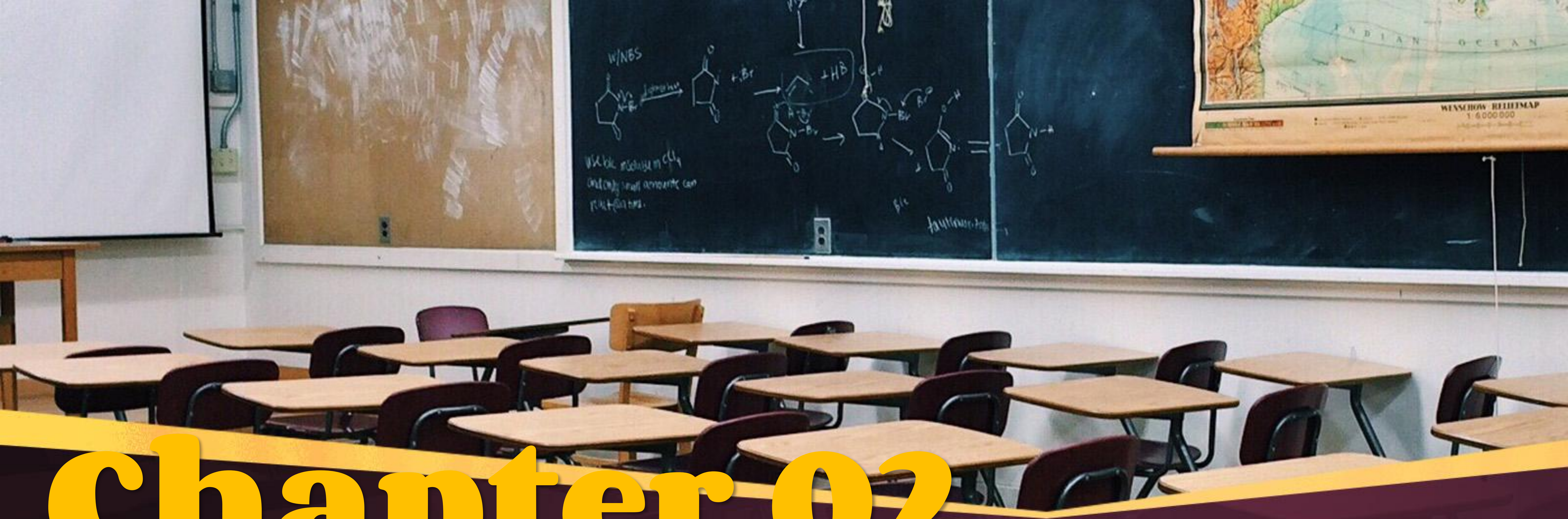


Chapter 02

Solution of Algebraic and Transcendental Equations

Solution of Algebraic and Transcendental equations

Chapter Outcomes :

After reading this chapter, you should be able to:

1. **Know about the Error, exact, relative, and percentage error.**
2. **Relate the absolute, relative, and approximate error to the number of significant digits.**
3. **Know that there are two inherent sources of error in numerical methods – round-off and truncation error.**
4. **Know the difference between round-off and truncation error.**
5. **Know the concept of significant digits.**

Solution of Algebraic and Transcendental equations

In this Lesson, we have discussed about the solution of equations , $f(x) = 0$

Where , f (X) is linear, nonlinear, **algebraic** or transcendental function.

We get the solution of the equation f (x) = 0 by using

(1) **Bisection method**,

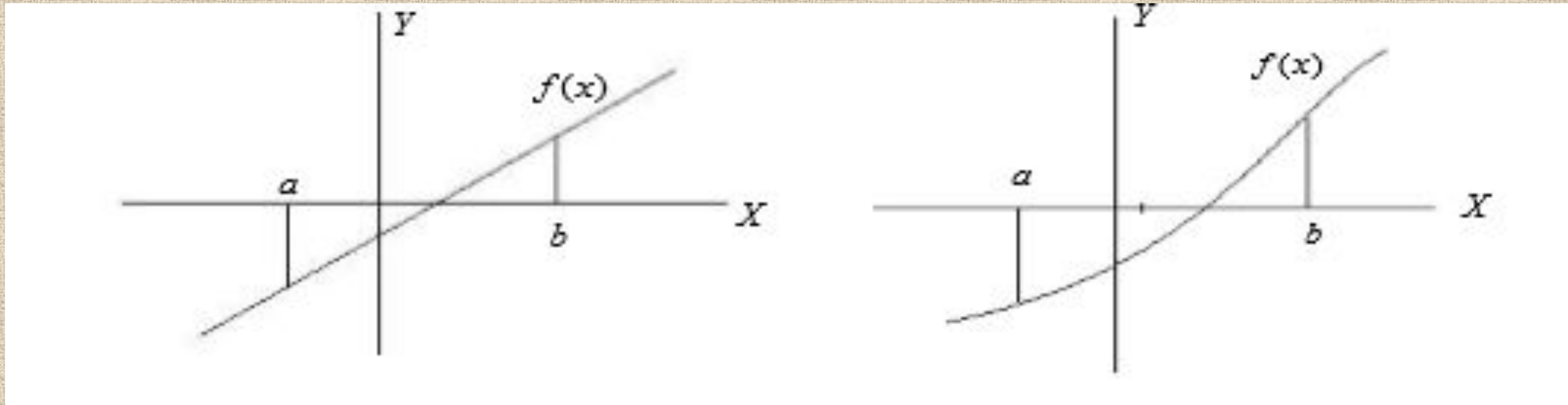
(2) Newton- Raphson method and

(3) **Method of false position.**

Those methods are established based on Intermediate Value Theorem.

Statement of Intermediate Value Theorem:

If f(x) is continuous in the interval (a, b) and if f (a) and f(b) are of opposite signs, then the equation f(x) = 0 will have at least one real root between a and b.



❖Algebraic Equation :

An algebraic equation is an equation that includes one or more variables such as

$$x^2 + xy - z = 0$$

❖Transcendental equation:

An equation together with algebraic, trigonometrical, exponential or logarithmic function etc. is called transcendental equation such as

$$e^x + 5\sin x - 2x = 0$$

❖Solution/root:

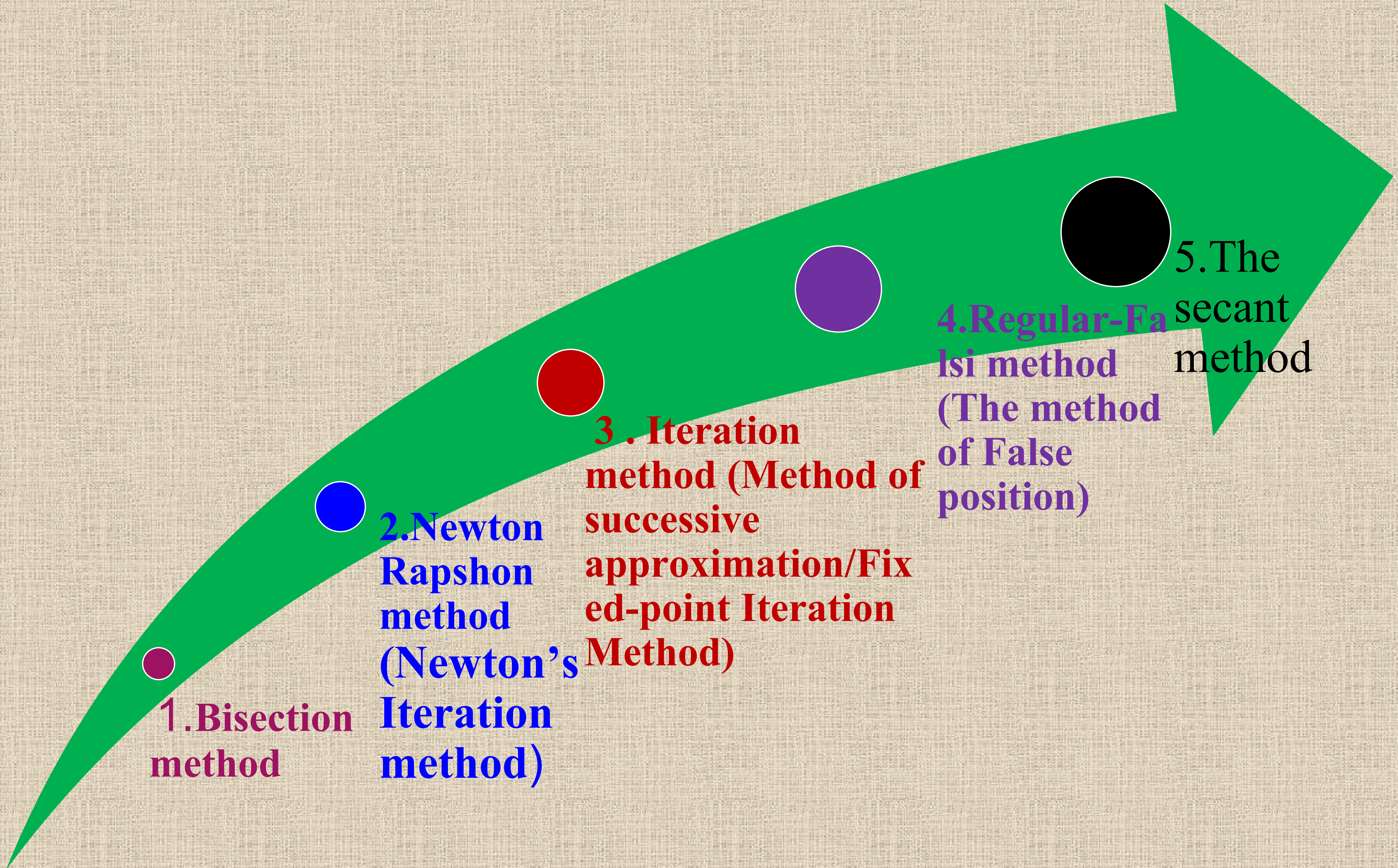
A solution/root of an equation is the value of the variable or variables that satisfies the equation.

Iteration:

- Iteration is the repeated process of calculation until the desired result or approximate numerical value has come.

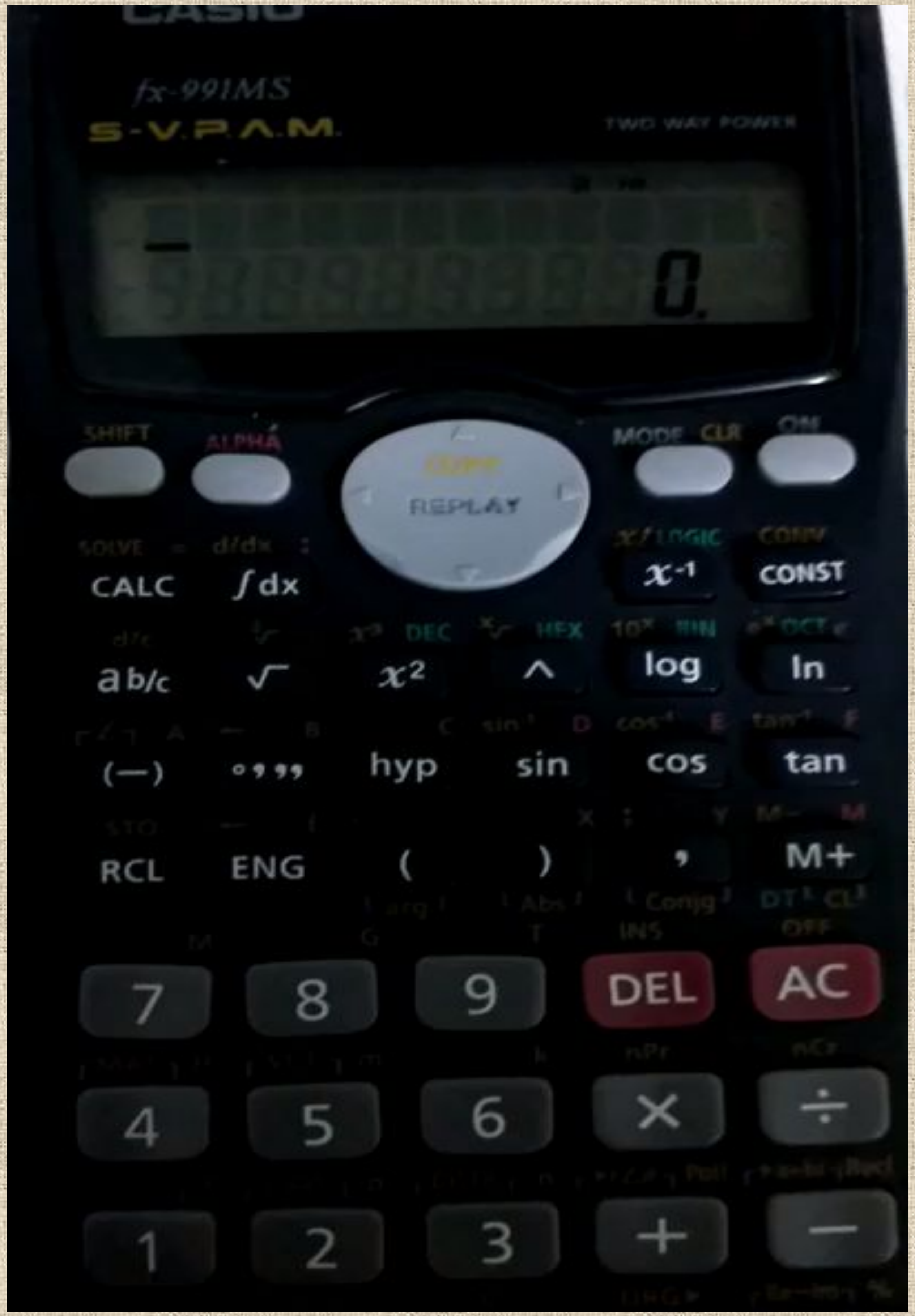
- Each repetition of the process is also called iteration and the result of one iteration is used as the starting point for the next iteration.

❖ We are capable to find the root of algebraic or transcendental function by using following methods:



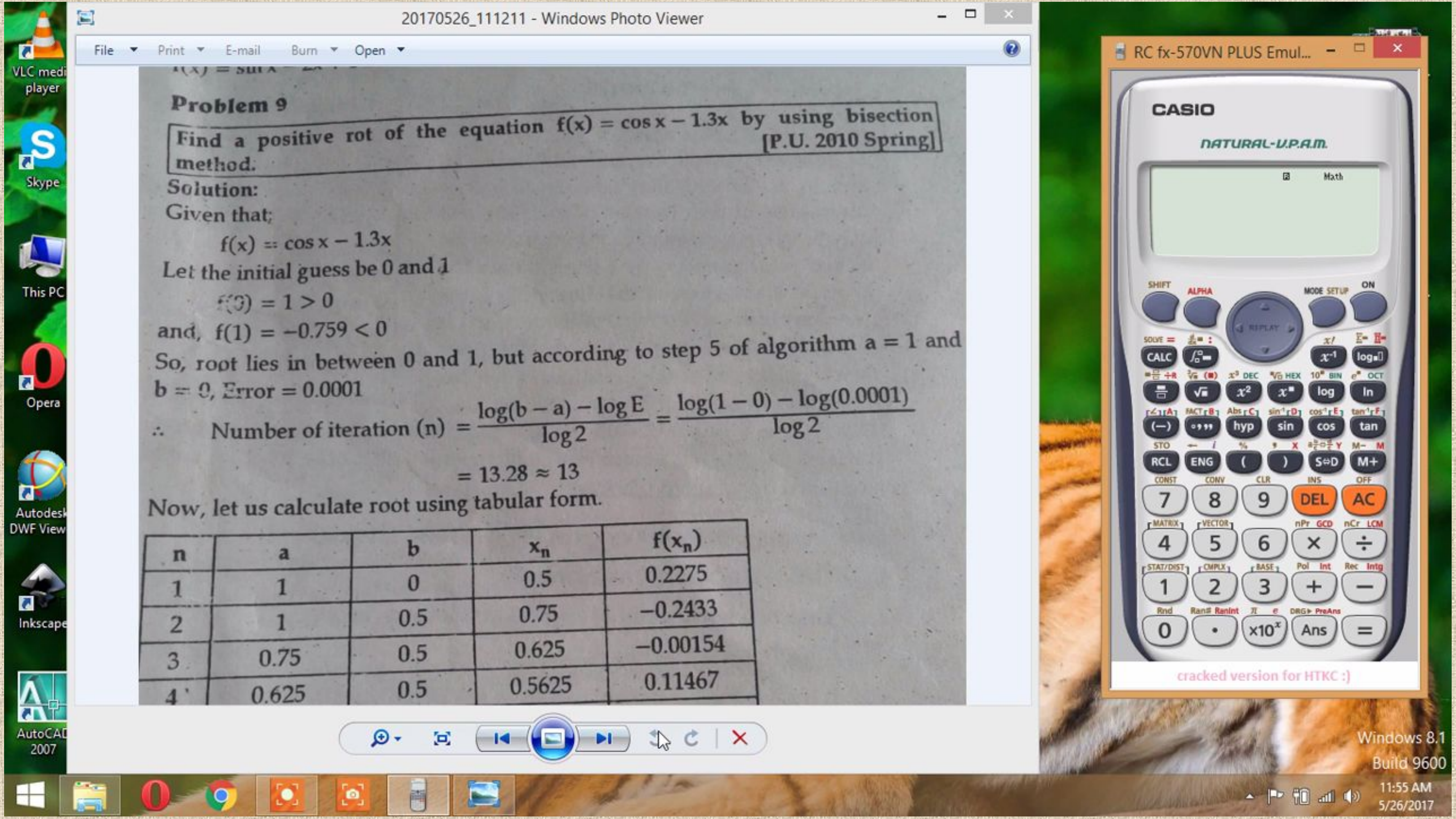
*** You have to know , how to use calculator for solving the mathematical problem

*** No. 01 : To determine the value of the trigonometrical function f(x), we have to change our calculator in radian mode.



Use of Calculator

*** You have to know , how to use calculator for solving the mathematical problem.



Problem 9

Find a positive rot of the equation $f(x) = \cos x - 1.3x$ by using bisection method. [P.U. 2010 Spring]

Solution:

Given that:

$f(x) = \cos x - 1.3x$

Let the initial guess be 0 and 1

$f(0) = 1 > 0$

and, $f(1) = -0.759 < 0$

So, root lies in between 0 and 1, but according to step 5 of algorithm a = 1 and b = 0, Error = 0.0001

$\therefore$ Number of iteration (n) $= \dfrac{\log(b-a) - \log E}{\log 2} = \dfrac{\log(1-0) - \log(0.0001)}{\log 2}$

$= 13.28 \approx 13$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	0	0.5	0.2275
2	1	0.5	0.75	−0.2433
3	0.75	0.5	0.625	−0.00154
4	0.625	0.5	0.5625	0.11467

Bisection Method

Procedure of Bisection Method :

Step 1 : Given function, f (x)

$$Let, x^3 + x^2 - 1 = 0$$

$$Given, f(x) = x^3 + x^2 - 1$$

Find f (x) in the step 1

$$f(x) = x^3 + x^2 - 1$$

For, $a = x = -1,\ f(a) = a^3 + a^2 - 1$

$\therefore f(-1) = (-1)^3 + (-1)^2 - 1 = -1 < 0$

Step 2 : Choose , two real numbers a and b

Such that , $f(a) \times f(b) < 0$

For, $b = x = 2 \therefore f(b) = b^3 + b^2 - 1$

$\therefore f(2) = 2^3 + 2^2 - 2 = +10 > 0$

For, $a = 0,\ f(a) = a^3 + a^2 - 1$

$\therefore f(0) = 0^3 + 0^2 - 1 = -1$

For, $b = 1 \therefore f(b) = b^3 + b^2 - 1$

$\therefore f(1) = 1^3 + 1^2 - 1 = +1$

$\therefore f(a) \times f(b) = f(0) \times f(1) = (-1) \times 1 = -1 < 0$

Since , f (a) = f(0) is negative and f (b) = f (1) is positive
So at least one real root lies between 0 and 1.

Find the Interval between (a , b) in the step 2

Find out the mid point of a & b and give it name c.

Step 3:

$Find\ the\ mid\ point\ of\ a\ and\ b\ say,\ c\ .$

$$\therefore\ c=\frac{a+b}{2}$$

C is the root of the given function if f (c) = 0; else follow the next step.

$a=0, b=1$

$$root, c=\frac{a+b}{2}=\frac{0+1}{2}=\frac{1}{2}=0.5$$

Find out f (c) by using the mid point c .

Step 4 : $Find\ f(c).$

$$f(x)=x^3+x^2-1$$

$$\therefore f(c)=c^3+c^2-1$$

$$=(0.5)^3+(0.5)^2-1$$

$$=0.125+0.25-1$$

$$=-0.625$$

Step 4 : Case I : If f (c) is negative , Then b = c

Case II : If f (c) is positive , Then a = c

f (c) is negative

Case I : If f (c) is negative , Then b = c

Case II : If f (c) is positive , Then a = c

No of Iterations	Value of a [(+) ve]	Value of b [(-) ve]	$c=\frac{a+b}{2}$	f (c)	Sign of f (c)	
01	1	0	0.5	- 0.625	<0	b = c
02	1	0.5	0.75	-0.015	<0	b = c
03	1	0.75	0.875	0.43555	>0	a = c
04	0.875	0.75	0.8125	0.19653	>0	a = c
05	0.8125	0.75	0.78125	0.0871	>0	a = c

f (c) is positive

Step 5 : **Repeat steps 2, 3 , 4 until the last two iterations are equal or the difference between the last two iterations are near to zero.**

$f(c) = c^3 + c^2 - 1$ **Steps 3, 4 , 5**

No of Iterations	Value of a [(+) ve]	Value of b [(-) ve]	$c = \frac{a+b}{2}$	f (c)	Sign of f (c)	Change
05	0.8125	0.75	0.78125	0.0871	>0	a = c
06	0.78125	0.75	0.765625	0.03497	>0	a = c
07	0.765625	0.75	0.7578125	0.00947	>0	a = c
08	0.7578125	0.75	0.75390625	-0.0031	<0	b = c
09	0.7578125	0.75390625	0.755859375	0.0035	>0	a = c
10	0.755859375	0.75390625	0.75488281255	1.65...	>0	a = c
11	0.75488281255	0.75390625	0.754394531275			

The difference between the last two iterations are near to zero.

It is evident that from the above table, the difference between last two successive iterative values of x is

$$\left|0.75488281255 - 0.754394531275\right| \approx 0.0005 \, \textit{or near to zero.}$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.754

Problem 1 :

Find the root of the equation $xe^x = 1$ *by using Bisection Method correct up to three decimal places on the interval* $(0,1)$.

Solution: $Let,\ f(x) = xe^x - 1$ **Step : 1**

Here, let , a = 0 , b = 1 then ,

$f(a) = a.e^a - 1$

$\therefore f(0) = 0.e^0 - 1 = -1 < 0$

$Again,\ f(b) = be^b - 1$

$\therefore f(1) = 1e^1 - 1 = 1.7182 > 0$

Since f (0) and f(1) are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore mid\ point, c = \frac{a+b}{2}$$

Step : 2

$$= \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

No. of iterations	Value of a [(+) ve]	Value of a [(-) ve]	$c=\dfrac{a+b}{2}$	$f(c)=ce^{c}-1$	Sign of f (c)
01	1	0	0.5	-0.1756	<0
02	1	0.5	0.75	0.5877	>0
03	0.75	0.5	0.625	0.1676	>0
04	0.625	0.5	0.5625	- 0.0127	<0
05	0.625	0.5625	0.59375	0.0751	>0
06	0.59375	0.5625	0.578125	0.0306	>0
07	0.578125	0.5625	0.5703125	0.00877	>0
08	0.5703125	0.5625	0.56640625	- 0.0023	<0
09	0.5703125	0.56640625	0.5683594	0.00336	>0
10	0.5683594	0.56640625	0.5673828	0.000662	

It is evident that from the above table, the difference between last two successive iterative values of x is

$$|0.5683594 - 0.5673828| \approx 0.001$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.567

Problem 2: Find the root of the equation $4\sin x - e^x = 0$ by using Bisection method correct up to four decimal places.

Solution: Consider that, $f(x) = 4\sin x - e^x$

Her, let , a = 0 , b = 1 then ,

$$f(a) = 4\sin a - e^a$$

$$\therefore f(0) = 4\sin 0 - e^0$$

$$= 0 - 1$$

$$= -1.000 < 0$$

$$Again, f(b) = 4\sin b - e^b$$

$$\therefore f(1) = 4\sin 1 - e^1$$

$$= 3.3658839392 - 2.7182818285$$

$$= 0.6476021107 > 0$$

Since f (0) and f (1) are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore mid\ point, c = \frac{a+b}{2}$$

$$= \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

No of Iterations	Value of a [(+) ve]	Value of b [(-) ve]	$c = \frac{a+b}{2}$	f (c)	Sign of f (c)
01	1	0	0.5	0.268	>0
02	0.5	0	0.25	-0.294	<0
03	0.5	0.25	0.375	0.0101	>0
04	0.375	0.25	0.3125	- 0.1371	<0
05	0.375	0.3125	0.34375	-0.0621	<0
06	0.375	0.34375	0.359375	-0.0256	<0
07	0.375	0.359375	0.3671875	-0.0077	<0
08	0.375	0.3671875	0.3710937	- 0.00122	<0
09	0.375	0.3710937	0.373046	0.00566	>0
10	0.373046	0.3710937	0.372070	-0.00344	<0

No of Iterations	Value of a [(+) ve]	Value of b [(-) ve]	$c = \frac{a+b}{2}$	f (c)	Sign of f (c)
10	0.373046	0.3710937	0.372070	-0.00344	<0
11	0.373046	0.372070	0.372558	0.00455	>0
12	0.372558	0.372070	0.372279	0.0039	>0
13	0.372279	0.372070	0.372174	0.0036	>0
14	0.372174	0.372070	0.372122	0.0036	

It is evident that from the above table, the difference between last two successive iterative values of x is

$$|0.372174 - 0.372122| = 0.0000520000 \approx 0.00001$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.3721

Algorithm of Bisection Method:

Steps	Task
01	Define $f(x)$
02	Read a 'The lower bound of the desired roots'
03	Read b 'The upper bound of the desired roots'
04	Set $k=1$
05	Calculate $x_k = \dfrac{a+b}{2}$
06	Calculate $f_k = f(x_k)$
07	Print k, x_k, f_k

08	If $\lvert x_k - x_{k-1} \rvert \approx 0.0001$ then
	GOTO Step 11
	elseif
	$f(a).f_k < 0$ then $b = x_k$.
	Else
	$f(b).f_k < 0$ then $a = x_k$.
	Endif

09	Set $k = k+1$
10	GOTO Step 05
11	Print 'Required root, x_k'
12	STOP

Practice Work

Find the root of the following equation by using Bisection method correct up to four decimal places :

1. $2^{x} - 5x + 2 = 0$

2. $2x + \cos x - 3 = 0$

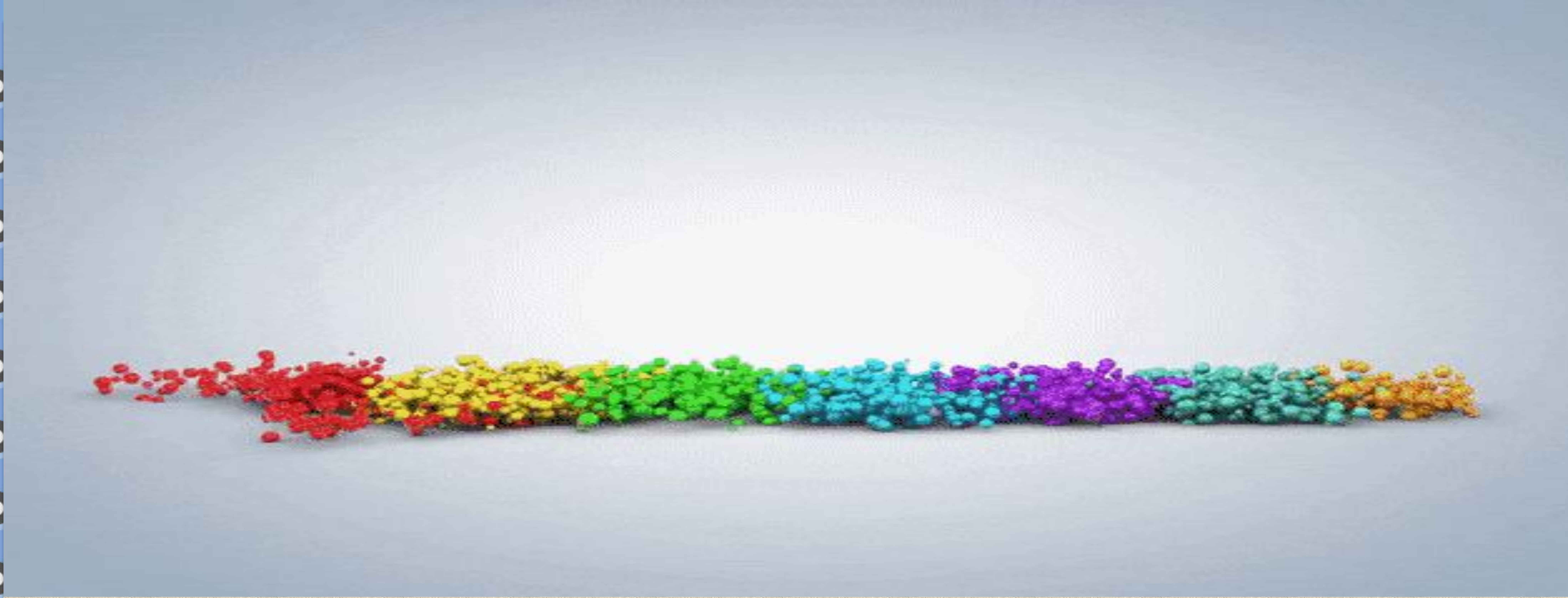
3. $x^{2} - 4x - 10 = 0$

4. $2x = 1 + \sin x$

5. $\cos x - x e^{x} = 0$

6. $e^{x} \tan x = 1$

7. $\cos(x) - \log(x) = 0$



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