NUMERICAL DIFFERENTIATION

Numerical Differentiation:

Consider Newton's forward difference formula:

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots, \quad (5.1)$$

where

$$x = x_0 + uh. \quad (5.2)$$

- $u = \frac{x - x_0}{h}$
- $\frac{du}{dx} = \frac{1}{h}$

Now from equation (5.1) we obtain,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \cdots\right). \quad (5.3)$$

This formula can be used for computing the value of dy/dx for *non-tabular values* of x. For tabular values of x, the formula takes a simpler form, for by setting $x = x_0$ we obtain $u = 0$ from (5.2), and hence (5.3) gives

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right). \quad (5.4)$$

Differentiating (5.3) once again, we obtain

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12u^2 - 36u + 22}{24}\Delta^4 y_0 + \cdots\right), \quad (5.5)$$

from which we obtain

$$\left[\frac{d^2 y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right). \quad (5.6)$$

Formulae for computing higher derivatives may be obtained by successive differentiation. In a similar way, different formulae can be derived by starting with other interpolation formuale. Thus,

(a) Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right) \qquad (5.7)$$

and

$$\left[\frac{d^2 y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right). \qquad (5.8)$$

If a derivative is required near the end of a table, one of the following formulae may be used to obtain better accuracy

$$hy_0' = \left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \frac{1}{6}\Delta^6 + \cdots\right)y_0 \qquad (5.11)$$

$$= \left(\Delta + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \frac{1}{12}\Delta^4 - \frac{1}{20}\Delta^5 + \frac{1}{30}\Delta^6 - \cdots\right)y_{-1} \qquad (5.12)$$

$$h^2 y_0'' = \left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \frac{363}{560}\Delta^8 - \cdots\right)y_0 \qquad (5.13)$$

$$= \left(\Delta^2 - \frac{1}{12}\Delta^4 + \frac{1}{12}\Delta^5 - \frac{13}{180}\Delta^6 + \frac{11}{180}\Delta^7 - \frac{29}{560}\Delta^8 + \cdots\right)y_{-1} \qquad (5.14)$$

$$hy_n' = \left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \frac{1}{5}\nabla^5 + \frac{1}{6}\nabla^6 + \frac{1}{7}\nabla^7 + \frac{1}{8}\nabla^8 + \cdots\right)y_n \qquad (5.15)$$

$$= \left(\nabla - \frac{1}{2}\nabla^2 - \frac{1}{6}\nabla^3 - \frac{1}{12}\nabla^4 - \frac{1}{20}\nabla^5 - \frac{1}{30}\nabla^6 - \frac{1}{42}\nabla^7 - \frac{1}{56}\nabla^8 - \cdots\right)y_{n+1} \qquad (5.16)$$

$$h^2 y_n'' = \left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + \cdots\right)y_n \qquad (5.17)$$

$$= \left(\nabla^2 - \frac{1}{12}\nabla^4 - \frac{1}{12}\nabla^5 - \frac{13}{180}\nabla^6 - \frac{11}{180}\nabla^7 - \frac{29}{560}\nabla^8 - \cdots\right)y_{n+1}. \qquad (5.18)$$

Example 5.1 From the following table of values of x and y, obtain dy/dx and d^2y/dx^2 for $x = 1.2$:

x	y	x	y
1.0	2.7183	1.8	6.0496
1.2	3.3201	2.0	7.3891
1.4	4.0552	2.2	9.0250
1.6	4.9530		

Solution:

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

For $x = 1.2$, here we have

$$x_0 = 1.2,\ y_0 = 3.3201 \text{ and } h = 0.2.$$

Hence we know that,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right).$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205.$$

Again we know that,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right).$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318.$$

Example-☺. The values of x and y are given in the following table :-

x	0	1	2	3	4	5
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when x = 1.

Solution : Since the derivatives are required at x = 1, which is beginning of the table, therefore we shall use the first and second derivatives of Newtor's forward formula at $x = x_0$. The difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	6.9897					
		0.4139				
1	7.4036		– 0.036			
		0.3779		0.0057		
2	7.7815		**– 0.0303**		– 0.0011	
		0.3476		**0.0046**		– 0.0001
3	8.1291		– 0.0257		**– 0.0012**	
		0.3219		0.0034		
4	8.4510		– 0.0223			
		0.2996				
5	8.7506					

Here h = 1 and x_0 = 1, the first derivative of Newton's forward formula at $x = x_0$ is :

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \quad \ldots\ldots (1)$$

and the second derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \quad \ldots\ldots (2)$$

Putting x_0 = 1, h = 1 and corresponding values of the differences in (1) and (2), we get

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{1}\left[0.3779 - \frac{1}{2}\times(-0.0303) + \frac{1}{3}(0.0046) - \frac{1}{4}(-0.0012)\right]$$

$$= 0.39488$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{1}{(1)^2}\left[-0.0303 - 0.0046 + \frac{11}{12}(-0.0012)\right]$$

$$= -0.036$$

Example-☺. From the following table of values of x and y find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when x = 5.5 :

x	4.5	5	5.5	6	6.5	7.0	7.5
y	9.69	12.90	16.71	21.18	26.36	32.34	39.15

Solution : Since x = 5.5 towards the beginning of the table, we use Newton forward difference formula. The difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
4.5	9.69						
		3.21					
5.0	12.90		0.6				
		3.81		0.06			
5.5	**16.71**		0.66		–0.01		
		4.47		0.05		0.05	
6.0	21.18		**0.71**		0.04		–0.15
		5.18		**0.09**		–0.1	
6.5	26.36		0.8		**–0.06**		
		5.98		0.03			
7.0	32.34		0.83				
		6.81					
7.5	39.15						

Here h = 0.5 and $x_0 = 5.5$

the first derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right]$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=5.5} = \frac{1}{0.5}\left[4.47 - \frac{1}{2}\times 0.71 + \frac{1}{3}\times 0.09 - \frac{1}{4}(-0.06)\right]$$

$$= 8.32$$

and the second derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{x=5.5} = \frac{1}{(0.5)^2}\left[0.71 - 0.09 + \frac{11}{12}(-0.06)\right]$$

$$= 2.26$$

Example 5.2 Calculate the first and second derivatives of the function tabulated in the preceding example at the point $x = 2.2$ and also dy/dx at $x = 2.0$.

Solution:

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

For $x = 2.2$, we have

$$x_n = 2.2,\ y_n = 9.0250, \text{ and } h = 0.2.$$

We know that,

Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right)$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014) + \frac{1}{6}(0.001)\right]$$

$$= \mathbf{9.0229}$$

Again for second derivative at $x = 2.2$ we obtain,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right).$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014) + \frac{137}{180}(0.0001\right]$$

= 8.994

<u>Now at $x = 2.0$</u> ,we obtain,

Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right)$$

$$\left[\frac{dy}{dx}\right]_{x=2.0} = \frac{1}{0.2}\left[1.3395 + \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) + \frac{1}{4}(0.0080) + \frac{1}{5}(0.0013)\right]$$

= 7.3896

Practice:–

☺ Given that

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) x = 1.1 (ii) x = 1.6

Ans. (i) 3.941; –3.3167; (ii) 2.732; –1.475

☺ From the following table find the first and second derivatives of sin x at (i) $x = 0^0$ (ii) $x = 40^0$ (iii) $x = 20^0$

x^0 :	0	10	20	30	40
$y = \sin x^0$:	0.000	0.1736	0.3420	0.5000	0.6428

Calculate f′(3) and f″(3) from the following table :

x	3.0	3.2	3.4	3.6	3.8	4.0
f(x)	−14.000	−10.032	−5.296	0.256	6.672	14.000

Ans. f′(3) = 18, f″(3) = 18

From the following values of x and y, find dy/dx when $x = 6$:

x	y	x	y
4.5	9.69	6.5	26.37
5.0	12.90	7.0	32.34
5.5	16.71	7.5	39.15
6.0	21.18		

The following table of values of x and y is given:

x	y	x	y
0	6.9897	4	8.4510
1	7.4036	5	8.7506
2	7.7815	6	9.0309
3	8.1291		

Find dy/dx when (i) $x = 1$, (ii) $x = 3$, and (iii) $x = 6$. Also find d^2y/dx^2 when $x = 3$.

A function $y = f(x)$ is defined as follows:

x	$y = f(x)$	x	$y = f(x)$
1.0	1.0	1.20	1.095
1.05	1.025	1.25	1.118
1.10	1.049	1.30	1.140
1.15	1.072		

Compute the values of dy/dx and d^2y/dx^2 at $x = 1.05$.

Runge-Kutta Method of 4th order

Consider $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

The Runge-Kutta method of 4th order is given by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Algorithm of Runge-Kutta Method of 4'th Order:

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N + 1)$ values of t.

Step 1 Set $h = (b - a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5.

Step 3 Set $K_1 = hf(t, w)$;
$K_2 = hf(t + h/2, w + K_1/2)$;
$K_3 = hf(t + h/2, w + K_2/2)$;
$K_4 = hf(t + h, w + K_3)$.

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 5 OUTPUT (t, w).

Step 6 STOP.

Problem(1):
By employing Runge-Kutta method of fourth order solve the differential equation $2y' - 6x = y$,y(0)=1 for x=0.2 in steps of 0.1 correct to four decimal places.

Soln: Given data: $f(x,y) = 3x + \frac{y}{2}$, h=0.1

$x_0 = 0$	$x_1 = 0.1$	$x_1 = 0.2$
$y_0 = 1$	$y_1 = 1.0664$	$y_2 = 1.1670$

The Runge-Kutta method of 4th order is given by

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\ \text{where } k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}\right\} \quad \text{--------------------- (1)}$$

Put n=0 in Eqn(1)

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{-------------------------- (2)}$$

$$\text{where } \left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3) \end{aligned}\right\}$$

$$k_1 = hf(x_0, y_0)$$

$$= h\left[3x_0 + \frac{y_0}{2}\right] = 0.1\left[3(0) + \frac{1}{2}\right] = 0.05$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left[3\left(x_0 + \frac{h}{2}\right) + \frac{1}{2}\left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1 + \frac{0.05}{2}\right)\right] = 0.0662$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h\left[3\left(x_0 + \frac{h}{2}\right) + \frac{1}{2}\left(y_0 + \frac{k_2}{2}\right)\right]$$

$$= 0.1\left[3\left(0 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1 + \frac{0.0662}{2}\right)\right] = 0.0666$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= h\left[3(x_0 + h) + \frac{1}{2}(y_0 + k_3)\right]$$
$$= 0.1\left[3(0 + 0.1) + \frac{1}{2}(1 + 0.0666)\right] = 0.0833$$

Substituting all these values in Eqn(2), we get

$$y_1 = 1 + \frac{1}{6}[0.05 + 2(0.0662) + 2(0.0666) + 0.0833] = 1.0664$$

Put n=1 in Eqn(1)

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{----------------------} \ (3)$$

Where

$$k_1 = hf(x_1, y_1)$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$k_1 = hf(x_1, y_1)$$
$$= h\left[3x_1 + \frac{y_1}{2}\right] = 0.1\left[3(0.1) + \frac{1.0664}{2}\right] = 0.0833$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= h\left[3\left(x_1 + \frac{h}{2}\right) + \frac{1}{2}\left(y_1 + \frac{k_1}{2}\right)\right]$$
$$= 0.1\left[3\left(0.1 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1.0664 + \frac{0.0833}{2}\right)\right] = 0.1004$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= h\left[3\left(x_1 + \frac{h}{2}\right) + \frac{1}{2}\left(y_1 + \frac{k_2}{2}\right)\right]$$
$$= 0.1\left[3\left(0.1 + \frac{0.1}{2}\right) + \frac{1}{2}\left(1.0644 + \frac{0.1004}{2}\right)\right] = 0.1008$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= h\left[3(x_1 + h) + \frac{1}{2}(y_1 + k_3)\right]$$
$$= 0.1\left[3(0.1 + 0.1) + \frac{1}{2}(1.0664 + 0.1008)\right] = 0.1183$$

Substituting all these values in Eqn(3), we get

$$y_2 = 1.0664 + \frac{1}{6}[0.0833 + 2(0.1004) + 2(0.1008) + 0.1183]$$
$$= 1.1670$$

Problem(2):
Apply Runge-Kutta method of fourth order to find an approximate value of y(0.1) and y(0.2) of $\frac{dy}{dx} = x + y^2$, y(0)=1 correct to three decimal places.

Soln: Given data: $f(x,y) = x + y^2$, h=0.1

$x_0 = 0$ $\quad$ $x_1 = 0.1$ $\quad$ $x_2 = 0.2$

$y_0 = 1$ $\quad$ $y_1 = ?$ $\quad$ $y_2 = ?$

The Runge-Kutta method of 4^{th} order is given by

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \\ \text{where } k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}\right\} \text{------------------------(1)}$$

Put n=0 in Eqn(1)

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{-------------------------(2)}$$

$$\text{where } k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_1 = hf(x_0, y_0)$$

$$= h\left[x_0 + (y_0)^2\right] = 0.1\left[0 + (1)^2\right] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.1152$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1152}{2}\right)^2\right] = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= h\left[(x_0 + h) + (y_0 + k_3)^2\right]$$
$$= 0.1\left[(0 + 0.1) + (1 + 0.1168)^2\right] = 0.1347$$

Substituting all these values in Eqn(2), we get

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1152) + 2(0.1168) + 0.1347] = 1.1164$$

Put n=1 in Eqn(1) $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$ -----------(3)

Where

$$k_1 = hf(x_1, y_1)$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$k_1 = hf(x_1, y_1)$$
$$= h\left[x_1 + (y_1)^2\right] = 0.1\left[(0.1) + (1.1164)^2\right] = 0.1346$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$
$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1346}{2}\right)^2\right] = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1551}{2}\right)^2\right] = 0.1575$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h\left[(x_1 + h) + (y_1 + k_3)^2\right]$$

$$= 0.1\left[(0.1 + 0.1) + (1.1164 + 0.1575)^2\right] = 0.1822$$

Substituting all these values in Eqn(3), we get

$$y_2 = 1.1164 + \frac{1}{6}[0.1346 + 2(0.1551) + 2(0.1575) + 0.1822]$$

$$= 1.2734$$

EXERCISE–

1. Solve $y' = 1 + y^2$ with $y(0) = 0$ for $x = 0.2\ (0.2)\ 0.6$ by Runge-Kutta method of fourth order. **[NUH-1999]**

   Ans : $y(0.6) = 0.6841$

2. Solve the following initial value problem using Runge-Kutta method of fourth order :

(i) $\frac{dy}{dx} = (1 + x)\,y$ with $y(0) = 1$ for $x = 0(0.2)\ 0.6$, $y(0.2) = 1.2247$, $y(0.4) = 1.5240$, $y(0.6) = 1.9581$

(ii) $y' = \frac{1}{x + y}$ with $y(0) = 1$ for $x = 0.5\ (0.5)\ 2$.

Ans : $y(0.5) = 1.3571$, $y(1.0) = 1.5873$, $y(1.5) = 1.7555$, $y(2.0) = 1.8956$