Hamilton Paths
Discrete Mathematics

Born on August 4, 1805 in Dublin, Ireland.

Died on September 2, 1865 in Dublin, Ireland.

Mathematician and astronomer, inventor of quaternions.

`www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Hamilton.html`

Hamilton's "A Voyage Round the World" Puzzle

www.puzzlemuseum.com/month/picm02/200207icosian.htm

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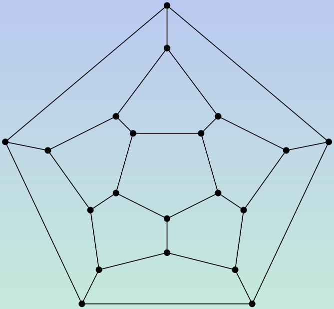

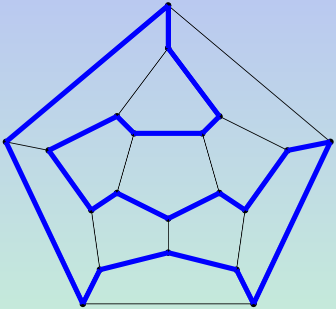

Hamilton Paths and Circuits

Definition

A *simple path* in a graph G that passes through every vertex exactly once is called a **Hamilton path**. In other words, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$.

A *simple circuit* in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**. In other words, the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

Necessary Conditions for Hamilton Circuits

A graph with a vertex of degree one cannot have a Hamilton circuit.

If a vertex in a graph has degree two, then both edge that are incident to this vertex must be part of any Hamilton circuit.

A Hamilton circuit cannot contain a smaller circuit within it.

Dirac's Theorem (1952)

Theorem

If G is a simple graph with n vertices, $n \geq 3$, such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

This theorem provides a sufficient condition, but not a necessary condition, for a connected simple graph to have a Hamilton circuit.

Gabriel Andrew Dirac (Budapest, March 13, 1925 – Arlesheim, Switzerland, June 20, 1984) was a mathematician who mainly worked in graph theory. Dirac was professor of mathematics in the University of Aarhus in Denmark.

Ore's Theorem (1960)

Theorem

If G is a simple graph with n vertices, $n \geq 3$, such that $\deg(u) + \deg(v) \geq n$ for every pair of non adjacent vertices u and v in G, then G has a Hamilton circuit.

This theorem provides a sufficient condition, but not a necessary condition, for a connected simple graph to have a Hamilton circuit.

Øystein Ore (7 October 1899 in Oslo, Norway – 13 August 1968 in Oslo) was a Norwegian mathematician. He was professor of mathematics at Yale.

Bondy and Chvátal Theorem (1972)

Theorem

Let $G = (V, E)$ be a simple graph with n vertices. Let u and v be two non adjacent vertices of G such that $\deg(u) + \deg(v) \geq n$. G has a Hamilton circuit if and only if $G' = (V, E \cup \{\{u, v\}\})$ has a Hamilton circuit.

John Adrian Bondy, a dual British and Canadian citizen, was a professor of graph theory at the University of Waterloo, in Canada. He is a faculty member of Université Lyon 1, France.

Václav (Vašek) Chvátal (born 1946 in Prague) is a professor in the Department of Computer Science and Software Engineering at Concordia University in Montreal, Canada, where he holds the Canada Research Chair in Combinatorial Optimization.

Systematic Determination whether a Graph is Hamiltonian

Theorem (The Golden Rule)

An Hamilton circuit must use exactly two edges at every vertex, and these two edges must link different vertices.

Corollary

- *If no edge is available at any vertex, the graph cannot have a Hamilton circuit.*
- *If only one edge is available at any vertex, the graph cannot have a Hamilton circuit.*
- *If a vertex belongs to exactly two edges, then these two edges must be part of the Hamilton circuit, if it exists.*
- *A loop cannot belong to an Hamilton circuit. Take them out.*
- *At most one edge linking two vertices can be used. Remove multiple copies of the same edge.*

Systematic Determination whether a Graph is Hamiltonian

Theorem (The Golden Rule)

An Hamilton circuit must use exactly two edges at every vertex, and these two edges must link different vertices.

Corollary

- *If two edges have already been used at a vertex, none of the other edges can be part of the corresponding Hamilton circuit, if it exists. Remove these other edges.*
- *If three edges (or more)* **must** *be used at a vertex, then the graph cannot have a Hamilton circuit.*
- *A Hamilton circuit cannot contain a proper subcircuit.*

The Method

- Remove all loops.
- Remove all extra copies of edges.
- At every step, try to find a new vertex for which the Golden Rule tells us unambiguously...
  - ... which edges must belong to a Hamilton circuit (if there is one)
  - ... which edges must not belong to a Hamilton circuit (if there is one)
  - ... that there is no Hamilton circuit.
- At every step, if the Golden Rule is not conclusive at any vertex. we choose one for which the number of cases to consider is the smallest. If the case chosen fails to produce a Hamilton circuit, we must step back and try the other cases.

(As usual, if at any moment, if there is a tie, choose in lexicographic order.)