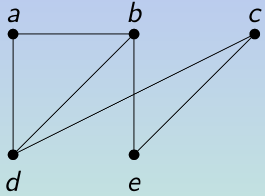
Representing Graphs
Discrete Mathematics

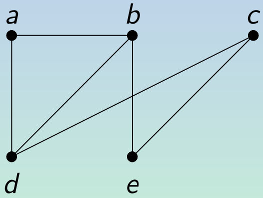
Representing Graphs

One way to represent a graph without multiple edges is to list all the edges of this graph.



$G = (V, E)$ with $V = \{a, b, c, d, e\}$ and
$E = \{\{a, b\}, \{a, d\}, \{b, d\}, \{b, e\}, \{d, c\}, \{e, c\}\}$.

An other way to represent a graph without multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.

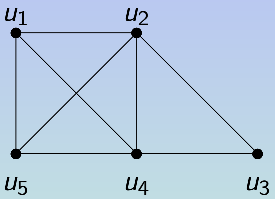


Vertex	Adjacent vertices
a	b, d
b	a, c, d, e
c	d, e
d	a, b, c
e	b, c

Definition: Adjacency Matrices

Definition

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily $v_1, v_2, \ldots, v_n$. The **adjacency matrix A** (or $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.



We order the vertices as u_1, u_2, u_3, u_4, u_5.

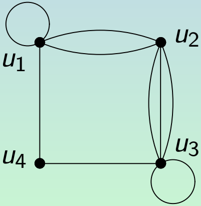
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Remarks on Adjacency Matrices

- Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there are as many as $n!$ different adjacency matrices for a graph with n vertices, because there are $n!$ different orderings of n vertices.
- The adjacency matrices of a simple graph is symmetric because if v_i is adjacent to v_j, then v_j is adjacent to v_i and if v_i is not adjacent to v_j, then v_j is not adjacent to v_i.
- Since a simple graph can not have a loop, $a_{ii} = 0$ for $i = 1, 2, ..., n$.

Adjacency Matrices for Pseudographs

- A loop on the vertex v_i is denoted by a 1 at the (i,i)th position of the adjacency matrix.
- When there are multiple edges between two vertices, the (i,j)th element of the adjacency matrix is equal to the number of edges between vertices v_i and v_j.
- All undirected graphs, including simple graphs, multigraphs and pseudographs, have symmetric adjacency matrices.



$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

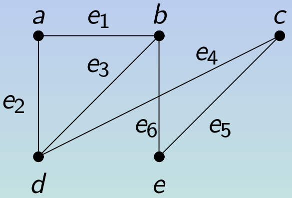
Definition: Incidence Matrices

Definition

Let $G = (V, E)$ be an undirected graph, in which $|V| = n$ and $|E| = m$. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the **incidence matrix** with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

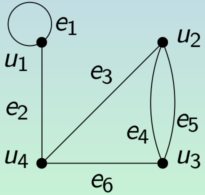
Example of an Incidence Matrix



	e_1	e_2	e_3	e_4	e_5	e_6
a	1	1	0	0	0	0
b	1	0	1	0	0	1
c	0	0	0	1	1	0
d	0	1	1	1	0	0
e	0	0	0	0	1	1

Incidence Matrix for Pseudographs

- Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop.
- Multiple edges are represented in the incidence matrix using columns with identical entries, because these edges are incident with the same pair of vertices.



	e_1	e_2	e_3	e_4	e_5	e_6
u_1	1	1	0	0	0	0
u_2	0	0	1	1	1	0
u_3	0	0	0	1	1	1
u_4	0	1	1	0	0	1