Sinusoids and Phasors

EEE 101: "Basic Electrical Technology"

Civil Engineering Department

Spring 2021

Sinusoids

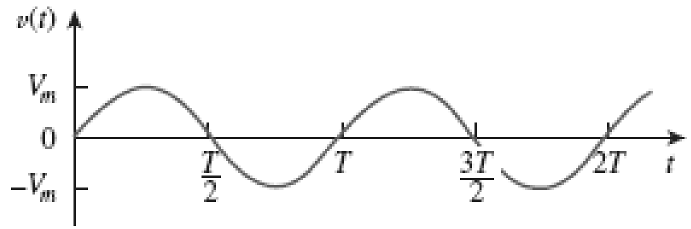
A sinusoid is a signal that has the form of the sine or cosine function.

Consider the sinusoidal voltage

$$v(t) = V_m \sin \omega t$$

where

V_m = the *amplitude* of the sinusoid
ω = the *angular frequency* in radians/s
ωt = the *argument* of the sinusoid



$$\boxed{T = \frac{2\pi}{\omega}}$$

$$v(t + T) = V_m \sin \omega(t + T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right)$$
$$= V_m \sin(\omega t + 2\pi) = V_m \sin \omega t = v(t)$$

Hence,

$$\boxed{v(t + T) = v(t)}$$

that is, v has the same value at $t + T$ as it does at t and $v(t)$ is said to be *periodic*. In general,

A periodic function is one that satisfies $f(t) = f(t + nT)$, for all t and for all integers n.

Sinusoids

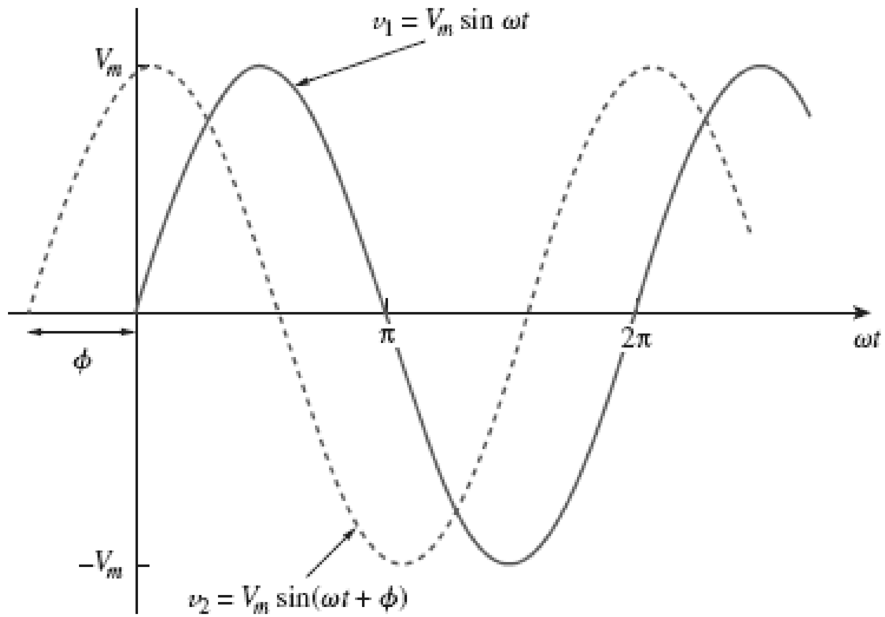
Let us now consider a more general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$

where $(\omega t + \phi)$ is the argument and ϕ is the *phase*. Both argument and phase can be in radians or degrees.

Let us examine the two sinusoids

$$v_1(t) = V_m \sin \omega t \qquad \text{and} \qquad v_2(t) = V_m \sin(\omega t + \phi)$$



Sinusoids

Find the amplitude, phase, period, and frequency of the sinusoid

Example 9.1

$$v(t) = 12\cos(50t + 10^\circ)$$

Solution:

The amplitude is $V_m = 12$ V.
The phase is $\phi = 10^\circ$.
The angular frequency is $\omega = 50$ rad/s.
The period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{50} = 0.1257$ s.
The frequency is $f = \dfrac{1}{T} = 7.958$ Hz.

Calculate the phase angle between $v_1 = -10\cos(\omega t + 50^\circ)$ and $v_2 = 12\sin(\omega t - 10^\circ)$. State which sinusoid is leading.

Example 9.2

Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.

A phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number z can be written in rectangular form as

$$z = x + jy$$

The complex number z can also be written in polar or exponential form as

$$z = r\angle\phi = re^{j\phi}$$

where r is the magnitude of z, and ϕ is the phase of z.
z can be represented in three ways:

$z = x + jy$	Rectangular form
$z = r\angle\phi$	Polar form
$z = re^{j\phi}$	Exponential form

Phasors

The relationship between the rectangular form and the polar form is shown in Fig. 9.6, where the x axis represents the real part and the y axis represents the imaginary part of a complex number. Given x and y, we can get r and ϕ as

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

On the other hand, if we know r and ϕ, we can obtain x and

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Thus, z may be written as

$$\boxed{z = x + jy = r\underline{/\phi} = r(\cos\phi + j\sin\phi)}$$

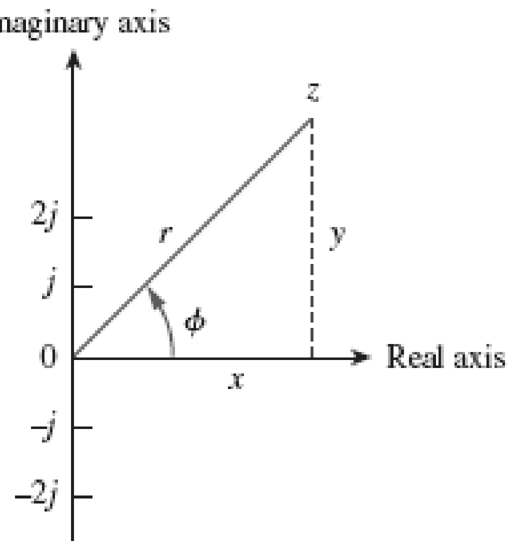


Figure 9.6
Representation of a complex number $z = x + jy = r\underline{/\phi}$.

Phasors

Addition and subtraction of complex numbers are better performed in rectangular form; multiplication and division are better done in polar form. Given the complex numbers

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$
$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{9.18a}$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{9.18b}$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2 \tag{9.18c}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2 \tag{9.18d}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r}\angle-\phi \tag{9.18e}$$

Square Root:

$$\sqrt{z} = \sqrt{r}\angle\phi/2 \tag{9.18f}$$

Complex Conjugate:

$$z^* = x - jy = r\angle-\phi = re^{-j\phi} \tag{9.18g}$$

Phasors

$$\mathbf{V} = V_m e^{j\phi} = V_m \underline{/\phi}$$

V is thus the *phasor representation* of the sinusoid $v(t)$, as we said earlier. In other words, a phasor is a complex representation of the magnitude and phase of a sinusoid.

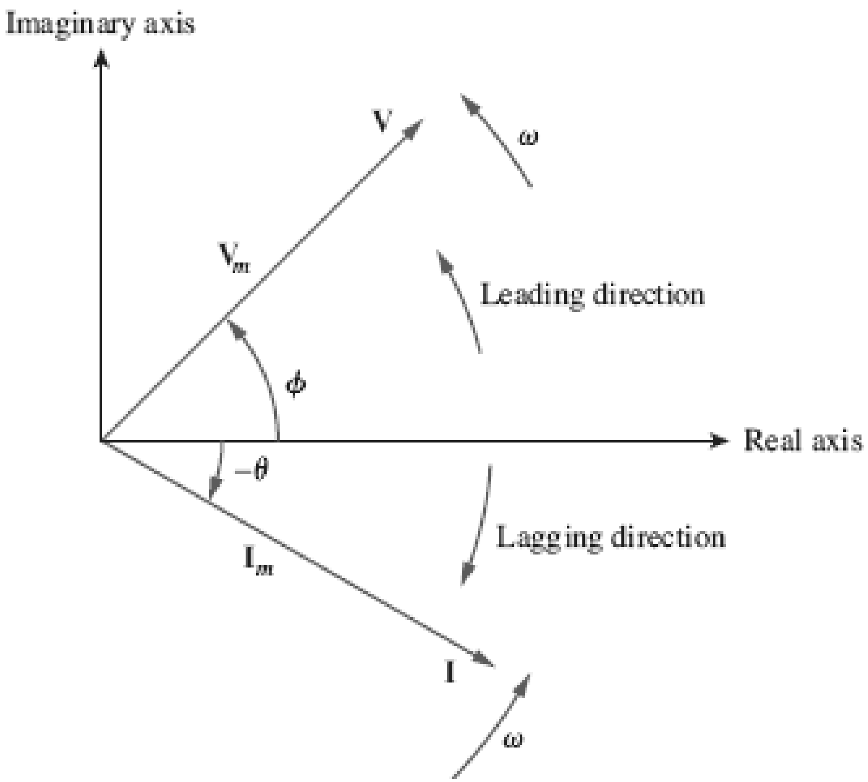


Figure 9.8
A phasor diagram showing $\mathbf{V} = V_m \underline{/\phi}$ and $\mathbf{I} = I_m \underline{/-\theta}$.

The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:

1. $v(t)$ is the *instantaneous or time domain* representation, while $\mathbf{V}$ is the *frequency or phasor domain* representation.
2. $v(t)$ is time dependent, while $\mathbf{V}$ is not. (This fact is often forgotten by students.)
3. $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

Example 9.4

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Solution:

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\underline{/-40^\circ} \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$\begin{aligned} v = -4\sin(30t + 50^\circ) &= 4\cos(30t + 50^\circ + 90^\circ) \\ &= 4\cos(30t + 140^\circ) \text{ V} \end{aligned}$$

The phasor form of v is

$$\mathbf{V} = 4\underline{/140^\circ} \text{ V}$$

Example 9.3

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$
$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$
$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$
$$= 0.565\angle -160.13^\circ$$

Phasors

Example 9.5

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

Solution:

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\ \text{A}$$

(b) Since $j = 1\angle 90^\circ$,

$$\begin{aligned}\mathbf{V} = j8\angle -20^\circ &= (1\angle 90^\circ)(8\angle -20^\circ)\\ &= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ\ \text{V}\end{aligned}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\ \text{V}$$

Phasors

Example 9.6

Given $i_1(t) = 4\cos(\omega t + 30^\circ)$ A and $i_2(t) = 5\sin(\omega t - 20^\circ)$ A, find their sum.

Solution:

Here is an important use of phasors—for summing sinusoids of the same frequency. Current $i_1(t)$ is in the standard form. Its phasor is

$$\mathbf{I}_1 = 4\underline{/30^\circ}$$

We need to express $i_2(t)$ in cosine form. The rule for converting sine to cosine is to subtract 90°. Hence,

$$i_2 = 5\cos(\omega t - 20^\circ - 90^\circ) = 5\cos(\omega t - 110^\circ)$$

and its phasor is

$$\mathbf{I}_2 = 5\underline{/-110^\circ}$$

If we let $i = i_1 + i_2$, then

$$\begin{aligned}\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 &= 4\underline{/30^\circ} + 5\underline{/-110^\circ}\\ &= 3.464 + j2 - 1.71 - j4.698 = 1.754 - j2.698\\ &= 3.218\underline{/-56.97^\circ}\text{ A}\end{aligned}$$

Phasor relationships for circuit elements

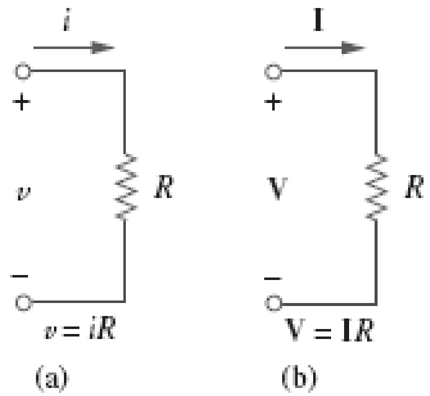


Figure 9.9
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain.

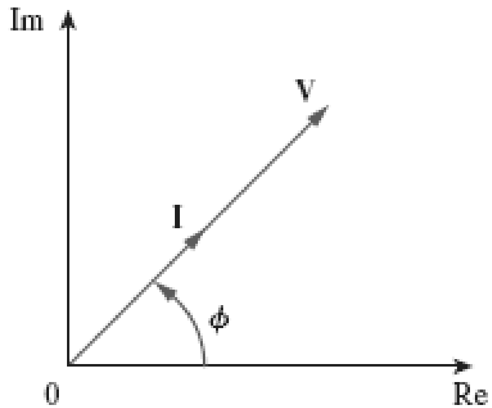


Figure 9.10
Phasor diagram for the resistor.

Phasor relationships for circuit elements

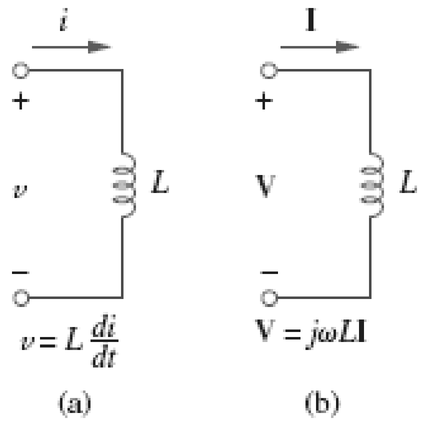


Figure 9.11
Voltage-current relations for an inductor in the: (a) time domain, (b) frequency domain.

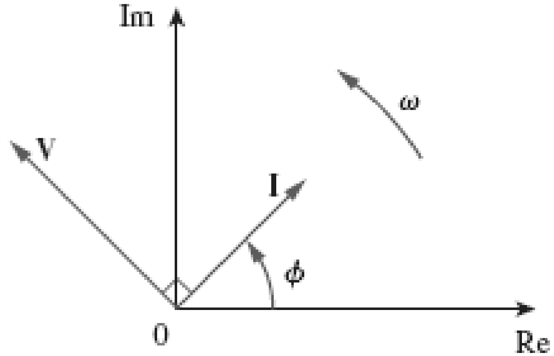


Figure 9.12
Phasor diagram for the inductor; **I** lags **V**.

Phasor relationships for circuit elements

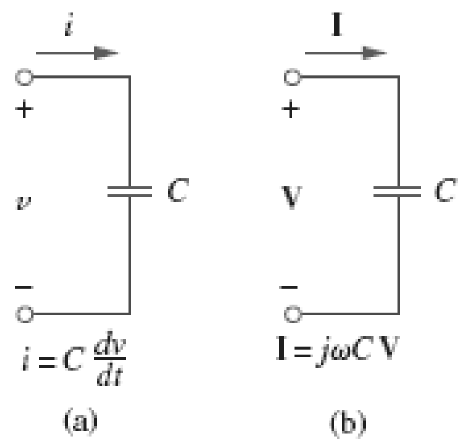


Figure 9.13
Voltage-current relations for a capacitor in the: (a) time domain, (b) frequency domain.

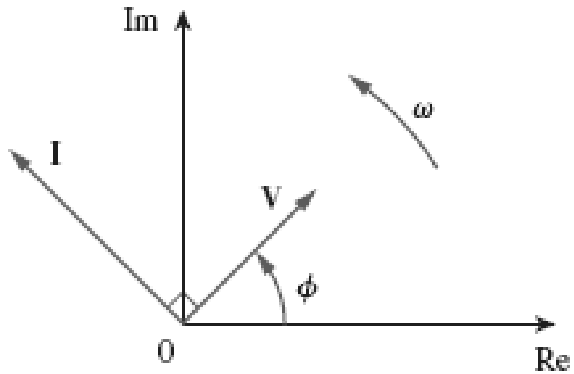


Figure 9.14
Phasor diagram for the capacitor; **I** leads **V**.

Phasor relationships for circuit elements

Example 9.8

The voltage $v = 12\cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

Solution:

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45°}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45°}}{j60 \times 0.1} = \frac{12\underline{/45°}}{6\underline{/90°}} = 2\underline{/-45°}\text{ A}$$

Converting this to the time domain,

$$i(t) = 2\cos(60t - 45°)\text{ A}$$

Impedance and admittance

The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω).

$$\boxed{\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{Z}\mathbf{I}}$$

$$\mathbf{Z} = R + jX$$

$$\mathbf{Z} = |\mathbf{Z}| \angle \theta$$

$$\boxed{\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta}$$

where

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

and

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

Impedance and admittance

The admittance **Y** is the reciprocal of impedance, measured in siemens (S).

$$\boxed{\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}}$$

$$\boxed{\mathbf{Y} = G + jB}$$

$$G + jB = \frac{1}{R + jX}$$

Impedance and admittance

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

Example 9.9

Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\underline{/0^\circ} \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5 \ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\underline{/0^\circ}}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \qquad \textbf{(9.9.1)}$$
$$= 1.6 + j0.8 = 1.789\underline{/26.57^\circ} \text{ A}$$

The voltage across the capacitor is

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\underline{/26.57^\circ}}{j4 \times 0.1} \qquad \textbf{(9.9.2)}$$
$$= \frac{1.789\underline{/26.57^\circ}}{0.4\underline{/90^\circ}} = 4.47\underline{/-63.43^\circ} \text{ V}$$

Converting **I** and **V** in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

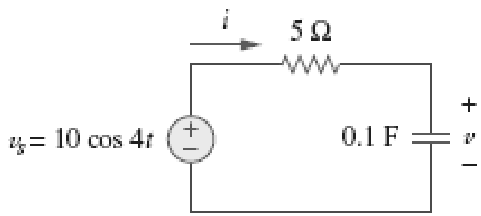


Figure 9.16
For Example 9.9.

Kirchhoff's laws (KVL and KCL) in frequency domain

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

indicating that Kirchhoff's voltage law holds for phasors.

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$$

which is Kirchhoff's current law in the frequency domain.

Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

Impedance combinations

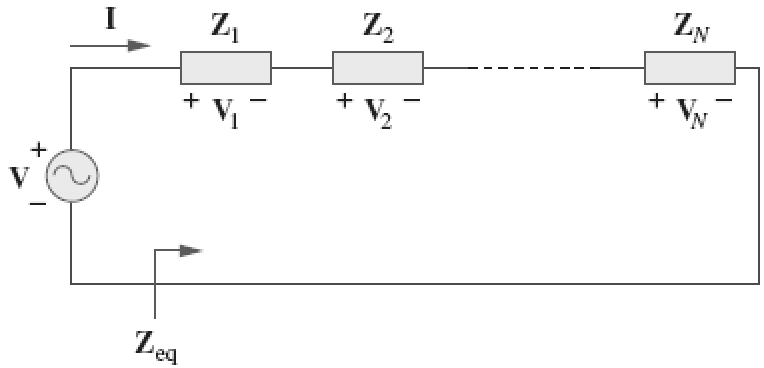


Figure 9.18
N impedances in series.

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

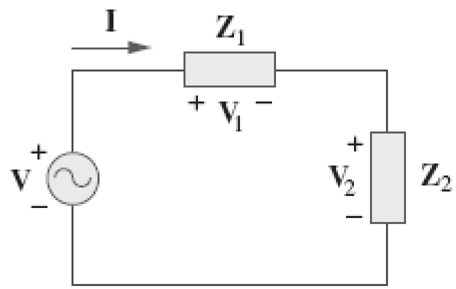


Figure 9.19
Voltage division.

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

Impedance combinations

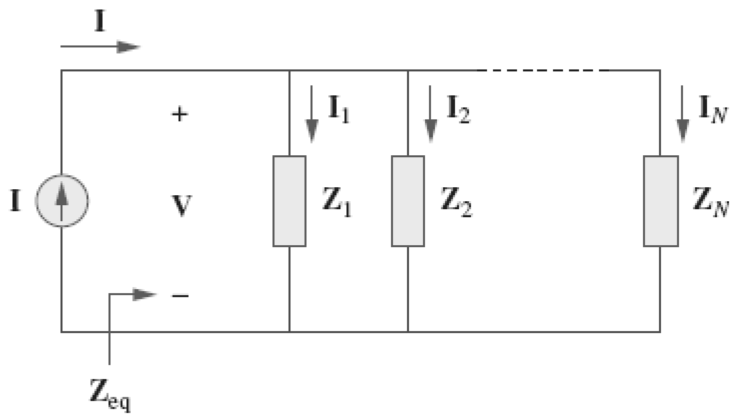


Figure 9.20
N impedances in parallel.

The equivalent impedance is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

and the equivalent admittance is

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

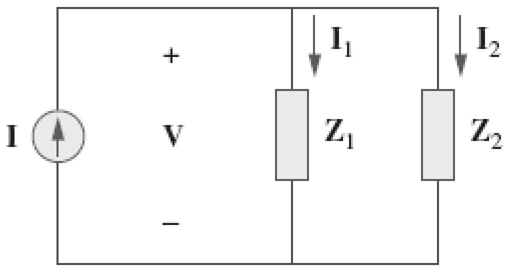


Figure 9.21
Current division.

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

$$\boxed{\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}}$$

Impedance combinations

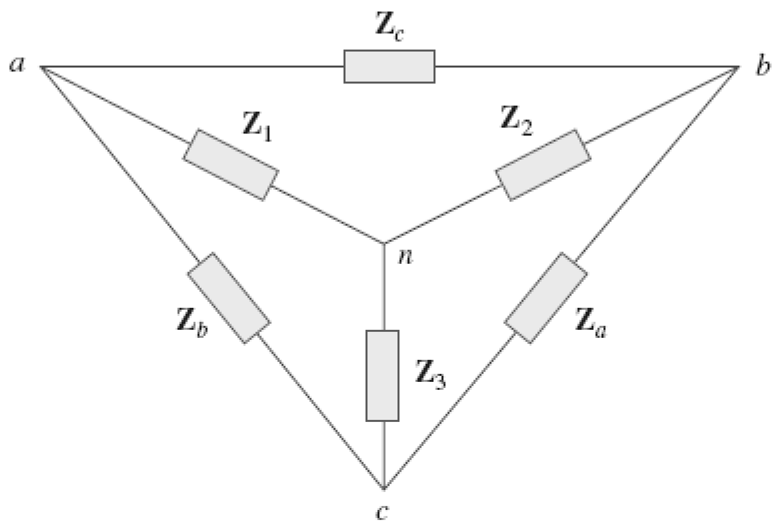


Figure 9.22
Superimposed Y and Δ networks.

Y-Δ Conversion:

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversion:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

A delta or wye circuit is said to be balanced if it has equal impedances in all three branches.

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

where $\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3$ and $\mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c$.

Impedance combinations

Example 9.10

Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at $\omega = 50$ rad/s.

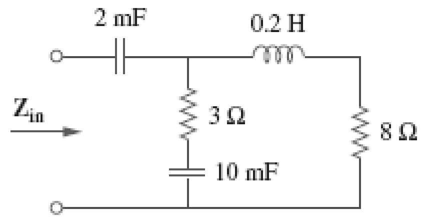


Figure 9.23
For Example 9.10.

Solution:
Let

$\mathbf{Z}_1$ = Impedance of the 2-mF capacitor
$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor
$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{\text{in}} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus,

$$\mathbf{Z}_{\text{in}} = 3.22 - j11.07\ \Omega$$

Example 9.11

Determine $v_o(t)$ in the circuit of Fig. 9.25.

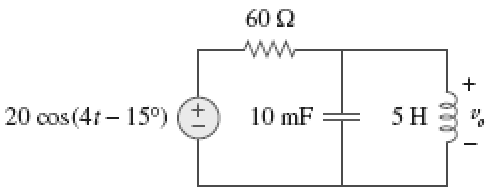


Figure 9.25
For Example 9.11.

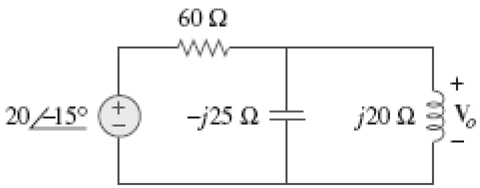


Figure 9.26
The frequency domain equivalent of the circuit in Fig. 9.25.

Solution:

To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor domain equivalent in Fig. 9.26. The transformation produces

$$v_s = 20\cos(4t - 15°) \quad \Rightarrow \quad \mathbf{V}_s = 20\underline{/-15°}\text{ V}, \qquad \omega = 4$$

$$10\text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

Let

$\mathbf{Z}_1$ = Impedance of the 60-Ω resistor

$\mathbf{Z}_2$ = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\underline{/-15°})$$
$$= (0.8575\underline{/30.96°})(20\underline{/-15°}) = 17.15\underline{/15.96°}\text{ V}$$

We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96°)\text{ V}$$

Impedance combinations

Find current **I** in the circuit of Fig. 9.28.

Example 9.12

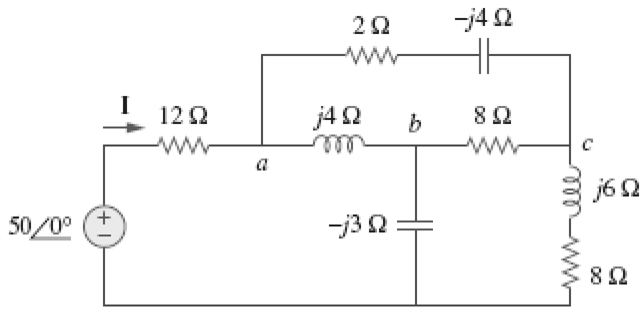


Figure 9.28
For Example 9.12.

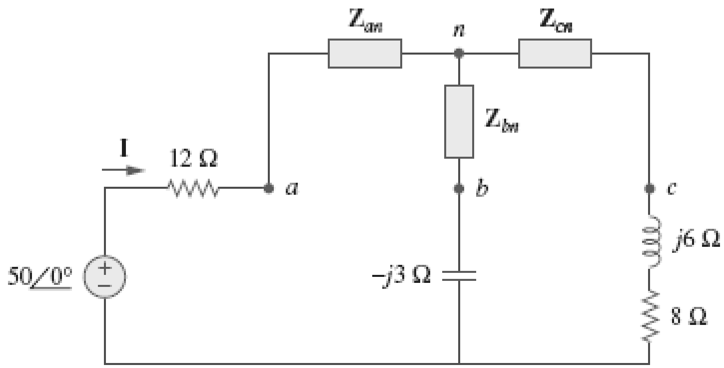


Figure 9.29
The circuit in Fig. 9.28 after delta-to-wye transformation.

Solution:
The delta network connected to nodes a, b, and c can be converted to the Y network of Fig. 9.29. We obtain the Y impedances as follows using Eq. (9.68):

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$

The total impedance at the source terminals is

$$\begin{aligned}\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8)\\ &= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8)\\ &= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3}\\ &= 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega\end{aligned}$$

The desired current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$$