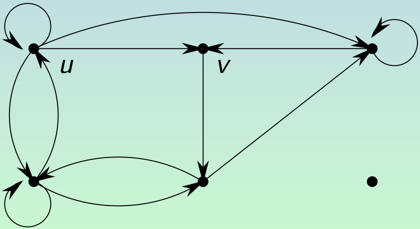
# Definition: Adjacent Vertex

**Definition**

When $(u, v)$ is an edge of the graph $G$ with directed edges, $u$ is said to be **adjacent** to $v$ and $v$ is said to be **adjacent** from $u$.

The vertex $u$ is called **initial vertex** of $(u, v)$ and $v$ is called the **terminal** or **end vertex** of $(u, v)$.

Remark: The initial vertex and and terminal vertex of a loop are the same.

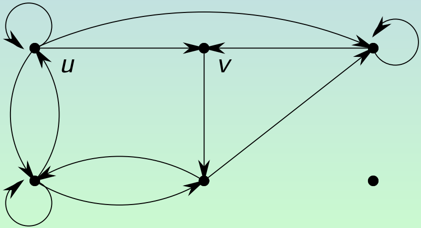


### Definition

In a graph with directed edges the **in-degree** of a vertex $v$, denoted by $\deg^{-}(v)$, is the number of edges with $v$ as their **terminal vertex**.

The **out-degree** of $v$, denoted by $\deg^{+}(v)$, is the number of edges with $v$ as their **initial vertex.**

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.



# Sum of In-Degrees and Out-Degrees

**Theorem**

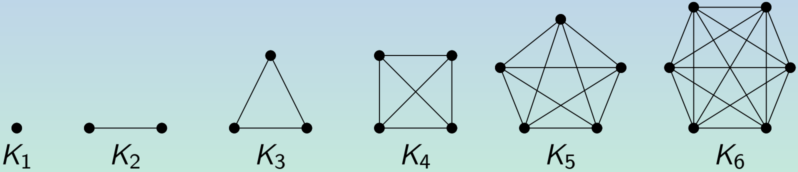
*Let $G = (V, E)$ be a graph with directed edges. Then*

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|.$$

# Definition: Complete Graph

**Definition**

The **complete graph** on $n$ vertices, denoted by $K_n$, is the simple graph that contains exactly one edge between each pair of distinct vertices.

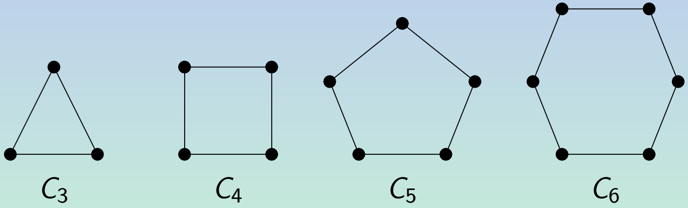



The graphs $K_n$ for $1 \leq n \leq 6$

# Definition: Cycle

**Definition**

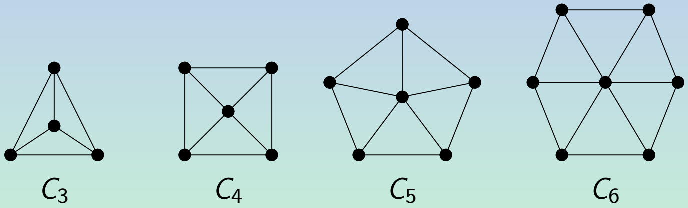
The **cycle** $C_n$, $n \geq 3$, consists of $n$ vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$.




The graphs $C_n$ , $3 \leq n \leq 6$

## Definition

We obtain the **wheel** $W_n$ when we add an additional vertex to the cycle $C_n$ for $n \geq 3$ and connect this new vertex to each of the $n$ vertices in $C_n$, by new edges.

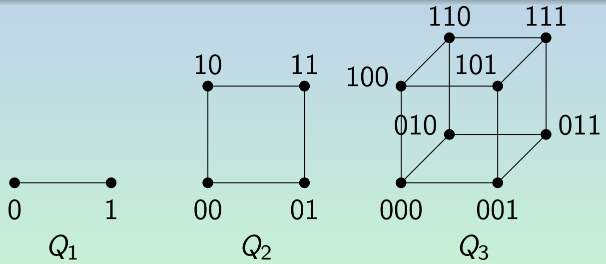



The graphs $W_n$ for $3 \leq n \leq 6$

# Definition: The $n$-Dimensional Hypercube

## Definition

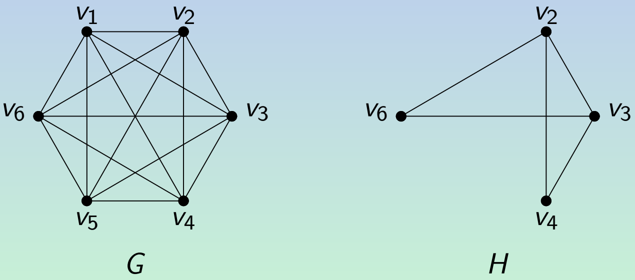
The $n$-**dimensional hypercube**, or $n$-**cube**, denoted by $Q_n$, is the graph that has vertices representing the $2^n$ bit strings of length $n$. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.




The graphs $Q_n$ for $1 \leq n \leq 3$

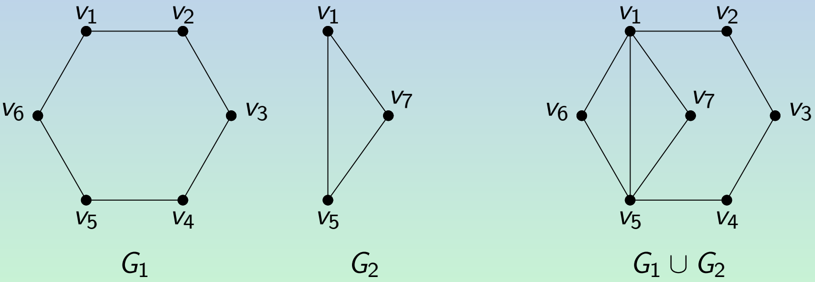
## Definition

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.



## Definition

The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$.



# Definition: Regular Graph

**Definition**

A simple graph is called **regular** if every vertex of this graph has the same degree.

A regular graph is called $n$-**regular** if every vertex in this graph has degree $n$.