Partial Orderings
Discrete Mathematics

Introduction Example

- Let the set $S = \{1, 2, 3, 4, 6\}$ and the relation $R = \{(a, b) \in S \times S \text{ such that } a|b\}$.
- Let the set $S = \{1, 2, 3, 4\}$ and the relation $R = \{(a, b) \in S \times S \text{ such that } a \leq b\}$.
- Let the set $S = \{a, b, c\}$, the power set $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and the relation $R = \{(A, B) \in P(S) \times P(S) \text{ such that } A \subseteq B\}$.

What are the common properties of these relations?

Partial Ordering

Definition

A relation R on a set S is called a **partial ordering** or **partial order** if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R). Members of S are called **elements** of the poset.

Notation

In a partially ordered set (S, R), the notation $a \preccurlyeq b$ denotes that $(a, b) \in R$.

This notation is used because the "less than or equal to" relation on a set of real numbers is the most familiar example of a partial ordering and the symbol $\preccurlyeq$ is similar to the $\leq$ symbol.

The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also we say "a is less than b" or "b is greater than a" if $a \prec b$.

Comparable Elements

Definition

The elements a and b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

Total Order

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, then S is called a **totally ordered set** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or a **linear order**. A totally ordered set is also called a **chain**.

Lexicographic Order

The words in the dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set constructed from a partial ordering on the set.

Definition

Let the two posets $(S_1, \preccurlyeq_1)$ and $(S_2, \preccurlyeq_2)$. The **lexicographic order** $\preccurlyeq$ on the Cartesian product $S_1 \times S_2$ is defined by specifying that one pair is less than the other pair, i.e.

$$(a_1, a_2) \prec (b_1, b_2)$$

if and only if

$$a_1 \prec_1 b_1$$

or

$$a_1 = b_1 \text{ and } a_2 \prec_2 b_2.$$

We obtain a partial ordering $\preccurlyeq$ by adding equality to the ordering $\prec$ on $A_1 \times A_2$.

Example of Lexicographic Order

Let S_1 be the alphabet and $\preccurlyeq_1$ be the usual alphabetic order. Let S_2 be the set $\{0, 1, 2, 3, \ldots, 9\}$ and $\preccurlyeq_2$ be the usual partial order $\leq$. Then

- $(A, 7) \prec (B, 1)$ because $A \prec_1 B$.
- $(C, 4) \prec (C, 7)$ because $C = C$ and $4 \prec_2 7$.

Lexicographic Order (n-tuple)

Definition

A lexicographic ordering can be defined on the Cartesian product of n posets $(A_1, \preccurlyeq_1)$, $(A_2, \preccurlyeq_2)$, ..., $(A_n, \preccurlyeq_n)$. Define the partial ordering $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, ..., a_n) \prec (b_1, b_2, ..., b_n)$$

if $a_1 \prec_1 b_1$, or if there is an integer $i > 0$ such that $a_1 = b_1$, ..., $a_i = b_i$ and $a_{i+1} \prec_{i+1} b_{i+1}$.

On other words, one n-tuple is less than a second n-tuple if the entry of the first n-tuple in the first position where the two n-tuples disagree is less than the entry in that position in the second n-tuple.

Example of Lexicographic Order

Let S_1 be the alphabet and $\preccurlyeq_1$ be the usual alphabetic order. Let S_2 be the set $\{0, 1, 2, 3, ..., 9\}$ and $\preccurlyeq_2$ be the usual partial order $\leq$. Let P, the set of postal codes. $P = S_1 \times S_2 \times S_1 \times S_2 \times S_1 \times S_2$. Then

- $(G, 9, X, 8, W, 7) \prec (H, 1, A, 2, B, 1)$ because $G \prec_1 H$.
- $(G, 1, K, 2, P, 4) \prec (G, 1, K, 7, A, 1)$ because $G = G$, $1 = 1$, $K = K$, $2 \prec_2 7$.

Lexicographic Order (Strings)

Definition

Consider the strings $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$ on a partially ordered set S. Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1 a_2 \cdots a_m$ is less than the string $b_1 b_2 \cdots b_n$ if and only if

$$(a_1, a_2, .., a_t) \prec (b_1, b_2, ..., b_t)$$

or

$$(a_1, a_2, .., a_t) = (b_1, b_2, ..., b_t)$$

and $m < n$, where $\prec$ in this inequality represents the lexicographic ordering of S^t.

Helmut Hasse

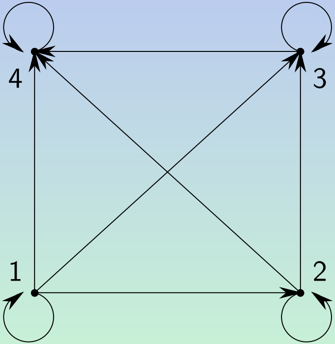
Born: 25 Aug 1898 in Kassel, Germany. Died: 26 Dec 1979 in Ahrensburg (near Hamburg), Germany

`www-groups.dcs.st-and.ac.uk/`

`~history/Mathematicians/Hasse.html`

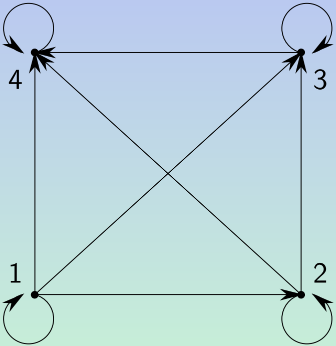
Example $(\{1,2,3,4\}, \leq)$

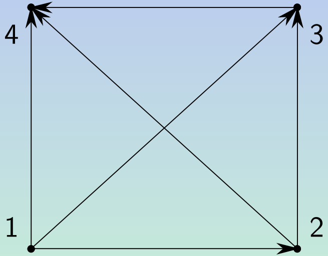
Let S be the set $S = \{1,2,3,4\}$ and the relation R be "$a \leq b$".
This relation is given by $R =$
$\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$.



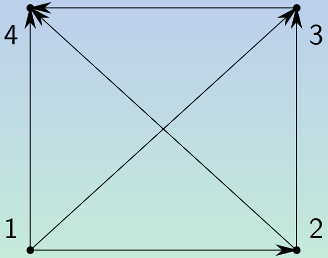
This relation is reflexive, antisymmetric and transitive.

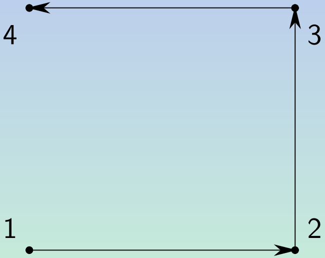
Step 1 of 4: We remove all loops caused by reflexivity.





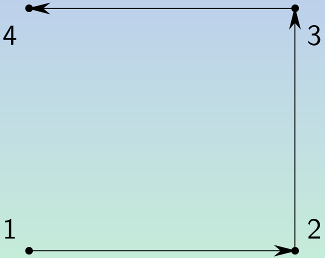
Step 2 of 4: We remove all edges implied by the transitivity property.





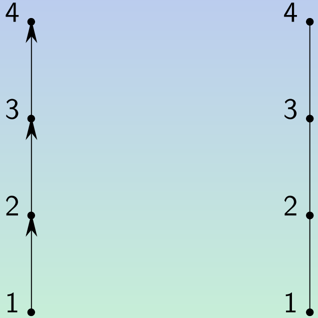
Hasse Diagram Construction

Step 3 of 4: We redraw edges and vertices such that the initial vertex of each edge is below its terminal vertex.



Hasse Diagram Construction

Step 4 of 4: Remove all arrows from the directed edges, since they are all upward. The diagram at right is the Hasse diagram.



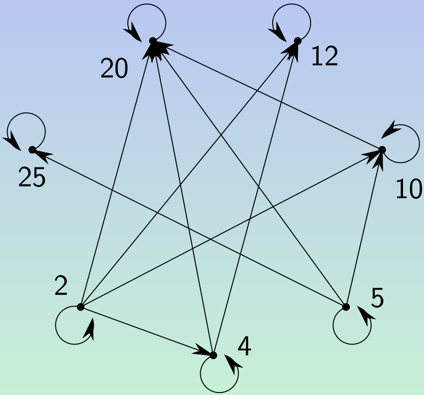
Example $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

Suppose the following poset $S = (\{2, 4, 5, 10, 12, 20, 25\}, R)$ where R is the partial order $a \mid b$.

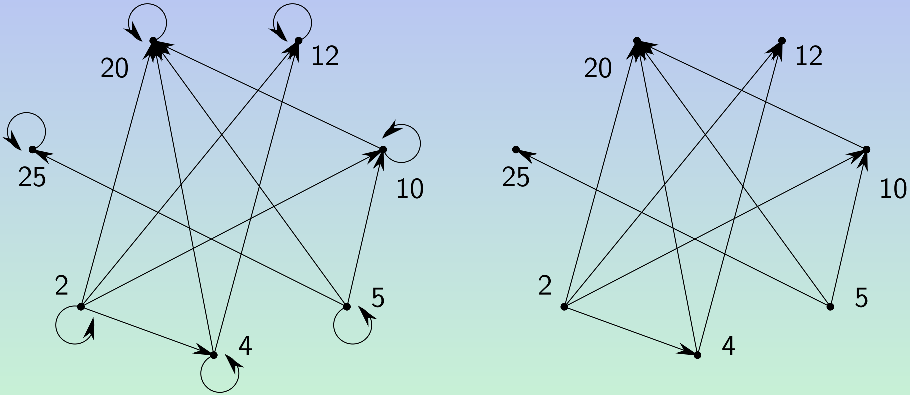
$R = \{(2, 2), (2, 4), (2, 10), (2, 12), (2, 20), (4, 4), (4, 12), (4, 20), (5, 5), (5, 20), (5, 25), (10, 10), (10, 20), (20, 20), (25, 25)\}$.

Example $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

This relation is reflexive, antisymmetric and transitive.

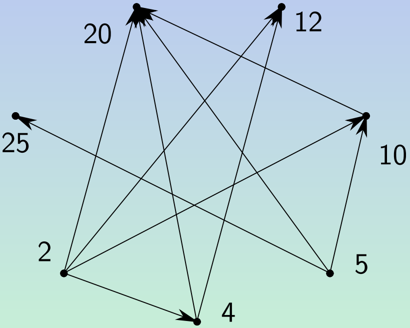


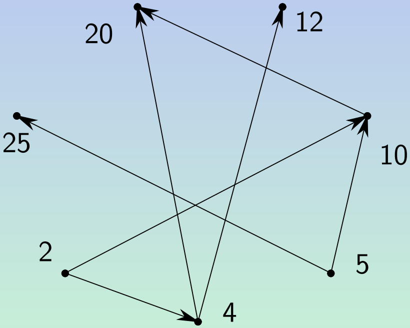
Step 1 of 4: We remove all loops caused by reflexivity.



Hasse Diagram Construction

Step 2 of 4: We remove all edges implied by the transitivity property.





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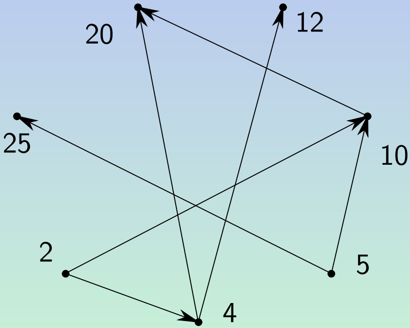

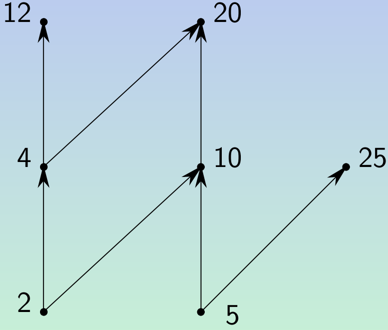

Hasse Diagram Construction

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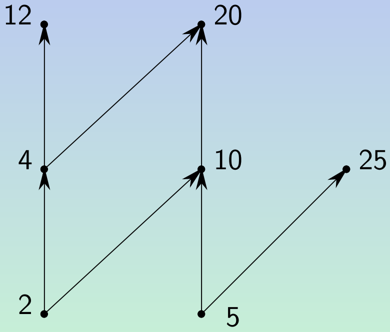
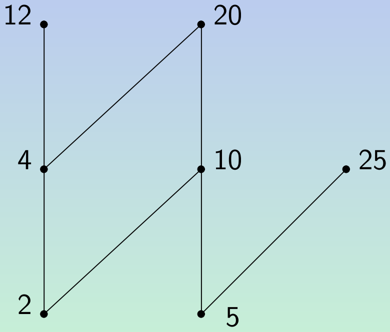




Maximal and Minimal Elements

Definition

An element a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no element $b \in S$ such that $a \preccurlyeq b$.

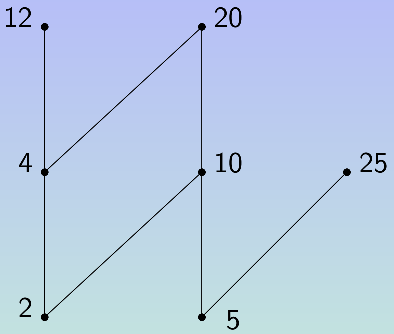
In other words, an element of a poset is called maximal if it is not less than any *comparable* element of the poset.

Definition

An element a is **minimal** in the poset $(S, \preccurlyeq)$ if there is no element $b \in S$ such that $b \preccurlyeq a$.

In other words, an element of a poset is called minimal if it is not greater than any *comparable* element of the poset.

Example $(\{2, 4, 5, 10, 12, 20, 25\}, |)$



- 2 and 5 are minimal elements.
- 12, 20 and 25 are maximal elements.
- The minimal and the maximal elements may not be unique.

Example $(\{1, 2, 3, 4\}, \leq)$

- 1 is the minimal element.
- 4 is the maximal element.
- There is at most one minimal element and one maximal element in a totally ordered set.

Greatest and Least Elements

Definition

The element a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists.

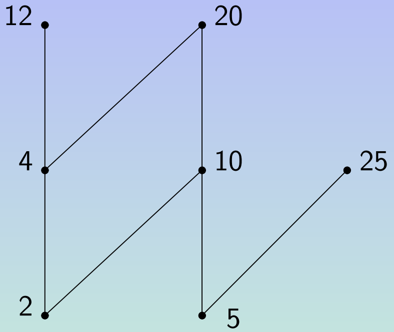
In other words, an element a in a poset $(S, \preccurlyeq)$ is the greatest element if it is greater than *every* other elements of S.

Definition

The element a is the **least element** of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists.

In other words, an element a in a poset $(S, \preccurlyeq)$ is the least element if it is less than *every* other elements of S.

Example $(\{2, 4, 5, 10, 12, 20, 25\}, |)$



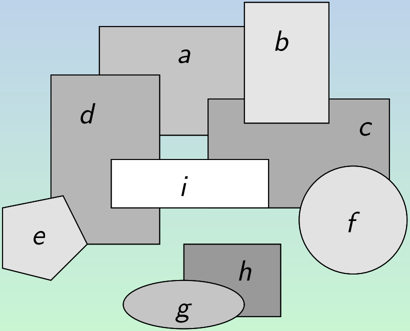
- There is no least element.
- There is no greatest element.

Example $(\{1, 2, 3, 4\}, \leq)$

- 1 is the least element.
- 4 is the greatest element.

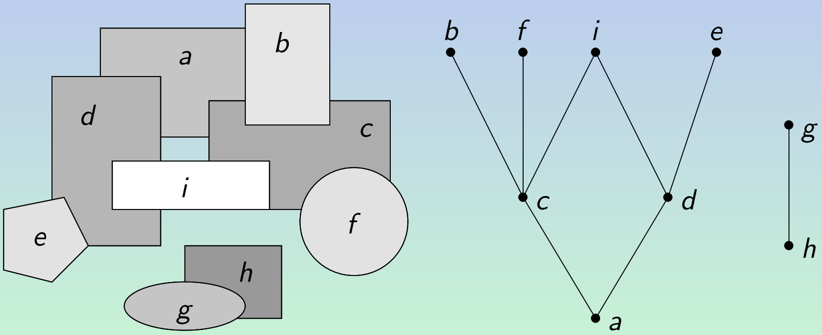
Let S be a set composed of the geometric shapes $\{a, b, c, d, e, f, g, h, i\}$. Let R be the relation "is more or as distant as. Then R is a partial ordering on S.

Two geometric shapes a and b are related, $a\,R\,b$, if a is more or as distant as b.



Introduction Example (cont.)

The relation $R = \{(a,a),(a,b),(a,c),(a,d),(a,e),(a,f),(a,i),$
$(b,b),(c,b),(c,c),(c,f),(c,i),(d,i),(d,d),(d,e),(e,e),$
$(f,f),(g,g),(h,g),(h,h),(i,i)\}$ and its Hasse diagram.



Compatible Ordering and Topological Sorting

Definition

A total ordering $\preccurlyeq$ is said to be **compatible** with the partial ordering R if $a \preccurlyeq b$ whenever $a \, R \, b$. Constructing a compatible total ordering from a partial ordering is called **topological sorting**.

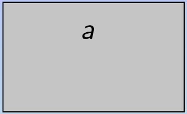
Lemma

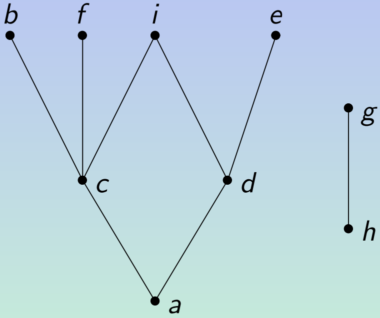
Every finite non empty poset $(S, \preccurlyeq)$ *has at least one minimal element.*

Topological Sorting Algorithm

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
begin
    $a_k :=$ a minimal element of S
        {such element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
end
$\{ a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

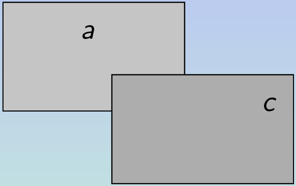
Step 1 of 9: We arbitrarily choose the minimal element a

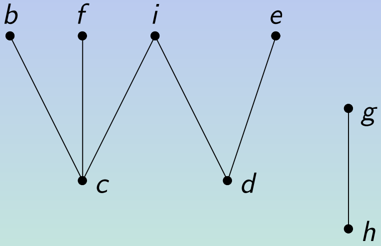




a

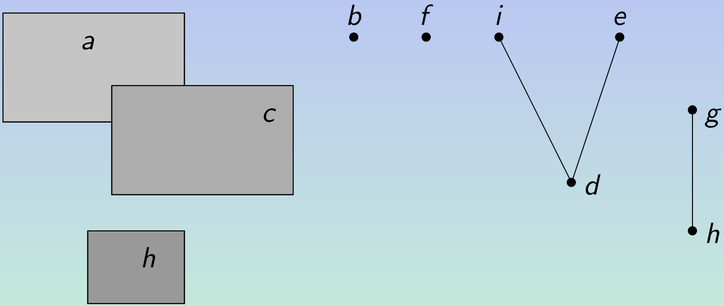
Step 2 of 9: We arbitrarily choose the minimal element c





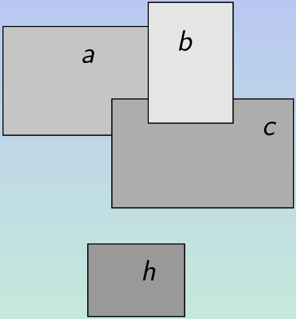
$a \preccurlyeq c$

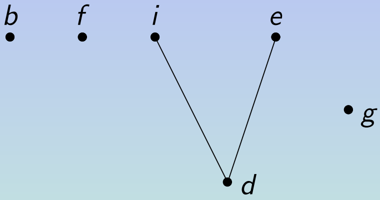
Step 3 of 9: We arbitrarily choose the minimal element h



$a \preccurlyeq c \preccurlyeq h$

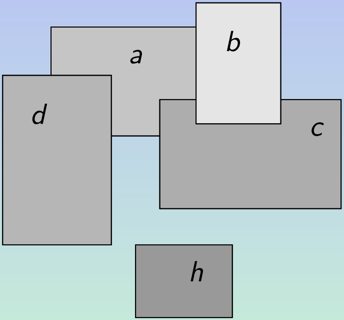
Step 4 of 9: We arbitrarily choose the minimal element b

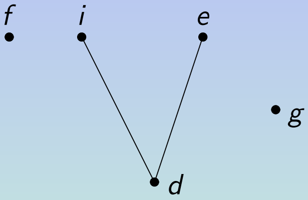




$a \preccurlyeq c \preccurlyeq h \preccurlyeq b$

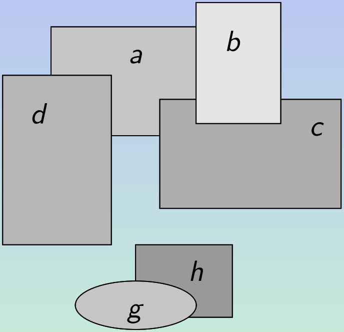
Step 5 of 9: We arbitrarily choose the minimal element d

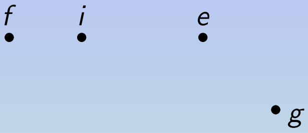




$a \preccurlyeq c \preccurlyeq h \preccurlyeq b \preccurlyeq d$

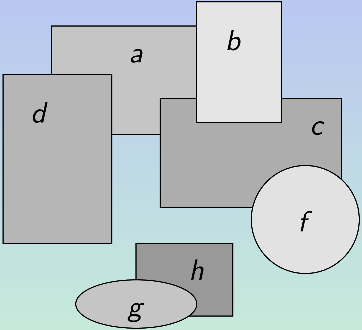
Step 6 of 9: We arbitrarily choose the minimal element g





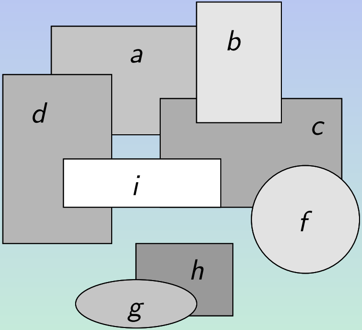
$a \preccurlyeq c \preccurlyeq h \preccurlyeq b \preccurlyeq d \preccurlyeq g$

Step 7 of 9: We arbitrarily choose the minimal element f



$a \preccurlyeq c \preccurlyeq h \preccurlyeq b \preccurlyeq d \preccurlyeq g \preccurlyeq f$

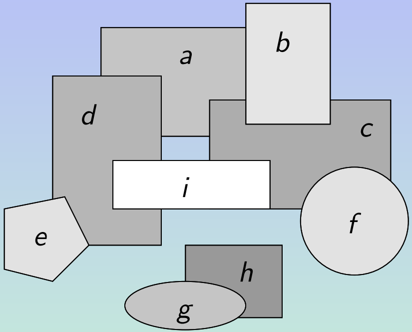
Step 8 of 9: We arbitrarily choose the minimal element i



$a \preccurlyeq c \preccurlyeq h \preccurlyeq b \preccurlyeq d \preccurlyeq g \preccurlyeq f \preccurlyeq i$

Topological Sorting Algorithm

Step 9 of 9: We arbitrarily choose the minimal element e



The total ordering $a \preccurlyeq c \preccurlyeq h \preccurlyeq b \preccurlyeq d \preccurlyeq g \preccurlyeq f \preccurlyeq i \preccurlyeq e$ is compatible with the partial ordering $R = \{(a,a),(a,b),(a,c),$ $(a,d),(a,e),(a,f),(a,i),(b,b),(c,b),(c,c),(c,f),(c,i),(d,i),$ $(d,d),(d,e),(e,e),(f,f),(g,g),(h,g),(h,h),(i,i)\}$.