The Initial Value Problem

Definition

The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions a, b and a constant $y_0 \in \mathbb{R}$, find a solution y of

$$y' = a(t)\, y + b(t), \qquad y(0) = y_0.$$

Remark: The initial condition selects one solution of the ODE.

Example

Find the solution to the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1.$$

Solution: Every solution of the ODE above is given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$

We conclude that $y(t) = \dfrac{5}{2}\, e^{2t} - \dfrac{3}{2}$. $\triangleleft$

Example

Find the solution y to the IVP $y' = -3y + 1$, $y(0) = 1$.

Solution: Write down the differential equation as $y' + 3\,y = 1$.
Key idea: The left-hand side above is a total derivative if we multiply it by the exponential e^{3t}. Indeed,

$$e^{3t} y' + 3\,e^{3t}\,y = e^{3t} \;\Leftrightarrow\; e^{3t} y' + \left(e^{3t}\right)' y = e^{3t}.$$

This is the key idea, because the derivative of a product implies

$$\left[e^{3t}\,y\right]' = e^{3t}.$$

The exponential e^{3t} is called an integrating factor. Integrating,

$$e^{3t}\,y = \frac{1}{3} e^{3t} + c \quad \Leftrightarrow \quad y(t) = c\,e^{-3t} + \frac{1}{3}.$$

$\square$

Example

Find the solution y to the IVP $y' = -3y + 1$, $y(0) = 1$.

Solution: Every solution of the ODE above is given by

$$y(t) = c\,e^{-3t} + \frac{1}{3}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c + \frac{1}{3} \quad \Rightarrow \quad c = \frac{2}{3}.$$

We conclude that $y(t) = \frac{2}{3}\,e^{-3t} + \frac{1}{3}$. $\triangleleft$

EXERCISE:

. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \; y''(0) = 1$

. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$