AC Power Analysis

EEE 101: "Basic Electrical Technology"

Civil Engineering Department

Spring 2021

Introduction

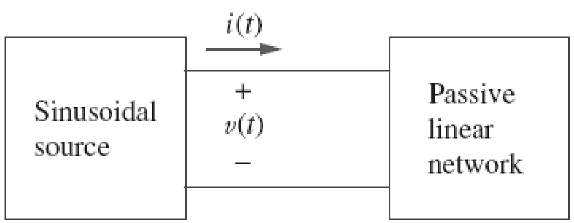
Our effort in ac circuit analysis so far has been focused mainly on calculating voltage and current. Our major concern in this chapter is power analysis.

Instantaneous Power and Average Power

The instantaneous power (in watts) is the power at any instant of time.

$$p(t) = v(t)i(t)$$

It is the rate at which an element absorbs energy.



$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

where V_m and I_m are the amplitudes (or peak values), and θ_v and θ_i are the phase angles of the voltage and current, respectively. The instantaneous power absorbed by the circuit is

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

Instantaneous Power and Average Power

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

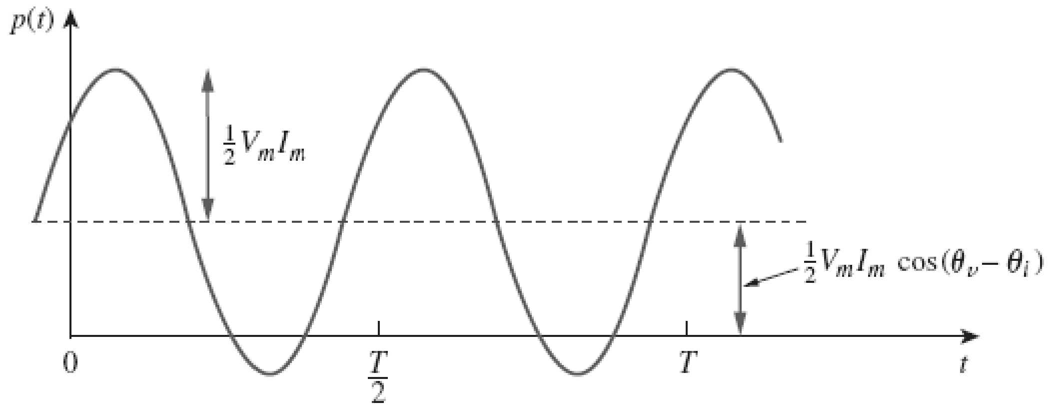
We apply the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

and express

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

This shows us that the instantaneous power has two parts. The first part is constant or time independent. Its value depends on the phase difference between the voltage and the current. The second part is a sinusoidal function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.



When $p(t)$ is negative, power is absorbed by the source; that is, power is transferred from the circuit to the source. This is possible because of the storage elements (capacitors and inductors) in the circuit.

Instantaneous Power and Average Power

The average power, in watts, is the average of the instantaneous power over one period.

Thus, the average power is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt$$

$$P = \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\,dt + \frac{1}{T}\int_0^T \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$= \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\frac{1}{T}\int_0^T dt \quad + \frac{1}{2}V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\,dt$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

Since $\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v)$, what is important is the difference in the phases of the voltage and current.

Note that $p(t)$ is time-varying while P does not depend on time.

Instantaneous Power and Average Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

The phasor forms of $v(t)$ and $i(t)$ are $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, respectively.

To use phasors, we notice that

$$\frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \underline{/\theta_v - \theta_i}$$

$$= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

$$\boxed{P = \frac{1}{2}\text{Re}[\mathbf{V}\mathbf{I}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)}$$

Instantaneous Power and Average Power

$$P = \frac{1}{2}\text{Re}[\mathbf{VI}^*] = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

Consider two special cases

When $\theta_v = \theta_i$, the voltage and current are in phase. This implies a purely resistive circuit or resistive load R, and

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R$$

where $|\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$. a purely resistive cir·cuit absorbs power at all times.

When $\theta_v - \theta_i = \pm 90°$, we have a purely reactive circuit, and

$$P = \frac{1}{2}V_m I_m \cos 90° = 0$$

showing that a purely reactive circuit absorbs no average power.

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power.

Instantaneous Power and Average Power

Example 11.1

Given that

$$v(t) = 120\cos(377t + 45^\circ)\ \text{V} \quad \text{and} \quad i(t) = 10\cos(377t - 10^\circ)\ \text{A}$$

find the instantaneous power and the average power

Solution:

The instantaneous power is given by

$$p = vi = 1200\cos(377t + 45^\circ)\cos(377t - 10^\circ)$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

gives

$$p = 600[\cos(754t + 35^\circ) + \cos 55^\circ]$$

or

$$p(t) = 344.2 + 600\cos(754t + 35^\circ)\ \text{W}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}120(10)\cos[45^\circ - (-10^\circ)]$$

$$= 600\cos 55^\circ = 344.2\ \text{W}$$

which is the constant part of $p(t)$ above.

Instantaneous Power and Average Power

Example 11.2

Calculate the average power absorbed by an impedance $\mathbf{Z} = 30 - j70\,\Omega$ when a voltage $\mathbf{V} = 120\underline{/0^\circ}$ is applied across it.

Solution:

The current through the impedance is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{120\underline{/0^\circ}}{30 - j70} = \frac{120\underline{/0^\circ}}{76.16\underline{/-66.8^\circ}} = 1.576\underline{/66.8^\circ}\text{ A}$$

The average power is

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2}(120)(1.576)\cos(0 - 66.8^\circ) = 37.24\text{ W}$$

Instantaneous Power and Average Power

Example 11.3

For the circuit shown in Fig. 11.3, find the average power supplied by the source and the average power absorbed by the resistor.

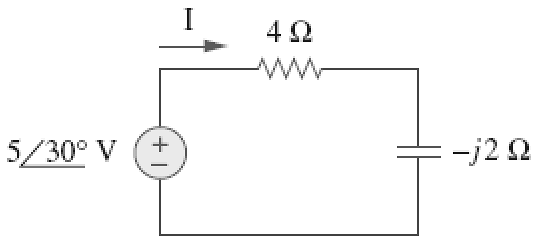


Figure 11.3
For Example 11.3.

Solution:

The current **I** is given by

$$\mathbf{I} = \frac{5\angle 30^\circ}{4 - j2} = \frac{5\angle 30^\circ}{4.472\angle -26.57^\circ} = 1.118\angle 56.57^\circ \text{ A}$$

The average power supplied by the voltage source is

$$P = \frac{1}{2}(5)(1.118)\cos(30^\circ - 56.57^\circ) = 2.5 \text{ W}$$

The current through the resistor is

$$\mathbf{I}_R = \mathbf{I} = 1.118\angle 56.57^\circ \text{ A}$$

and the voltage across it is

$$\mathbf{V}_R = 4\mathbf{I}_R = 4.472\angle 56.57^\circ \text{ V}$$

The average power absorbed by the resistor is

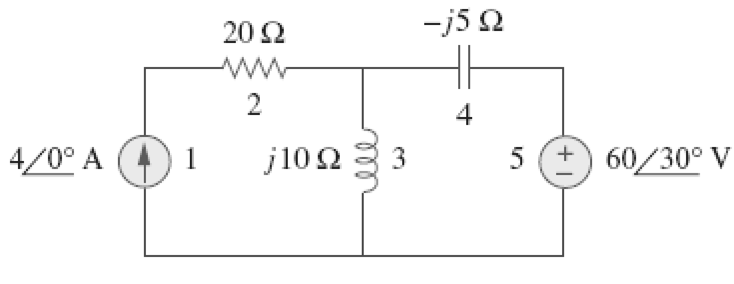
$$P = \frac{1}{2}(4.472)(1.118) = 2.5 \text{ W}$$

which is the same as the average power supplied. Zero average power is absorbed by the capacitor.

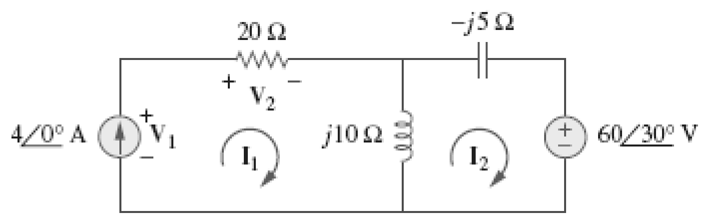
Instantaneous Power and Average Power

Example 11.4

Determine the average power generated by each source and the average power absorbed by each passive element in the circuit of Fig. 11.5(a).



(a)



(b)

Figure 11.5
For Example 11.4.

Solution:
We apply mesh analysis as shown in Fig. 11.5(b). For mesh 1,

$$\mathbf{I}_1 = 4 \text{ A}$$

For mesh 2,

$$(j10 - j5)\mathbf{I}_2 - j10\mathbf{I}_1 + 60\underline{/30^\circ} = 0, \qquad \mathbf{I}_1 = 4 \text{ A}$$

or

$$j5\mathbf{I}_2 = -60\underline{/30^\circ} + j40 \quad \Rightarrow \quad \mathbf{I}_2 = -12\underline{/-60^\circ} + 8$$
$$= 10.58\underline{/79.1^\circ} \text{ A}$$

For the voltage source, the current flowing from it is $\mathbf{I}_2 = 10.58\underline{/79.1^\circ}$ A and the voltage across it is $60\underline{/30^\circ}$ V, so that the average power is

$$P_5 = \frac{1}{2}(60)(10.58)\cos(30^\circ - 79.1^\circ) = 207.8 \text{ W}$$

Following the passive sign convention (see Fig. 1.8), this average power is absorbed by the source, in view of the direction of $\mathbf{I}_2$ and the polarity of the voltage source. That is, the circuit is delivering average power to the voltage source.

For the current source, the current through it is $\mathbf{I}_1 = 4\underline{/0^\circ}$ and the voltage across it is

$$\mathbf{V}_1 = 20\mathbf{I}_1 + j10(\mathbf{I}_1 - \mathbf{I}_2) = 80 + j10(4 - 2 - j10.39)$$
$$= 183.9 + j20 = 184.984\underline{/6.21^\circ} \text{ V}$$

Instantaneous Power and Average Power

The average power supplied by the current source is

$$P_1 = -\frac{1}{2}(184.984)(4)\cos(6.21^\circ - 0) = -367.8 \text{ W}$$

It is negative according to the passive sign convention, meaning that the current source is supplying power to the circuit.

For the resistor, the current through it is $\mathbf{I}_1 = 4\underline{/0^\circ}$ and the voltage across it is $20\mathbf{I}_1 = 80\underline{/0^\circ}$, so that the power absorbed by the resistor is

$$P_2 = \frac{1}{2}(80)(4) = 160 \text{ W}$$

For the capacitor, the current through it is $\mathbf{I}_2 = 10.58\underline{/79.1^\circ}$ and the voltage across it is $-j5\mathbf{I}_2 = (5\underline{/-90^\circ})(10.58\underline{/79.1^\circ}) = 52.9\underline{/79.1^\circ - 90^\circ}$. The average power absorbed by the capacitor is

$$P_4 = \frac{1}{2}(52.9)(10.58)\cos(-90^\circ) = 0$$

For the inductor, the current through it is $\mathbf{I}_1 - \mathbf{I}_2 = 2 - j10.39 = 10.58\underline{/-79.1^\circ}$. The voltage across it is $j10(\mathbf{I}_1 - \mathbf{I}_2) = 10.58\underline{/-79.1^\circ + 90^\circ}$. Hence, the average power absorbed by the inductor is

$$P_3 = \frac{1}{2}(105.8)(10.58)\cos 90^\circ = 0$$

Notice that the inductor and the capacitor absorb zero average power and that the total power supplied by the current source equals the power absorbed by the resistor and the voltage source, or

$$P_1 + P_2 + P_3 + P_4 + P_5 = -367.8 + 160 + 0 + 0 + 207.8 = 0$$

indicating that power is conserved.

Maximum Average Power Transfer

Consider the circuit in Fig. 11.7, where an ac circuit is connected to a load $\mathbf{Z}_L$ and is represented by its Thevenin equivalent. The load is usually represented by an impedance, which may model an electric motor, an antenna, a TV, and so forth.

In rectangular form, the Thevenin impedance $\mathbf{Z}_{Th}$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_{Th} = R_{Th} + jX_{Th}$$

$$\mathbf{Z}_L = R_L + jX_L$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L} = \frac{\mathbf{V}_{Th}}{(R_{Th} + jX_{Th}) + (R_L + jX_L)}$$

the average power delivered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L/2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

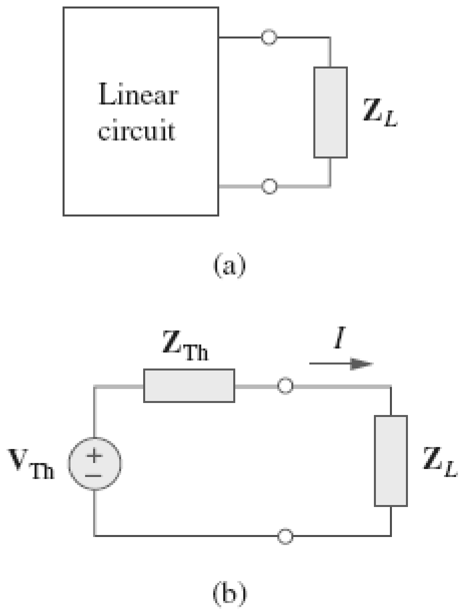


Figure 11.7
Finding the maximum average power transfer: (a) circuit with a load, (b) the Thevenin equivalent.

Our objective is to adjust the load parameters R_L and X_L so that P is maximum. To do this we set $\partial P/\partial R_L$ and $\partial P/\partial X_L$ equal to zero.

Maximum Average Power Transfer

Setting $\partial P/\partial X_L$ to zero gives

$$X_L = -X_{Th} \tag{11.17}$$

and setting $\partial P/\partial R_L$ to zero results in

$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2} \tag{11.18}$$

Combining Eqs. (11.17) and (11.18) leads to the conclusion that for maximum average power transfer, $\mathbf{Z}_L$ must be selected so that $X_L = -X_{Th}$ and $R_L = R_{Th}$, i.e.,

$$\boxed{\mathbf{Z}_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^*} \tag{11.19}$$

For maximum average power transfer, the load impedance $\mathbf{Z}_L$ must be equal to the complex conjugate of the Thevenin impedance $\mathbf{Z}_{Th}$.

This result is known as the *maximum average power transfer theorem* for the sinusoidal steady state. Setting $R_L = R_{Th}$ and $X_L = -X_{Th}$ in Eq. (11.15) gives us the maximum average power as

$$\boxed{P_{max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}}} \tag{11.20}$$

Maximum Average Power Transfer

In a situation in which the load is purely real, the condition for maximum power transfer is obtained from Eq. (11.18) by setting $X_L = 0$; that is,

$$R_L = \sqrt{R_{\text{Th}}^2 + X_{\text{Th}}^2} = |\mathbf{Z}_{\text{Th}}| \qquad \textbf{(11.21)}$$

This means that for maximum average power transfer to a purely resistive load, the load impedance (or resistance) is equal to the magnitude of the Thevenin impedance.

Maximum Average Power Transfer

Example 11.5

Determine the load impedance $\mathbf{Z}_L$ that maximizes the average power drawn from the circuit of Fig. 11.8. What is the maximum average power?

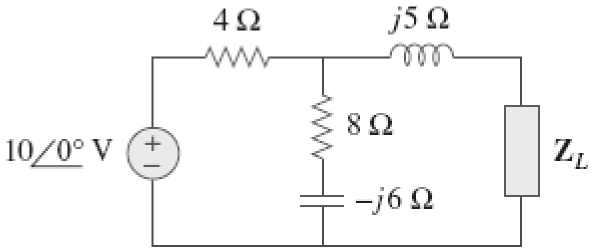


Figure 11.8
For Example 11.5.

Solution:

First we obtain the Thevenin equivalent at the load terminals. To get $\mathbf{Z}_{\text{Th}}$, consider the circuit shown in Fig. 11.9(a). We find

$$\mathbf{Z}_{\text{Th}} = j5 + 4 \parallel (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

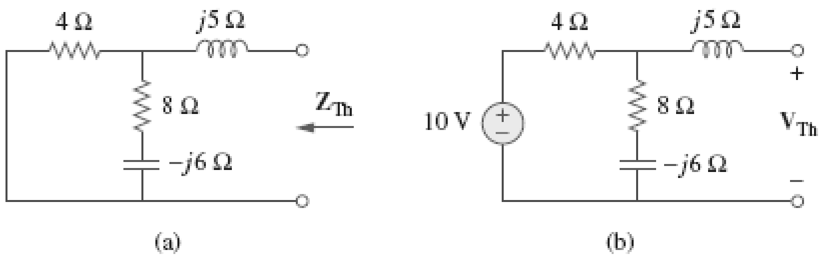


Figure 11.9
Finding the Thevenin equivalent of the circuit in Fig. 11.8.

To find $\mathbf{V}_{\text{Th}}$, consider the circuit in Fig. 11.8(b). By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

The load impedance draws the maximum power from the circuit when

$$\mathbf{Z}_L = \mathbf{Z}_{\text{Th}}^* = 2.933 - j4.467\ \Omega$$

According to Eq. (11.20), the maximum average power is

$$P_{\text{max}} = \frac{|\mathbf{V}_{\text{Th}}|^2}{8R_{\text{Th}}} = \frac{(7.454)^2}{8(2.933)} = 2.368\ \text{W}$$

Maximum Average Power Transfer

Example 11.6

In the circuit in Fig. 11.11, find the value of R_L that will absorb the maximum average power. Calculate that power.

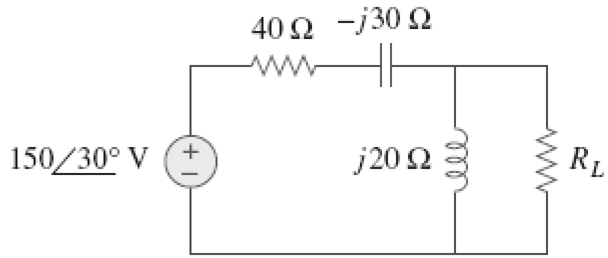


Figure 11.11
For Example 11.6.

Solution:
We first find the Thevenin equivalent at the terminals of R_L.

$$\mathbf{Z}_{\text{Th}} = (40 - j30) \| j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

By voltage division,

$$\mathbf{V}_{\text{Th}} = \frac{j20}{j20 + 40 - j30}(150\angle 30^\circ) = 72.76\angle 134^\circ\ \text{V}$$

The value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_{\text{Th}}| = \sqrt{9.412^2 + 22.35^2} = 24.25\ \Omega$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_{\text{Th}}}{\mathbf{Z}_{\text{Th}} + R_L} = \frac{72.76\angle 134^\circ}{33.66 + j22.35} = 1.8\angle 100.42^\circ\ \text{A}$$

The maximum average power absorbed by R_L is

$$P_{\text{max}} = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}(1.8)^2(24.25) = 39.29\ \text{W}$$

Effective or RMS Value

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current.

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \qquad V_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T v^2\,dt}$$

This indicates that the effective value is the (square) *root* of the *mean* (or average) of the *square* of the periodic signal. Thus, the effective value is often known as the *root-mean-square* value, or *rms* value for short; and we write

$$I_{\text{eff}} = I_{\text{rms}}, \qquad V_{\text{eff}} = V_{\text{rms}}$$

For any periodic function $x(t)$ in general, the rms value is given by

$$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

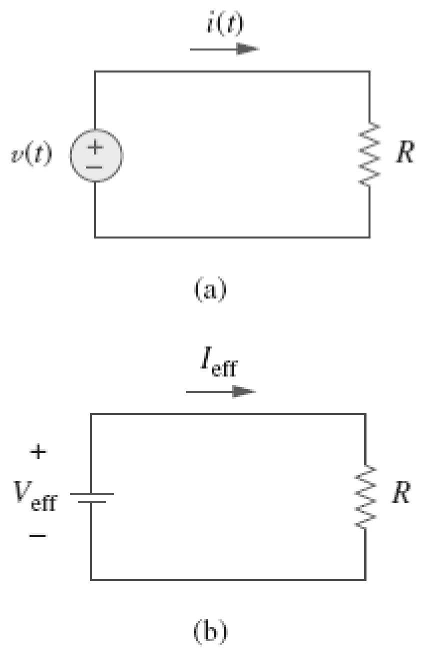


Figure 11.13
Finding the effective current: (a) ac circuit, (b) dc circuit.

Effective or RMS Value

$$X_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\,dt}$$

For the sinusoid $i(t) = I_m \cos\omega t$, the effective or rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2\omega t\,dt}$$

$$= \sqrt{\frac{I_m^2}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega t)\,dt} = \frac{I_m}{\sqrt{2}}$$

Similarly, for $v(t) = V_m \cos\omega t$,

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

The average power in Eq. (11.8) can be written in terms of the rms values.

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$
$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{11.30}$$

Effective or RMS Value

Similarly, the average power absorbed by a resistor R in Eq. (11.11) can be written as

$$P = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \qquad \textbf{(11.31)}$$

When a sinusoidal voltage or current is specified, it is often in terms of its maximum (or peak) value or its rms value, since its average value is zero. The power industries specify phasor magnitudes in terms of their rms values rather than peak values. For instance, the 110 V available at every household is the rms value of the voltage from the power company. It is convenient in power analysis to express voltage and current in their rms values. Also, analog voltmeters and ammeters are designed to read directly the rms value of voltage and current, respectively.

Effective or RMS Value

Example 11.7

Determine the rms value of the current waveform in Fig. 11.14. If the current is passed through a 2-Ω resistor, find the average power absorbed by the resistor.

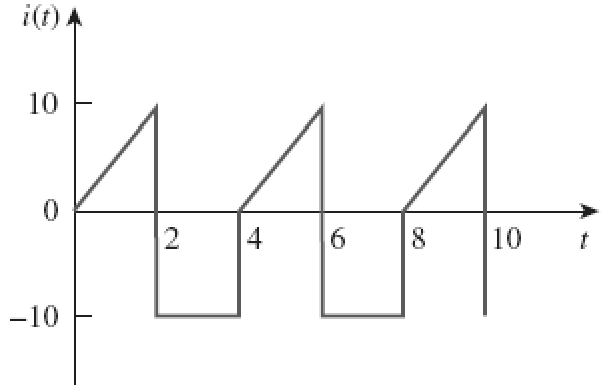


Figure 11.14
For Example 11.7.

Solution:

The period of the waveform is $T = 4$. Over a period, we can write the current waveform as

$$i(t) = \begin{cases} 5t, & 0 < t < 2 \\ -10, & 2 < t < 4 \end{cases}$$

The rms value is

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{4}\left[\int_0^2 (5t)^2\,dt + \int_2^4 (-10)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{4}\left[25\frac{t^3}{3}\bigg|_0^2 + 100t\bigg|_2^4\right]} = \sqrt{\frac{1}{4}\left(\frac{200}{3} + 200\right)} = 8.165\text{ A}$$

The power absorbed by a 2-Ω resistor is

$$P = I_{\text{rms}}^2 R = (8.165)^2(2) = 133.3\text{ W}$$

Effective or RMS Value

Example 11.8

The waveform shown in Fig. 11.16 is a half-wave rectified sine wave. Find the rms value and the amount of average power dissipated in a 10-Ω resistor.

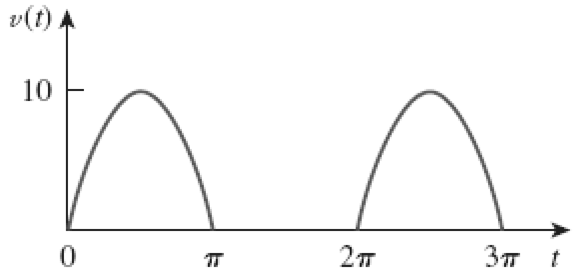


Figure 11.16
For Example 11.8.

Solution:

The period of the voltage waveform is $T = 2\pi$, and

$$v(t) = \begin{cases} 10 \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

The rms value is obtained as

$$V_{\text{rms}}^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} (10 \sin t)^2\,dt + \int_{\pi}^{2\pi} 0^2\,dt\right]$$

But $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$. Hence,

$$V_{\text{rms}}^2 = \frac{1}{2\pi}\int_0^{\pi} \frac{100}{2}(1 - \cos 2t)\,dt = \frac{50}{2\pi}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{\pi}$$

$$= \frac{50}{2\pi}\left(\pi - \frac{1}{2}\sin 2\pi - 0\right) = 25, \qquad V_{\text{rms}} = 5\text{ V}$$

The average power absorbed is

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{5^2}{10} = 2.5\text{ W}$$

Apparent Power and Power Factor

$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \theta_i)$$

or, in phasor form, $\mathbf{V} = V_m \underline{/\theta_v}$ and $\mathbf{I} = I_m \underline{/\theta_i}$, the average power is

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$$

We have added a new term to the equation:

$$\boxed{S = V_{\text{rms}} I_{\text{rms}}}$$

The average power is a product of two terms. The product $V_{\text{rms}} I_{\text{rms}}$ is known as the *apparent power* S. The factor $\cos(\theta_v - \theta_i)$ is called the *power factor* (pf).

Apparent Power and Power Factor

The apparent power is so called because it seems apparent that the power should be the voltage-current product, by analogy with dc resistive circuits. It is measured in volt-amperes or VA to distinguish it from the average or real power, which is measured in watts. The power factor is dimensionless, since it is the ratio of the average power to the apparent power,

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i) \tag{11.36}$$

The angle $\theta_v - \theta_i$ is called the *power factor angle*, since it is the angle whose cosine is the power factor. The power factor angle is equal to the angle of the load impedance if **V** is the voltage across the load and **I** is the current through it. This is evident from the fact that

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m\angle\theta_v}{I_m\angle\theta_i} = \frac{V_m}{I_m}\angle\theta_v - \theta_i = \frac{V_{\text{rms}}}{I_{\text{rms}}}\angle\theta_v - \theta_i$$

The power factor is the cosine of the phase difference between voltage and current. It is also the cosine of the angle of the load impedance.

Apparent Power and Power Factor

Example 11.9

A series-connected load draws a current $i(t) = 4\cos(100\pi t + 10^\circ)$ A when the applied voltage is $v(t) = 120\cos(100\pi t - 20^\circ)$ V. Find the apparent power and the power factor of the load. Determine the element values that form the series-connected load.

Solution:

The apparent power is

$$S = V_{\text{rms}} I_{\text{rms}} = \frac{120}{\sqrt{2}} \frac{4}{\sqrt{2}} = 240 \text{ VA}$$

The power factor is

$$\text{pf} = \cos(\theta_v - \theta_i) = \cos(-20^\circ - 10^\circ) = 0.866 \qquad \text{(leading)}$$

The pf is leading because the current leads the voltage. The pf may also be obtained from the load impedance.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{120\angle -20^\circ}{4\angle 10^\circ} = 30\angle -30^\circ = 25.98 - j15\ \Omega$$

$$\text{pf} = \cos(-30^\circ) = 0.866 \qquad \text{(leading)}$$

The load impedance $\mathbf{Z}$ can be modeled by a 25.98-Ω resistor in series with a capacitor with

$$X_C = -15 = -\frac{1}{\omega C} \qquad C = \frac{1}{15\omega} = \frac{1}{15 \times 100\pi} = 212.2\ \mu\text{F}$$

Apparent Power and Power Factor

Example 11.10

Determine the power factor of the entire circuit of Fig. 11.18 as seen by the source. Calculate the average power delivered by the source.

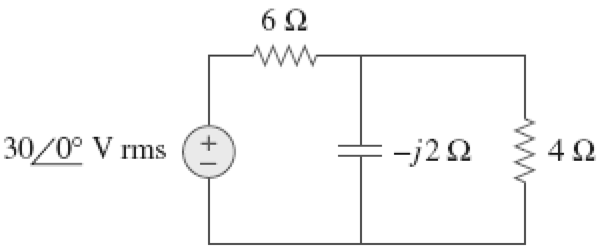


Figure 11.18
For Example 11.10.

Solution:
The total impedance is

$$\mathbf{Z} = 6 + 4 \,\|\, (-j2) = 6 + \frac{-j2 \times 4}{4 - j2} = 6.8 - j1.6 = 7\angle{-13.24^\circ}\ \Omega$$

The power factor is

$$\text{pf} = \cos(-13.24) = 0.9734 \qquad \text{(leading)}$$

since the impedance is capacitive. The rms value of the current is

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} = \frac{30\angle{0^\circ}}{7\angle{-13.24^\circ}} = 4.286\angle{13.24^\circ}\ \text{A}$$

The average power supplied by the source is

$$P = V_{\text{rms}} I_{\text{rms}}\, \text{pf} = (30)(4.286)0.9734 = 125\ \text{W}$$

or

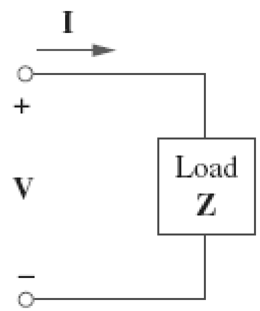
$$P = I_{\text{rms}}^2 R = (4.286)^2(6.8) = 125\ \text{W}$$

where R is the resistive part of $\mathbf{Z}$.

Complex Power

it contains *all* the information pertaining to the power absorbed by a given load.

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^*$$



$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}$$

where

$$\mathbf{V}_{\text{rms}} = \frac{\mathbf{V}}{\sqrt{2}} = V_{\text{rms}}\underline{/\theta_v}$$

and

$$\mathbf{I}_{\text{rms}} = \frac{\mathbf{I}}{\sqrt{2}} = I_{\text{rms}}\underline{/\theta_i}$$

Thus

$$\begin{aligned}\mathbf{S} &= V_{\text{rms}}I_{\text{rms}}\underline{/\theta_v - \theta_i} \\ &= V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i) + jV_{\text{rms}}I_{\text{rms}}\sin(\theta_v - \theta_i)\end{aligned}$$

the magnitude of the complex power is the apparent power; hence, the complex power is measured in volt-amperes (VA). Also, we notice that the angle of the complex power is the power factor angle.

Complex Power

$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*$$

$\mathbf{S} = I_{\text{rms}}^2 (R + jX) = P + jQ$ where P and Q are the real and imaginary parts of the complex power

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X$$

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i), \qquad Q = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i)$$

P is the average or real power and it depends on the load's resistance R. Q depends on the load's reactance X and is called the *reactive* (or quadrature) power.

Complex Power

The real power P is the average power in watts delivered to a load; it is the only useful power. It is the actual power dissipated by the load.

The reactive power Q is a measure of the energy exchange between the source and the reactive part of the load. The unit of Q is the *volt-ampere reactive* (VAR) to distinguish it from the real power, whose unit is the watt. We know from Chapter 6 that energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, the reactive power is being transferred back and forth between the load and the source. It represents a lossless interchange between the load and the source. Notice that:

1. $Q = 0$ for resistive loads (unity pf).
2. $Q < 0$ for capacitive loads (leading pf).
3. $Q > 0$ for inductive loads (lagging pf).

$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$

$$= |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| \underline{/\theta_v - \theta_i}$$

$$\text{Apparent Power} = S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}|\,|\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$

$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$

$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

This shows how the complex power contains *all* the relevant power information in a given load.

Complex Power

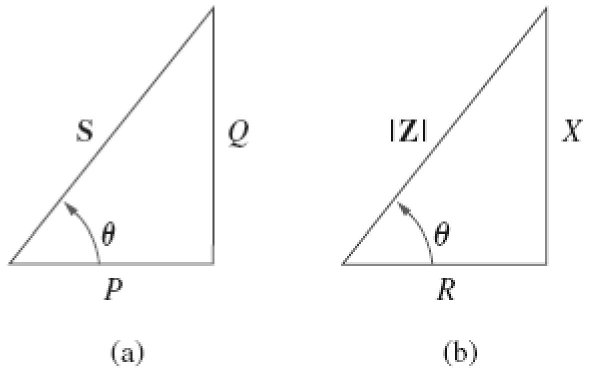


Figure 11.21
(a) Power triangle, (b) impedance triangle.

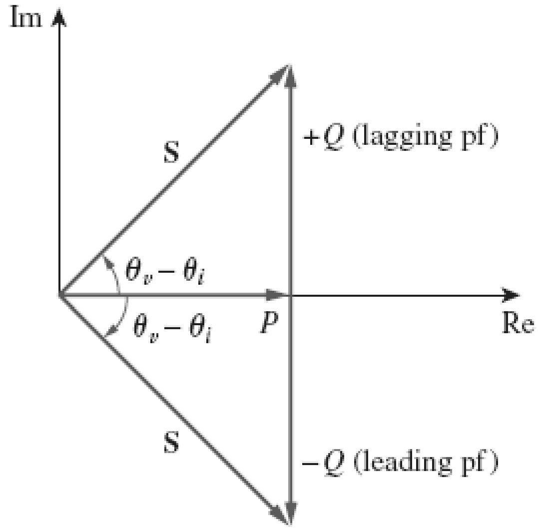


Figure 11.22
Power triangle.

Complex Power

Example 11.11

The voltage across a load is $v(t) = 60\cos(\omega t - 10^\circ)$ V and the current through the element in the direction of the voltage drop is $i(t) = 1.5\cos(\omega t + 50^\circ)$ A. Find: (a) the complex and apparent powers, (b) the real and reactive powers, and (c) the power factor and the load impedance.

Solution:

(a) For the rms values of the voltage and current, we write

$$\mathbf{V}_{\text{rms}} = \frac{60}{\sqrt{2}}\angle -10^\circ, \qquad \mathbf{I}_{\text{rms}} = \frac{1.5}{\sqrt{2}}\angle +50^\circ$$

The complex power is

$$\mathbf{S} = \mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}} = \left(\frac{60}{\sqrt{2}}\angle -10^\circ\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^\circ\right) = 45\angle -60^\circ \text{ VA}$$

The apparent power is

$$S = |\mathbf{S}| = 45 \text{ VA}$$

(b) We can express the complex power in rectangular form as

$$\mathbf{S} = 45\angle -60^\circ = 45[\cos(-60^\circ) + j\sin(-60^\circ)] = 22.5 - j38.97$$

Since $\mathbf{S} = P + jQ$, the real power is

$$P = 22.5 \text{ W}$$

while the reactive power is

$$Q = -38.97 \text{ VAR}$$

(c) The power factor is

$$\text{pf} = \cos(-60^\circ) = 0.5 \text{ (leading)}$$

It is leading, because the reactive power is negative. The load impedance is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^\circ}{1.5\angle +50^\circ} = 40\angle -60^\circ\ \Omega$$

which is a capacitive impedance.

Complex Power

Example 11.12

A load **Z** draws 12 kVA at a power factor of 0.856 lagging from a 120-V rms sinusoidal source. Calculate: (a) the average and reactive powers delivered to the load, (b) the peak current, and (c) the load impedance.

Solution:

(a) Given that pf = $\cos\theta = 0.856$, we obtain the power angle as $\theta = \cos^{-1} 0.856 = 31.13°$. If the apparent power is $S = 12{,}000$ VA, then the average or real power is

$$P = S\cos\theta = 12{,}000 \times 0.856 = 10.272 \text{ kW}$$

while the reactive power is

$$Q = S\sin\theta = 12{,}000 \times 0.517 = 6.204 \text{ kVA}$$

(b) Since the pf is lagging, the complex power is

$$\mathbf{S} = P + jQ = 10.272 + j6.204 \text{ kVA}$$

From $\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$, we obtain

$$\mathbf{I}^*_{rms} = \frac{\mathbf{S}}{\mathbf{V}_{rms}} = \frac{10{,}272 + j6204}{120\angle 0°} = 85.6 + j51.7 \text{ A} = 100\angle 31.13° \text{ A}$$

Thus $\mathbf{I}_{rms} = 100\angle -31.13°$ and the peak current is

$$I_m = \sqrt{2}I_{rms} = \sqrt{2}(100) = 141.4 \text{ A}$$

(c) The load impedance

$$\mathbf{Z} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{120\angle 0°}{100\angle -31.13°} = 1.2\angle 31.13° \ \Omega$$

which is an inductive impedance.

Complex Power

Example 11.13

Figure 11.24 shows a load being fed by a voltage source through a transmission line. The impedance of the line is represented by the $(4 + j2)\ \Omega$ impedance and a return path. Find the real power and reactive power absorbed by: (a) the source, (b) the line, and (c) the load.

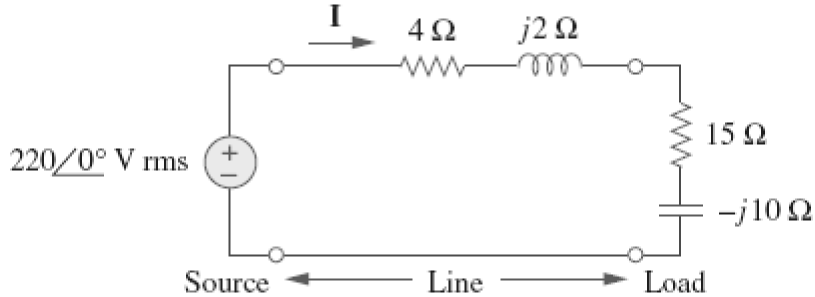


Figure 11.24
For Example 11.13.

Solution:

The total impedance is

$$\mathbf{Z} = (4 + j2) + (15 - j10) = 19 - j8 = 20.62\angle{-22.83^\circ}\ \Omega$$

The current through the circuit is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{220\angle{0^\circ}}{20.62\angle{-22.83^\circ}} = 10.67\angle{22.83^\circ} \text{ A rms}$$

(a) For the source, the complex power is

$$\begin{aligned} \mathbf{S}_s = \mathbf{V}_s\mathbf{I}^* &= (220\angle{0^\circ})(10.67\angle{-22.83^\circ}) \\ &= 2347.4\angle{-22.83^\circ} = (2163.5 - j910.8) \text{ VA} \end{aligned}$$

From this, we obtain the real power as 2163.5 W and the reactive power as 910.8 VAR (leading).

(b) For the line, the voltage is

$$\begin{aligned} \mathbf{V}_{\text{line}} = (4 + j2)\mathbf{I} &= (4.472\angle{26.57^\circ})(10.67\angle{22.83^\circ}) \\ &= 47.72\angle{49.4^\circ} \text{ V rms} \end{aligned}$$

The complex power absorbed by the line is

$$\begin{aligned} \mathbf{S}_{\text{line}} = \mathbf{V}_{\text{line}}\mathbf{I}^* &= (47.72\angle{49.4^\circ})(10.67\angle{-22.83^\circ}) \\ &= 509.2\angle{26.57^\circ} = 455.4 + j227.7 \text{ VA} \end{aligned}$$

That is, the real power is 455.4 W and the reactive power is 227.76 VAR (lagging).

(c) For the load, the voltage is

$$\begin{aligned} \mathbf{V}_L = (15 - j10)\mathbf{I} &= (18.03\angle{-33.7^\circ})(10.67\angle{22.83^\circ}) \\ &= 192.38\angle{-10.87^\circ} \text{ V rms} \end{aligned}$$

The complex power absorbed by the load is

$$\begin{aligned} \mathbf{S}_L = \mathbf{V}_L\mathbf{I}^* &= (192.38\angle{-10.87^\circ})(10.67\angle{-22.83^\circ}) \\ &= 2053\angle{-33.7^\circ} = (1708 - j1139) \text{ VA} \end{aligned}$$

The real power is 1708 W and the reactive power is 1139 VAR (leading). Note that $\mathbf{S}_s = \mathbf{S}_{\text{line}} + \mathbf{S}_L$, as expected. We have used the rms values of voltages and currents.

Complex Power

Example 11.14

In the circuit of Fig. 11.26, $\mathbf{Z}_1 = 60\angle -30^\circ\ \Omega$ and $\mathbf{Z}_2 = 40\angle 45^\circ\ \Omega$. Calculate the total: (a) apparent power, (b) real power, (c) reactive power, and (d) pf, supplied by the source and seen by the source.

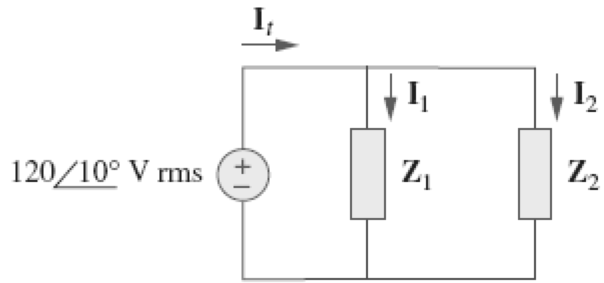


Figure 11.26
For Example 11.14.

Solution:

The current through $\mathbf{Z}_1$ is

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{120\angle 10^\circ}{60\angle -30^\circ} = 2\angle 40^\circ \text{ A rms}$$

while the current through $\mathbf{Z}_2$ is

$$\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{120\angle 10^\circ}{40\angle 45^\circ} = 3\angle -35^\circ \text{ A rms}$$

The complex powers absorbed by the impedances are

$$\mathbf{S}_1 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_1^*} = \frac{(120)^2}{60\angle 30^\circ} = 240\angle -30^\circ = 207.85 - j120 \text{ VA}$$

$$\mathbf{S}_2 = \frac{V_{\text{rms}}^2}{\mathbf{Z}_2^*} = \frac{(120)^2}{40\angle -45^\circ} = 360\angle 45^\circ = 254.6 + j254.6 \text{ VA}$$

The total complex power is

$$\mathbf{S}_t = \mathbf{S}_1 + \mathbf{S}_2 = 462.4 + j134.6 \text{ VA}$$

(a) The total apparent power is

$$|\mathbf{S}_t| = \sqrt{462.4^2 + 134.6^2} = 481.6 \text{ VA.}$$

(b) The total real power is

$$P_t = \text{Re}(\mathbf{S}_t) = 462.4 \text{ W or } P_t = P_1 + P_2.$$

(c) The total reactive power is

$$Q_t = \text{Im}(\mathbf{S}_t) = 134.6 \text{ VAR or } Q_t = Q_1 + Q_2.$$

(d) The pf $= P_t/|\mathbf{S}_t| = 462.4/481.6 = 0.96$ (lagging).

We may cross check the result by finding the complex power $\mathbf{S}_s$ supplied by the source.

$$\begin{aligned}\mathbf{I}_t = \mathbf{I}_1 + \mathbf{I}_2 &= (1.532 + j1.286) + (2.457 - j1.721)\\ &= 4 - j0.435 = 4.024\angle -6.21^\circ \text{ A rms}\\ \mathbf{S}_s = \mathbf{V}\mathbf{I}_t^* &= (120\angle 10^\circ)(4.024\angle 6.21^\circ)\\ &= 482.88\angle 16.21^\circ = 463 + j135 \text{ VA}\end{aligned}$$

which is the same as before.