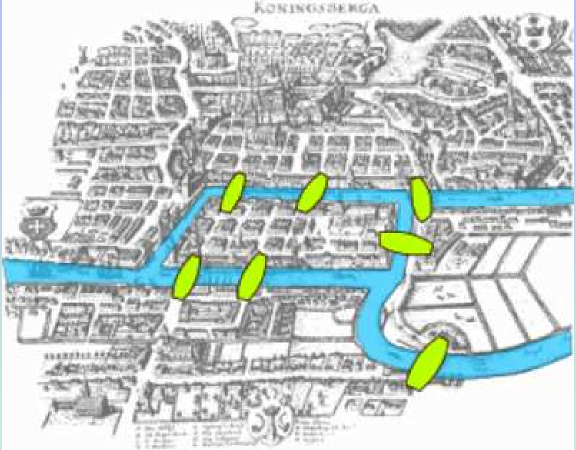
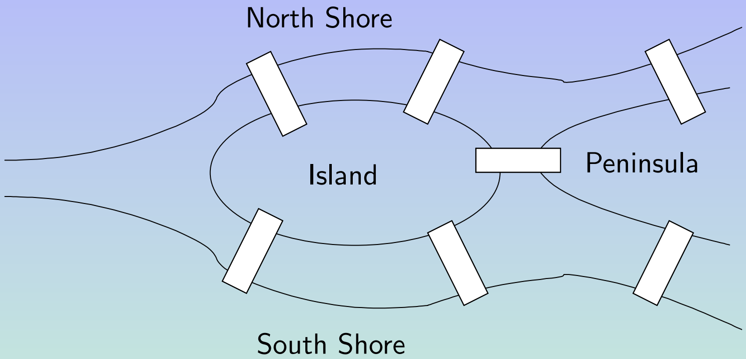
Euler Paths
Discrete Mathematics

The Seven Bridges of Königsberg



fr.wikipedia.org/wiki/Sept_ponts_de_K%C3%B6nigsberg

The Seven Bridges of Königsberg



Problem: The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, travel across all the bridges without crossing any bridge twice, and return to the starting point.

Historical Note: Leonhard Euler

Born April 15, 1707 in Basel, Switzerland.

Died on September 18, 1783 in St-Petersburg, Russia.

`www-groups.dcs.st-and.ac.uk/`

`~history/Mathematicians/ Euler.html`

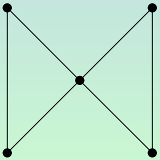
Euler Paths and Circuits

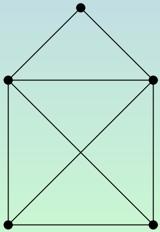
Definition

An **Euler circuit** in a graph G is a simple *circuit* containing every edge of G.

Definition

An **Euler path** in a graph G is a simple *path* containing every edge of G.



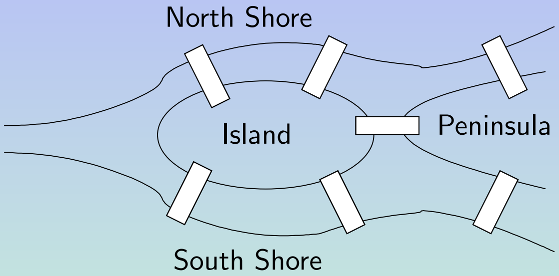


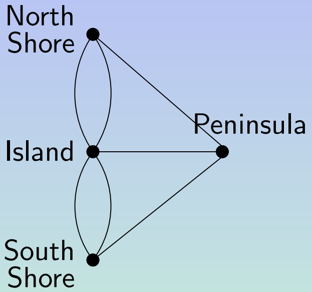
Necessary and Sufficient Conditions for Euler Circuits and Path

Theorem

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

The Seven Bridges of Königsberg



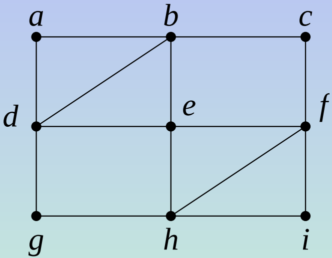


This problem has no solution because some vertices have odd degree (in fact, all vertices have odd degree).

Algorithm for Constructing Euler Circuits

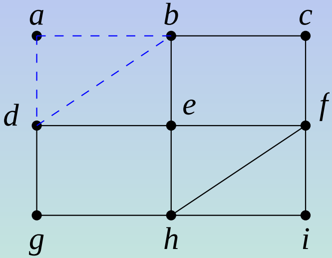
procedure *Euler*(G: Connected multigraph with all vertices
of even degree)
WIPEulerCircuit := a circuit in G beginning at an arbitrarily chosen
vertex with edges successively added to form a path
that returns to this vertex.
H := G with the edges of this *WIPEulerCircuit* removed.
while H has edges
begin
subcircuit := a circuit in H beginning at a vertex in H
that also is an endpoint of an edge of *WIPEulerCircuit*.
H := H with edges of *subcircuit* and
all isolated vertices removed.
WIPEulerCircuit := *WIPEulerCircuit* with *subcircuit* inserted
at the appropriate vertex.
end {*WIPEulerCircuit* is an Euler circuit}

Initial Problem:



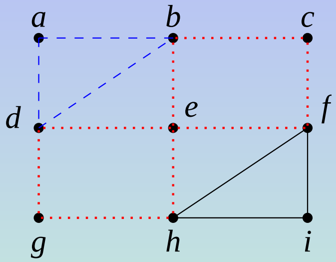
All vertices have even degree. An Euler circuit exists.

Step 1 of 3:



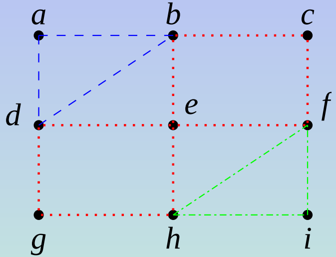
$WIPEulerCircuit := a, d, b, a$

Step 2 of 3:



$WIPEulerCircuit := a, d, b, a$
$subcircuit := d, e, f, c, b, e, h, g, d$
$WIPEulerCircuit := a, \ d, e, f, c, b, e, h, g, d, \ b, a$

Step 3 of 3:



$WIPEulerCircuit := a, d, e, f, c, b, e, h, g, d, b, a$

$subcircuit := f, i, h, f$

$WIPEulerCircuit := a, d, e, f, i, h, f, c, b, e, h, g, d, b, a$

Existence Condition of an Euler Path

Theorem

A connected multigraph has an Euler path, but not an Euler circuit, if and only if it has exactly two vertices of odd degree.

In this case, the Euler path starts and ends at these two vertices of odd degree.

Euler Circuits for Directed Graphs

Theorem

A weakly connected directed multigraph with at least two vertices has an Euler circuit if and only if each of its vertices satisfies $\deg^+(v) = \deg^-(v)$.

Theorem

A weakly connected directed multigraph with at least two vertices has an Euler path, but not an Euler circuit, if and only if each of its vertices satisfies $\deg^+(v) = \deg^-(v)$, *except for two vertices, one with* $\deg^+(v) = \deg^-(v) + 1$ *and one with* $\deg^+(v) = \deg^-(v) - 1$.

In this case, the Euler path starts at the vertex with $\deg^+(v) = \deg^-(v) + 1$ and ends at the vertex with $\deg^+(v) = \deg^-(v) - 1$.