Sinusoidal Steady-State Analysis

EEE 101 - "Basic Electrical Technology"

Civil Engineering Department

Spring 2021

Introduction

Steps to Analyze AC Circuits:

1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

Nodal Analysis

Example 10.1 Find i_x in the circuit of Fig. 10.1 using nodal analysis.

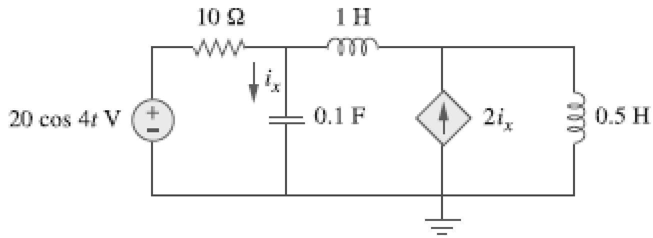


Figure 10.1
For Example 10.1.

Solution:
We first convert the circuit to the frequency domain:

$$20\cos 4t \quad \Rightarrow \quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s}$$
$$1 \text{ H} \quad \Rightarrow \quad j\omega L = j4$$
$$0.5 \text{ H} \quad \Rightarrow \quad j\omega L = j2$$
$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2.5$$

Thus, the frequency domain equivalent circuit is as shown in Fig. 10.2.

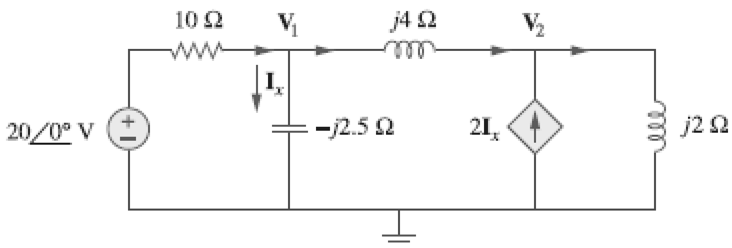


Figure 10.2
Frequency domain equivalent of the circuit in Fig. 10.1.

Nodal Analysis

Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20 \qquad \textbf{(10.1.1)}$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

By simplifying, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0 \qquad \textbf{(10.1.2)}$$

Equations (10.1.1) and (10.1.2) can be put in matrix form as

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$\Delta = \begin{vmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\underline{/18.43^\circ} \text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\underline{/198.3^\circ} \text{ V}$$

The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\underline{/18.43^\circ}}{2.5\underline{/-90^\circ}} = 7.59\underline{/108.4^\circ} \text{ A}$$

Transforming this to the time domain,

$$i_x = 7.59 \cos(4t + 108.4^\circ) \text{ A}$$

Nodal Analysis

Example 10.2 Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit of Fig. 10.4.

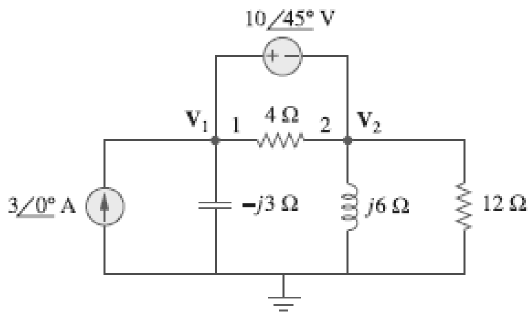


Figure 10.4
For Example 10.2.

Solution:
Nodes 1 and 2 form a supernode as shown in Fig. 10.5. Applying KCL at the supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12}$$

or

$$36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2 \qquad \textbf{(10.2.1)}$$

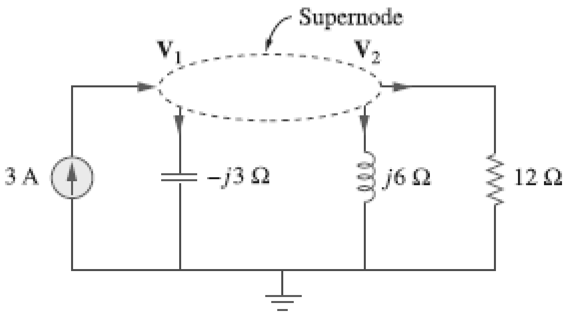


Figure 10.5
A supernode in the circuit of Fig. 10.4.

But a voltage source is connected between nodes 1 and 2, so that

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ} \qquad \textbf{(10.2.2)}$$

Substituting Eq. (10.2.2) in Eq. (10.2.1) results in

$$36 - 40\underline{/135^\circ} = (1 + j2)\mathbf{V}_2 \quad \Rightarrow \quad \mathbf{V}_2 = 31.41\underline{/-87.18^\circ} \text{ V}$$

From Eq. (10.2.2),

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\underline{/45^\circ} = 25.78\underline{/-70.48^\circ} \text{ V}$$

Mesh Analysis

Determine current $\mathbf{I}_o$ in the circuit of Fig. 10.7 using mesh analysis.

Example 10.3

Solution:

Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\underline{/90^\circ} = 0 \qquad \textbf{(10.3.2)}$$

For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

Equations (10.3.3) and (10.3.4) can be put in matrix form as

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

from which we obtain the determinants

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\underline{/-35.22^\circ}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\underline{/-35.22^\circ}}{68} = 6.12\underline{/-35.22^\circ}\text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\underline{/144.78^\circ}\text{ A}$$

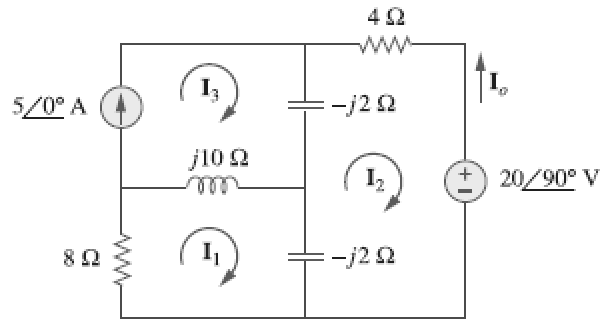


Figure 10.7
For Example 10.3.

Mesh Analysis

Solve for $\mathbf{V}_o$ in the circuit of Fig. 10.9 using mesh analysis. **Example 10.4**

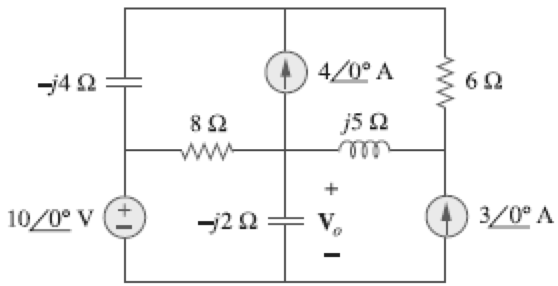


Figure 10.9
For Example 10.4.

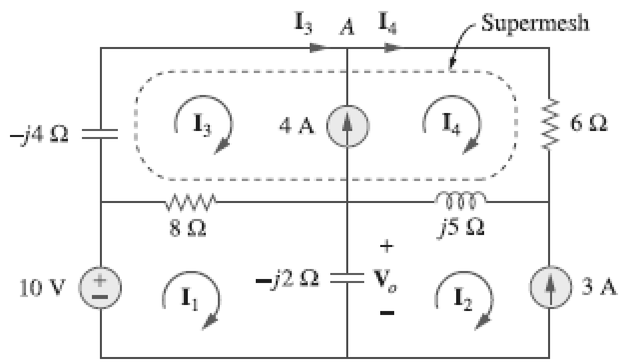


Figure 10.10
Analysis of the circuit in Fig. 10.9.

Solution:
As shown in Fig. 10.10, meshes 3 and 4 form a supermesh due to the current source between the meshes. For mesh 1, KVL gives

$$-10 + (8 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - 8\mathbf{I}_3 = 0$$

or

$$(8 - j2)\mathbf{I}_1 + j2\mathbf{I}_2 - 8\mathbf{I}_3 = 10 \tag{10.4.1}$$

For mesh 2,

$$\mathbf{I}_2 = -3 \tag{10.4.2}$$

For the supermesh,

$$(8 - j4)\mathbf{I}_3 - 8\mathbf{I}_1 + (6 + j5)\mathbf{I}_4 - j5\mathbf{I}_2 = 0 \tag{10.4.3}$$

Due to the current source between meshes 3 and 4, at node A,

$$\mathbf{I}_4 = \mathbf{I}_3 + 4 \tag{10.4.4}$$

The required voltage $\mathbf{V}_0$ is

$$\mathbf{V}_o = -j2(\mathbf{I}_1 - \mathbf{I}_2)$$

Superposition Theorem

Example 10.5

Use the superposition theorem to find $\mathbf{I}_o$ in the circuit in Fig. 10.7.

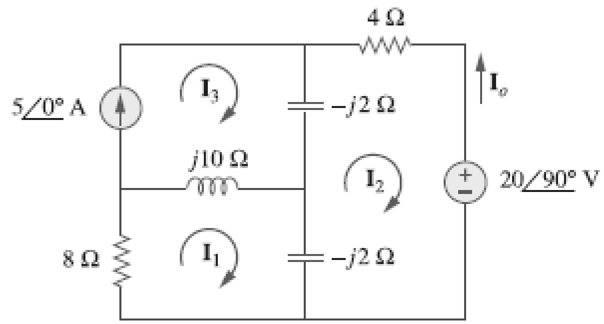


Figure 10.7

Solution:
Let

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o \tag{10.5.1}$$

where $\mathbf{I}'_o$ and $\mathbf{I}''_o$ are due to the voltage and current sources, respectively. To find $\mathbf{I}'_o$, consider the circuit in Fig. 10.12(a). If we let $\mathbf{Z}$ be the parallel combination of $-j2$ and $8 + j10$, then

$$\mathbf{Z} = \frac{-j2(8 + j10)}{-2j + 8 + j10} = 0.25 - j2.25$$

and current $\mathbf{I}'_o$ is

$$\mathbf{I}'_o = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

or

$$\mathbf{I}'_o = -2.353 + j2.353 \tag{10.5.2}$$

To get $\mathbf{I}''_o$, consider the circuit in Fig. 10.12(b). For mesh 1,

$$(8 + j8)\mathbf{I}_1 - j10\mathbf{I}_3 + j2\mathbf{I}_2 = 0 \tag{10.5.3}$$

For mesh 2,

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j2\mathbf{I}_3 = 0 \tag{10.5.4}$$

For mesh 3,

$$\mathbf{I}_3 = 5 \tag{10.5.5}$$

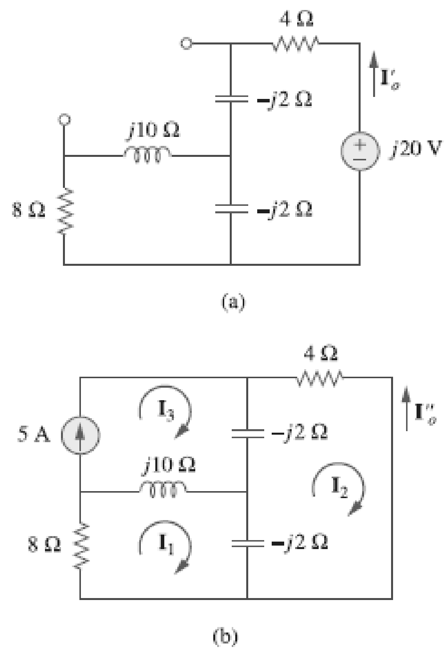


Figure 10.12
Solution of Example 10.5.

Superposition Theorem

From Eqs. (10.5.4) and (10.5.5),

$$(4 - j4)\mathbf{I}_2 + j2\mathbf{I}_1 + j10 = 0$$

Expressing $\mathbf{I}_1$ in terms of $\mathbf{I}_2$ gives

$$\mathbf{I}_1 = (2 + j2)\mathbf{I}_2 - 5 \qquad \textbf{(10.5.6)}$$

Substituting Eqs. (10.5.5) and (10.5.6) into Eq. (10.5.3), we get

$$(8 + j8)[(2 + j2)\mathbf{I}_2 - 5] - j50 + j2\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

Current $\mathbf{I}_o''$ is obtained as

$$\mathbf{I}_o'' = -\mathbf{I}_2 = -2.647 + j1.176 \qquad \textbf{(10.5.7)}$$

From Eqs. (10.5.2) and (10.5.7), we write

$$\mathbf{I}_o = \mathbf{I}_o' + \mathbf{I}_o'' = -5 + j3.529 = 6.12\underline{/144.78^\circ} \text{ A}$$

Superposition Theorem

Example 10.6 Find v_o of the circuit of Fig. 10.13 using the superposition theorem.

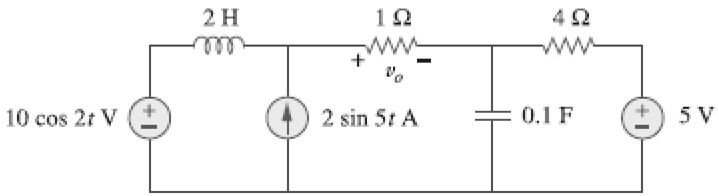


Figure 10.13
For Example 10.6.

Solution:
Since the circuit operates at three different frequencies ($\omega = 0$ for the dc voltage source), one way to obtain a solution is to use superposition, which breaks the problem into single-frequency problems. So we let

$$v_o = v_1 + v_2 + v_3 \qquad (10.6.1)$$

where v_1 is due to the 5-V dc voltage source, v_2 is due to the $10\cos 2t$ V voltage source, and v_3 is due to the $2\sin 5t$ A current source.

To find v_1, we set to zero all sources except the 5-V dc source. We recall that at steady state, a capacitor is an open circuit to dc while an inductor is a short circuit to dc. There is an alternative way of looking at this. Since $\omega = 0$, $j\omega L = 0$, $1/j\omega C = \infty$. Either way, the equivalent circuit is as shown in Fig. 10.14(a). By voltage division,

$$-v_1 = \frac{1}{1+4}(5) = 1 \text{ V} \qquad (10.6.2)$$

To find v_2, we set to zero both the 5-V source and the $2\sin 5t$ current source and transform the circuit to the frequency domain.

$$10\cos 2t \quad \Rightarrow \quad 10\angle 0^\circ, \qquad \omega = 2 \text{ rad/s}$$

$$2 \text{ H} \quad \Rightarrow \quad j\omega L = j4\ \Omega$$

$$0.1 \text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j5\ \Omega$$

Superposition Theorem

The equivalent circuit is now as shown in Fig. 10.14(b). Let

$$\mathbf{Z} = -j5 \parallel 4 = \frac{-j5 \times 4}{4 - j5} = 2.439 - j1.951$$

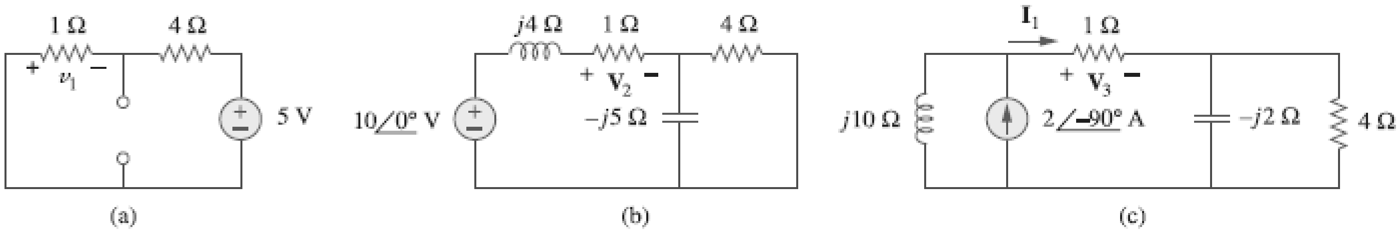


Figure 10.14
Solution of Example 10.6: (a) setting all sources to zero except the 5-V dc source, (b) setting all sources to zero except the ac voltage source, (c) setting all sources to zero except the ac current source.

By voltage division,

$$\mathbf{V}_2 = \frac{1}{1 + j4 + \mathbf{Z}}(10\underline{/0^\circ}) = \frac{10}{3.439 + j2.049} = 2.498\underline{/-30.79^\circ}$$

In the time domain,

$$v_2 = 2.498\cos(2t - 30.79^\circ) \qquad \textbf{(10.6.3)}$$

To obtain v_3, we set the voltage sources to zero and transform what is left to the frequency domain.

$$2\sin 5t \quad \Rightarrow \quad 2\underline{/-90^\circ}, \qquad \omega = 5 \text{ rad/s}$$

$$2\text{ H} \quad \Rightarrow \quad j\omega L = j10\ \Omega$$

$$0.1\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j2\ \Omega$$

The equivalent circuit is in Fig. 10.14(c). Let

$$\mathbf{Z}_1 = -j2 \parallel 4 = \frac{-j2 \times 4}{4 - j2} = 0.8 - j1.6\ \Omega$$

By current division,

$$\mathbf{I}_1 = \frac{j10}{j10 + 1 + \mathbf{Z}_1}(2\underline{/-90^\circ})\text{ A}$$

$$\mathbf{V}_3 = \mathbf{I}_1 \times 1 = \frac{j10}{1.8 + j8.4}(-j2) = 2.328\underline{/-80^\circ}\text{ V}$$

In the time domain,

$$v_3 = 2.33\cos(5t - 80^\circ) = 2.33\sin(5t + 10^\circ)\text{ V} \qquad \textbf{(10.6.4)}$$

Substituting Eqs. (10.6.2) to (10.6.4) into Eq. (10.6.1), we have

$$v_o(t) = -1 + 2.498\cos(2t - 30.79^\circ) + 2.33\sin(5t + 10^\circ)\text{ V}$$

Source Transformation

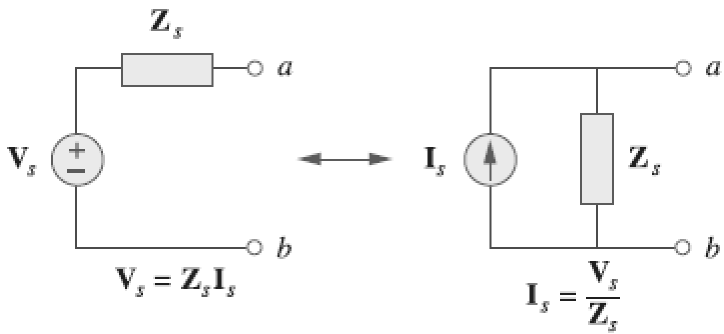


Figure 10.16
Source transformation.

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$

Source Transformation

Example 10.7

Calculate $\mathbf{V}_x$ in the circuit of Fig. 10.17 using the method of source transformation.

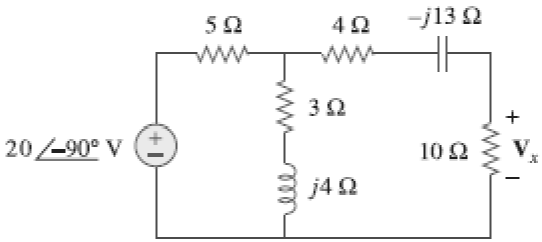


Figure 10.17
For Example 10.7.

Solution:
We transform the voltage source to a current source and obtain the circuit in Fig. 10.18(a), where

$$\mathbf{I}_s = \frac{20\angle -90^\circ}{5} = 4\angle -90^\circ = -j4 \text{ A}$$

The parallel combination of 5-Ω resistance and $(3 + j4)$ impedance gives

$$\mathbf{Z}_1 = \frac{5(3 + j4)}{8 + j4} = 2.5 + j1.25\ \Omega$$

Converting the current source to a voltage source yields the circuit in Fig. 10.18(b), where

$$\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_1 = -j4(2.5 + j1.25) = 5 - j10 \text{ V}$$

By voltage division,

$$\mathbf{V}_x = \frac{10}{10 + 2.5 + j1.25 + 4 - j13}(5 - j10) = 5.519\angle -28^\circ \text{ V}$$

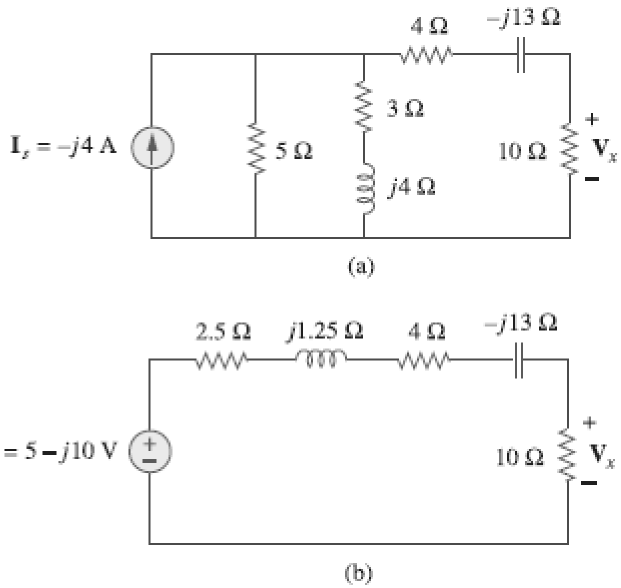


Figure 10.18
Solution of the circuit in Fig. 10.17.

Thevenin and Norton Equivalent Circuits

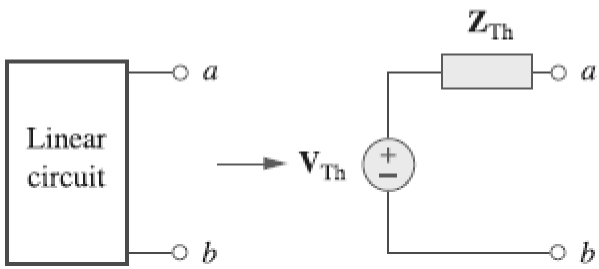


Figure 10.20
Thevenin equivalent.

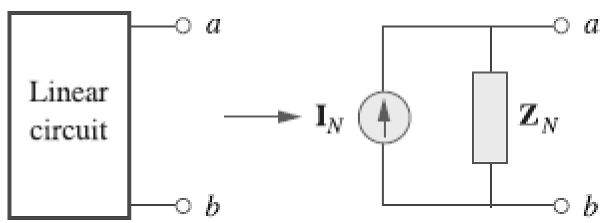


Figure 10.21
Norton equivalent.

$$\mathbf{V}_{Th} = \mathbf{Z}_N \mathbf{I}_N, \qquad \mathbf{Z}_{Th} = \mathbf{Z}_N$$

$\mathbf{V}_{Th}$ is the open-circuit voltage while $\mathbf{I}_N$ is the short-circuit current.

Thevenin and Norton Equivalent Circuits

Example 10.8 Obtain the Thevenin equivalent at terminals *a-b* of the circuit in Fig. 10.22.

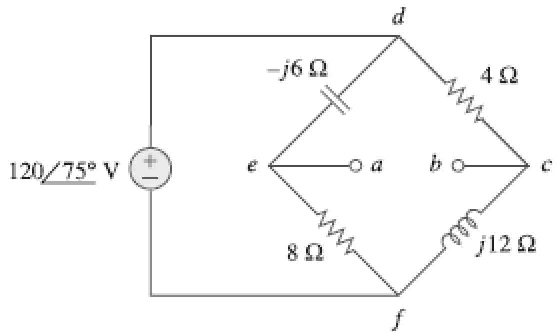
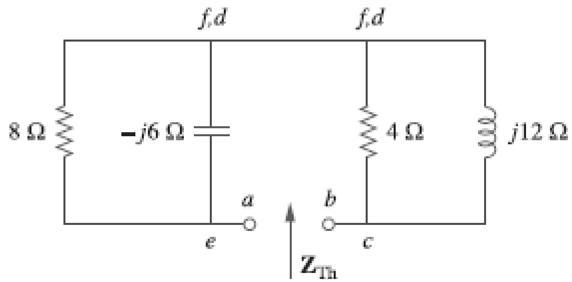


Figure 10.22
For Example 10.8.

Solution:
We find $\mathbf{Z}_{Th}$ by setting the voltage source to zero. As shown in Fig. 10.23(a), the 8-Ω resistance is now in parallel with the $-j6$ reactance, so that their combination gives

$$\mathbf{Z}_1 = -j6 \parallel 8 = \frac{-j6 \times 8}{8 - j6} = 2.88 - j3.84\ \Omega$$



Similarly, the 4-Ω resistance is in parallel with the $j12$ reactance, and their combination gives

$$\mathbf{Z}_2 = 4 \parallel j12 = \frac{j12 \times 4}{4 + j12} = 3.6 + j1.2\ \Omega$$

The Thevenin impedance is the series combination of $\mathbf{Z}_1$ and $\mathbf{Z}_2$; that is,

$$\mathbf{Z}_{Th} = \mathbf{Z}_1 + \mathbf{Z}_2 = 6.48 - j2.64\ \Omega$$

Thevenin and Norton Equivalent Circuits

To find $\mathbf{V}_{Th}$, consider the circuit in Fig. 10.23(b). Currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are obtained as

$$\mathbf{I}_1 = \frac{120\angle 75^\circ}{8 - j6}\ \text{A}, \qquad \mathbf{I}_2 = \frac{120\angle 75^\circ}{4 + j12}\ \text{A}$$

Applying KVL around loop *bcdeab* in Fig. 10.23(b) gives

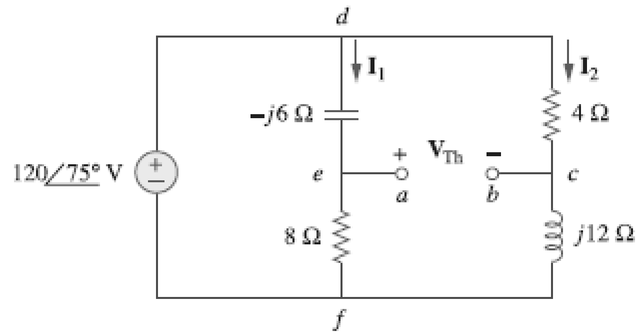
$$\mathbf{V}_{Th} - 4\mathbf{I}_2 + (-j6)\mathbf{I}_1 = 0$$

or

$$\mathbf{V}_{Th} = 4\mathbf{I}_2 + j6\mathbf{I}_1 = \frac{480\angle 75^\circ}{4 + j12} + \frac{720\angle 75^\circ + 90^\circ}{8 - j6}$$

$$= 37.95\angle 3.43^\circ + 72\angle 201.87^\circ$$

$$= -28.936 - j24.55 = 37.95\angle 220.31^\circ\ \text{V}$$



Thevenin and Norton Equivalent Circuits

Example 10.9

Find the Thevenin equivalent of the circuit in Fig. 10.25 as seen from terminals *a*-*b*.

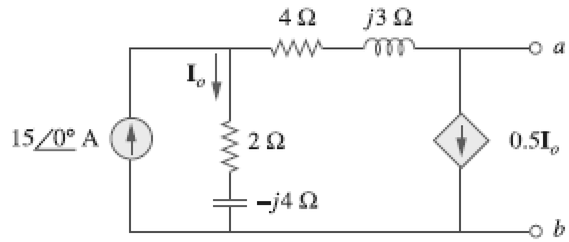


Figure 10.25
For Example 10.9.

Solution:
To find $\mathbf{V}_{Th}$, we apply KCL at node 1 in Fig. 10.26(a).

$$15 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 10 \text{ A}$$

Applying KVL to the loop on the right-hand side in Fig. 10.26(a), we obtain

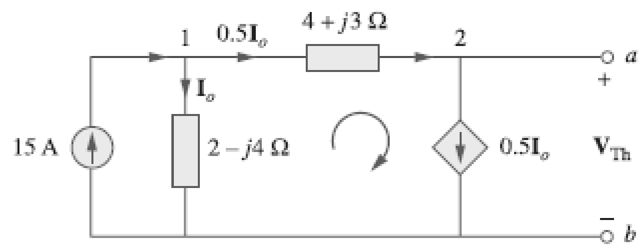
$$-\mathbf{I}_o(2 - j4) + 0.5\mathbf{I}_o(4 + j3) + \mathbf{V}_{Th} = 0$$

or

$$\mathbf{V}_{Th} = 10(2 - j4) - 5(4 + j3) = -j55$$

Thus, the Thevenin voltage is

$$\mathbf{V}_{Th} = 55\angle{-90^\circ} \text{ V}$$



Thevenin and Norton Equivalent Circuits

To obtain $\mathbf{Z}_{Th}$, we remove the independent source. Due to the presence of the dependent current source, we connect a 3-A current source (3 is an arbitrary value chosen for convenience here, a number divisible by the sum of currents leaving the node) to terminals *a-b* as shown in Fig. 10.26(b). At the node, KCL gives

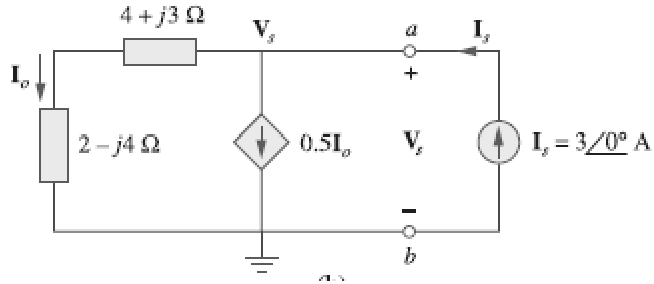
$$3 = \mathbf{I}_o + 0.5\mathbf{I}_o \quad \Rightarrow \quad \mathbf{I}_o = 2 \text{ A}$$

Applying KVL to the outer loop in Fig. 10.26(b) gives

$$\mathbf{V}_s = \mathbf{I}_o(4 + j3 + 2 - j4) = 2(6 - j)$$

The Thevenin impedance is

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6 - j)}{3} = 4 - j0.6667\ \Omega$$



Example 10.10

Obtain current $\mathbf{I}_o$ in Fig. 10.28 using Norton's theorem.

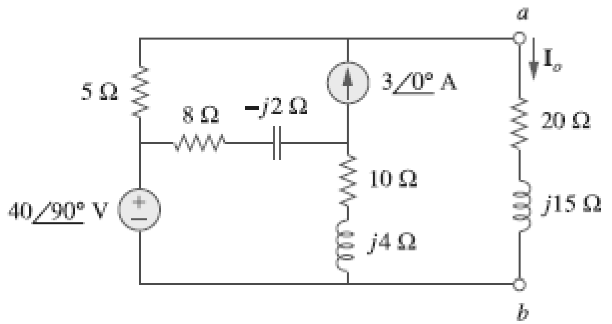


Figure 10.28
For Example 10.10.

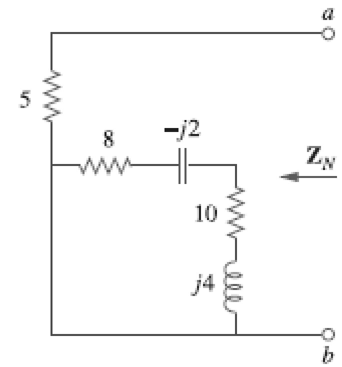
Solution:

Our first objective is to find the Norton equivalent at terminals *a-b*. $\mathbf{Z}_N$ is found in the same way as $\mathbf{Z}_{Th}$. We set the sources to zero as shown in Fig. 10.29(a). As evident from the figure, the $(8 - j2)$ and $(10 + j4)$ impedances are short-circuited, so that

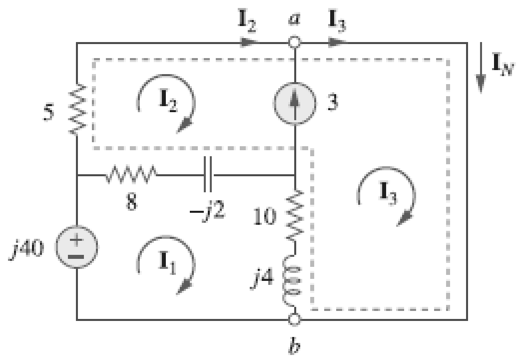
$$\mathbf{Z}_N = 5\ \Omega$$

To get $\mathbf{I}_N$, we short-circuit terminals *a-b* as in Fig. 10.29(b) and apply mesh analysis. Notice that meshes 2 and 3 form a supermesh because of the current source linking them. For mesh 1,

$$-j40 + (18 + j2)\mathbf{I}_1 - (8 - j2)\mathbf{I}_2 - (10 + j4)\mathbf{I}_3 = 0 \qquad \textbf{(10.10.1)}$$



(a)



(b)

Thevenin and Norton Equivalent Circuits

For the supermesh,

$$(13 - j2)\mathbf{I}_2 + (10 + j4)\mathbf{I}_3 - (18 + j2)\mathbf{I}_1 = 0 \qquad \mathbf{(10.10.2)}$$

At node a, due to the current source between meshes 2 and 3,

$$\mathbf{I}_3 = \mathbf{I}_2 + 3 \qquad \mathbf{(10.10.3)}$$

Adding Eqs. (10.10.1) and (10.10.2) gives

$$-j40 + 5\mathbf{I}_2 = 0 \quad \Rightarrow \quad \mathbf{I}_2 = j8$$

From Eq. (10.10.3),

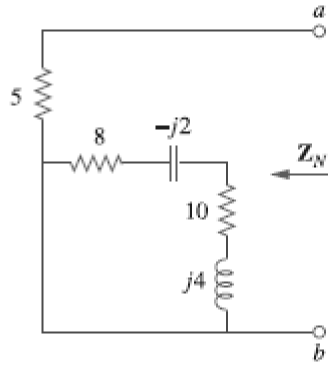
$$\mathbf{I}_3 = \mathbf{I}_2 + 3 = 3 + j8$$

The Norton current is

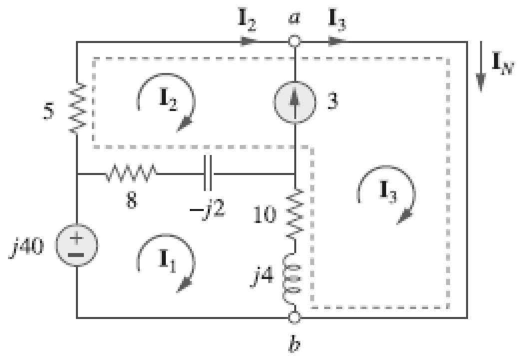
$$\mathbf{I}_N = \mathbf{I}_3 = (3 + j8)\text{ A}$$

Figure 10.29(c) shows the Norton equivalent circuit along with the impedance at terminals a-b. By current division,

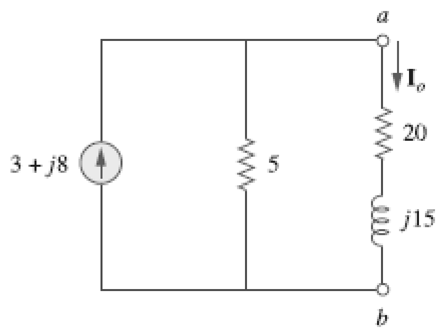
$$\mathbf{I}_o = \frac{5}{5 + 20 + j15}\mathbf{I}_N = \frac{3 + j8}{5 + j3} = 1.465\underline{/38.48^\circ}\text{ A}$$



(a)



(b)



(c)