Relations, Their Properties and Representations
Discrete Mathematics

Review: Ordered n-tuple

Definition

The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Two n-tuples are equal if and only if each corresponding pair of their elements is equal.

We call 2-tuples **couples** or **ordered pairs**.

Definition: Cartesian Product

Definition

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Note: The Cartesian products $A \times B$ and $B \times A$ are, in general, not equal.

Definition

Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$.

Example: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then
$R = \{(0, a), (0, b), (1, a), (2, b)\} \subseteq A \times B$
is a binary relation from A to B.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover, when $(a, b) \in R$, a is said to be **related to** b by R.

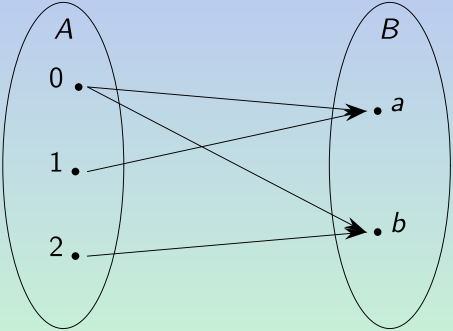
Representing Relations Using Digraphs

Definition

A **directed graph** $G = (V, E)$, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of **edges** (or **arcs**). The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

Example of a Digraph

Example: Let $A = \{0, 1, 2\}$, $B = \{a, b\}$ and the relation $R = \{(0, a), (0, b), (1, a), (2, b)\}$ from A to B.



Representing Relations Using Tables

Let R be a binary relation from A to B. Rows of a table representing the relation are an enumeration of the elements of the set A and the columns, an enumeration of the elements of the set B. There is a $\times$ at a given row and column of this table if the corresponding element of this row is related to the corresponding element of this column.

Example of a Table Representing a Relation

Example: Let $A = \{0, 1, 2\}$, $B = \{a, b\}$ and the relation $R = \{(0, a), (0, b), (1, a), (2, b)\}$ from A to B.

R	a	b
0	$\times$	$\times$
1	$\times$	
2		$\times$

Note: This table is more or less a matrix.

Representing Relations Using Matrices

Suppose that R is a relation from $A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$. Here, the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$, we use the same ordering for A and B. The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R, \end{cases}$$

In other words, the zero-one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. Such a representation depends on the ordering used for A and B.

Example of a Matrix Representing a Relation

Example: Let $A = \{0, 1, 2\}$, $B = \{a, b\}$ and the relation $R = \{(0, a), (0, b), (1, a), (2, b)\}$ from A to B.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Definition

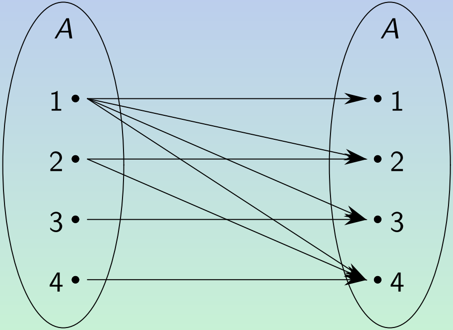
A **relation on a set** A is a relation from A to A.

Example: Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(a, b) \mid a \text{ divides } b\} \subseteq A \times A$.

The ordered pairs of this relation are given by
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

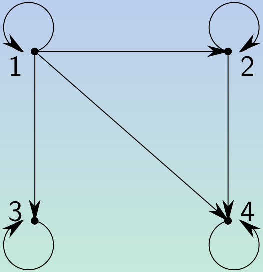
Representing a Relation Using a Directed Graph

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(a, b) \mid a \text{ divides } b\} \subseteq A \times A$. This relation is the set $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.



Representing a Relation on a Set Using a Directed Graph

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation on the set A given by $R = \{(a, b) \mid a \text{ divides } b\} \subseteq A \times A$. This relation is the set $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.



An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

Representing Relations on a Set Using Tables

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation on the set A given by $R = \{(a, b) \mid a \text{ divides } b\} \subseteq A \times A$. This relation is the set $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

R	1	2	3	4
1	$\times$	$\times$	$\times$	$\times$
2		$\times$		$\times$
3			$\times$	
4				$\times$

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation on the set A given by $R = \{(a, b) \mid a \text{ divides } b\} \subseteq A \times A$. This relation is the set $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Definition: Reflexive Relation

Definition

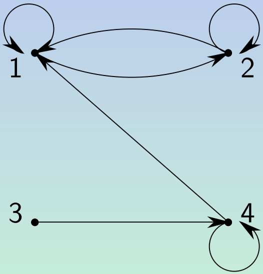
A relation R on a set A is called **reflexive** if $(a, a) \in R$ for all element $a \in A$.

Remark: Using quantifiers, a relation R on a set A is reflexive if $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

Example: Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation reflexive?

Representing a Relation on a Set Using a Directed Graph

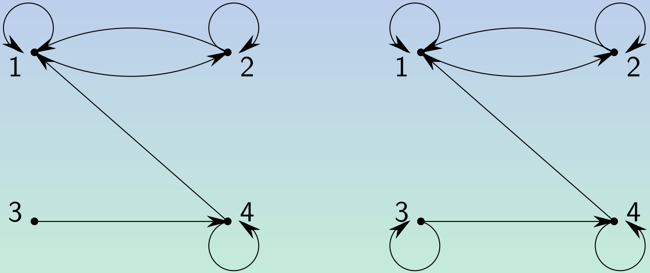
Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation reflexive?



If not, what is needed to made this relation reflexive?

Representing a Relation on a Set Using a Directed Graph

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation reflexive?



To be reflexive, a loop at each vertex is necessary.

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation reflexive?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

If not, what is needed to made this relation reflexive?

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation reflexive?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

To be reflexive, 1 on each entry of the main diagonal is necessary.

Definition: Symmetric Relation

Definition

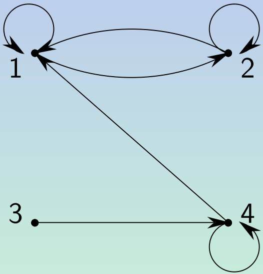
A relation R on a set A is called **symmetric** if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$.

Remark: Using quantifiers, a relation R on a set A is symmetric if $\forall a \forall b \, ((a, b) \in R \rightarrow (b, a) \in R)$.

Example: Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation symmetric?

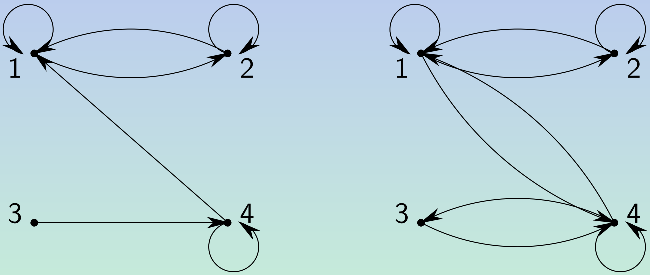
Representing a Relation on a Set Using a Directed Graph

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation symmetric?



If not, what is needed to made this relation symmetric?

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation symmetric?



To be symmetric, if there is an edge between two elements of the set, then there must be an edge in both directions.

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation symmetric?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

If not, what is needed to made this relation symmetric?

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation symmetric?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & \mathbf{1} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \mathbf{1} & 1 \end{bmatrix}.$$

To be symmetric, the matrix need to be symmetric.

Definition: Antisymmetric Relation

Definition

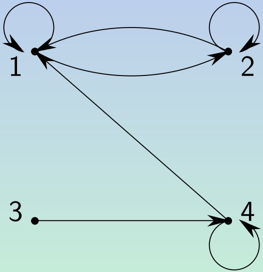
A relation R on a set A is called **antisymmetric** if, for all $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

Remark: Using quantifiers, a relation R on a set A is antisymmetric if $\forall a \forall b (((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$. The contrapositive is $\forall a \forall b ((a \neq b) \rightarrow ((a, b) \notin R \vee (b, a) \notin R))$.

Example: Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation antisymmetric?

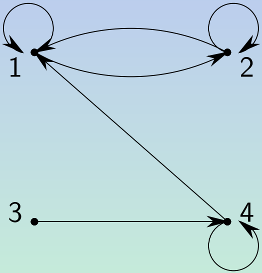
Representing a Relation on a Set Using a Directed Graph

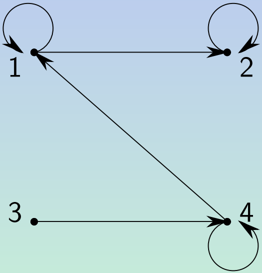
Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation antisymmetric?



If not, what is needed to made this relation antisymmetric?

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation antisymmetric?





To be antisymmetric, there should not be edges in both directions between two vertices.

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation antisymmetric?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

If not, what is needed to made this relation antisymmetric?

Representing Relations on a Set Using Matrices

Let A be the set $\{1,2,3,4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation antisymmetric?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \mathbf{0} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \mathbf{0} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

To be antisymmetric, for each 1 out of the diagonal, there should be a 0 at the corresponding transposed position.

Definition: Transitive Relation

Definition

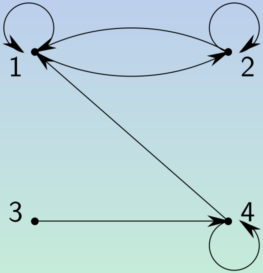
A relation R on a set A is called **transitive** if, whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers, a relation R on a set A is transitive if $\forall a \forall b \forall c (((a, b) \in R \land (b, c) \in R) \to (a, c) \in R)$.

Example: Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation transitive?

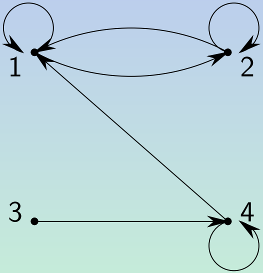
Representing a Relation on a Set Using a Directed Graph

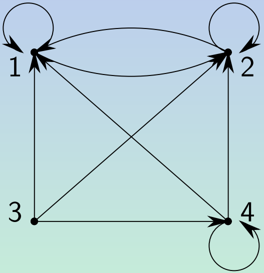
Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$. Is this relation transitive?



If not, what is needed to made this relation transitive?

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation transitive?





To be transitive, "triangular paths" must be closed.

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$. Is this relation transitive?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

If not, what is needed to made this relation transitive?

Representing Relations on a Set Using Matrices

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation transitive?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 1 \\ 1 & \mathbf{1} & 0 & 1 \end{bmatrix}.$$

To be transitive, the method used is quite complex...

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations form A to B can be combined in any way two sets can be combined.

Example: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$.
- $R_1 \cap R_2 = \{(1, 1)\}$.
- $R_1 - R_2 = \{(2, 2), (3, 3)\}$.
- $R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.