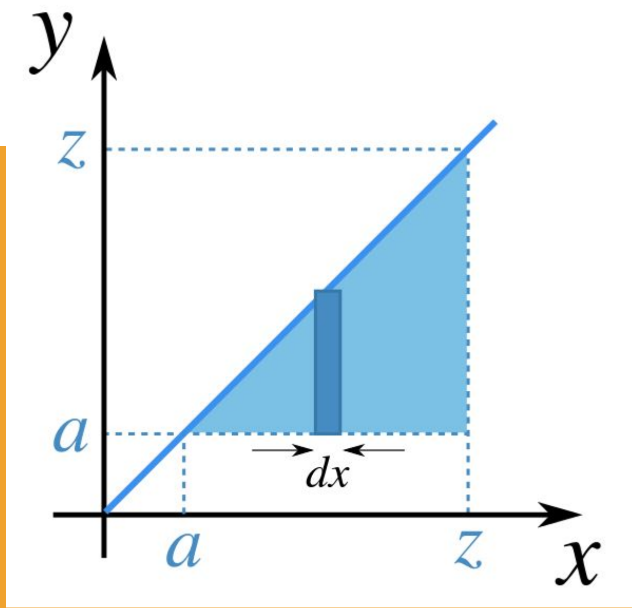
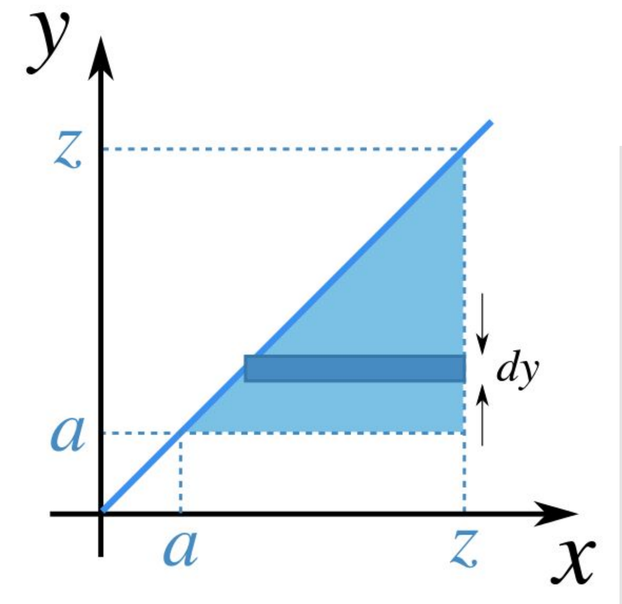
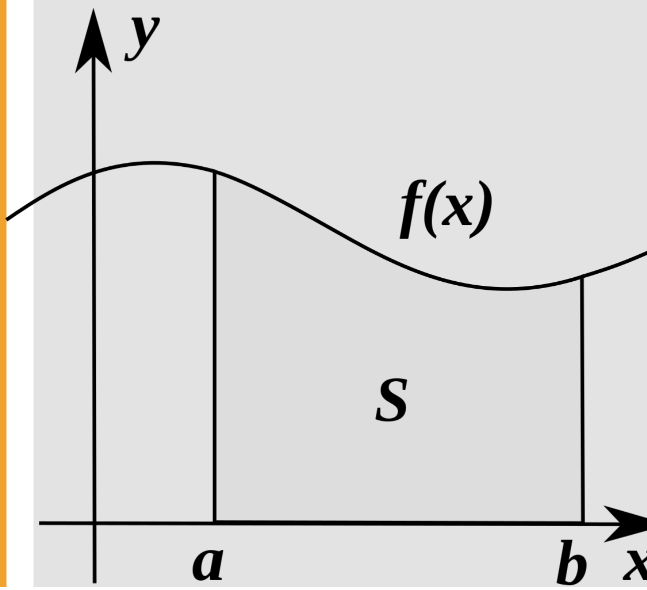




Numerical Integration

3rd Part : Simpson's three-Eight and Weddle's Rule



Simpson's $\frac{3}{8}$ Rule

The general Integration Formula ,

$$I = \int_{a}^{b} y \quad dx = \int_{x_0}^{x_0+nh} y \, dx$$

Simpson's $\frac{3}{8}$ Rule

$$I = h\left[n\,y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots\ldots\ldots + upto\ (n+1)\ terms\right]$$

Setting $n = 3$ in above equation and neglecting the fourth and higher order, we get

Simpson's $\frac{3}{8}$ Rule

$$\int_{x_0}^{x_0+3h} y\,dx = h\left(3y_0 + \frac{3^2}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!}\right)$$

$$= h\left(3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left(8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + \Delta^3 y_0\right)$$

Simpson's $\frac{3}{8}$ Rule

$$= \frac{3h}{8}\left(8y_0 + 12\left(y_1 - y_0\right) + 6\left(\Delta y_1 - \Delta y_0\right) + \Delta\left(\Delta y_1 - \Delta y_0\right)\right)$$

$$= \frac{3h}{8}\left(8y_0 + 12\left(y_1 - y_0\right) + 6\left(y_2 - 2y_1 + y_0\right) + \left(\Delta y_2 - 2\Delta y_1 + \Delta y_0\right)\right)$$

$$= \frac{3h}{8}\left(8y_0 + 12\left(y_1 - y_0\right) + 6\left(y_2 - 2y_1 + y_0\right) + \left(y_3 - y_2 - 2\left(y_2 - y_1\right) + y_1 - y_0\right)\right)$$

Simpson's $\frac{3}{8}$ Rule

$$= \frac{3h}{8}\left(8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0\right)$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

$$\therefore \int_{x_0}^{x_0+3h} y\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Simpson's $\frac{3}{8}$ Rule

Similarly, we can write,

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\int_{x_0+6h}^{x_0+9h} y\,dx = \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

Simpson's $\frac{3}{8}$ Rule

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Simpson's $\frac{3}{8}$ Rule

Adding all these above integrals, we can write

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right) + \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right) + \cdots\cdots + \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Simpson's $\frac{3}{8}$ Rule

$$= \frac{3h}{8}\{y_0 + 3y_1 + 3y_2 + y_3 + y_3 + 3y_4$$

$$+ 3y_5 + y_6 + y_6 + 3y_7 + 3y_8 + y_9 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\}$$

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\{(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots\cdots + y_{n-2} + y_{n-1})$$

$$+ 2(y_3 + y_6 + y_9 + \cdots\cdots\cdots + y_{n-3})\}$$

Simpson's $\frac{3}{8}$ Rule

The above formula is known as the Simpson's 3/8 rule for numerical integration.

Shortly we can write,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left\{ (y_0 + y_n) + 3 \sum_{\substack{k=1 \\ k \neq 3,6,9,\ldots}}^{n-1} y_k + 2 \sum_{k=3,6,9,\ldots\ldots}^{n-3} y_k \right\}$$

Note: This formula is used only when the number of partitions of the interval of integration is a multiple of the number 3.

Weddle's Rule

Similarly if we put n=6 in general integration formula then we get Weddle's formula

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \dots\dots\dots]$$

Shortly we can write,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\left\{\sum_{k=0,2,4,6,\dots\dots}^{n} y_k + 5\sum_{k=1,3,5,\dots}^{n-1} y_k + \sum_{k=3,6,9\dots}^{n-3} y_k\right\}$$

Weddle's Rule

Note:

1. This formula requires at least seven consecutive values of the function.
2. This formula is used only when the number of partitions of the interval of integration is a multiple of the number 6.

Mathematical Problems

Determine $\int_{4}^{5.2} \ln x \, dx$ by Simpson's 3/8 rule and Weddle's rule considering the number of intervals six. Find true value and then compare and comment on it.

Solution:

Given that the function is, $\int_{4}^{5.2} \ln x \, dx$

Here upper limit is $b = 5.2$, lower limit is $a = 4$ and No. of subintervals $n = 6$

Mathematical Problems

Let ,

$$y = f(x) = \ln x$$

Now,

$$h = \frac{5.2 - 4}{6} = \frac{1.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

Mathematical Problems

x	$x_0 = 4$	$x_1 = 4.2$	$x_2 = 4.4$	$x_3 = 4.6$	$x_4 = 4.8$	$x_5 = 5.0$	$x_6 = 5.2$
$y = f(x) = \ln x$	$y_0 = 1.3862$	$y_1 = 1.4350$	$y_2 = 1.4816$	$y_3 = 1.5260$	$y_4 = 1.5686$	$y_5 = 1.6094$	$y_6 = 1.6486$

Simpson's 3/8 rule:

We know that

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\{(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots\cdots + y_{n-2} + y_{n-1})$$

$$+ 2(y_3 + y_6 + y_9 + \cdots\cdots + y_{n-3})\}$$

Mathematical Problems

Now for $n = 6$ the above formula reduces to the following form,

$$= \frac{3 \times 0.2}{8} \{ (y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 \}$$

$$= \frac{3 \times 0.2}{8} \{ (1.3862 + 1.6486) + 3(1.4350 + 1.4816 + 1.5686 + 1.6094) + 2 \times 1.5260 \}$$

$$= \frac{3 \times 0.2}{8} \{ 3.0348 + 3 \times 6.0946 + 2 \times 1.5260 \}$$

Mathematical Problems

$$=1.827795$$

$$\int_4^{5.2} \ln x \, dx \quad = 1.827795$$

$$\int u\, v\, dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx \right] dx$$

Exact value is $\int_4^{5.2} \ln x\, dx = [x \ln x]_4^{5.2} - \int_4^{5.2} \left[\frac{d}{dx} (\ln x) \int dx \right] dx$

$$= [x \ln x]_4^{5.2} - \int_4^{5.2} \left[\frac{1}{x} . x \right] dx$$

Mathematical Problems

$$= [x \ln x]_4^{5.2} - \int_4^{5.2} dx$$

$$= [x \ln x]_4^{5.2} - [x]_4^{5.2}$$

$$= (5.2 \ln 5.2 - 4 \ln 4) - (5.2 - 4)$$

$$= 1.827847409$$

Mathematical Problems

Weddle's Rule:

We know that

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{10}\left\{ \sum_{k=0,2,4,6,......}^{n} y_k + 5 \sum_{k=1,3,5,...}^{n-1} y_k + \sum_{k=3,6,9...}^{n-3} y_k \right\}$$

Mathematical Problems

Now for $n = 6$ the above formula reduces to the following form,

$$\int_4^{5.2} \ln x \, dx = \frac{3\times 0.2}{10}\left\{ \sum_{k=0,2,4,6,......}^{6} y_k + 5 \sum_{k=1,3,5,...}^{5} y_k + \sum_{k=3,6,9...}^{3} y_k \right\}$$

$$\text{Or, } \int_4^{5.2} \ln x \, dx = \frac{3\times 0.2}{10}\{y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_3 + y_5) + y_3\}$$

$$\text{Or, } \int_4^{5.2} \ln x \, dx = \frac{3\times 0.2}{10}\{6.085 + 5\times 4.5704 + 1.5260\}$$

Mathematical Problems

$$\int_{4}^{5.2} \ln x \, dx = 1.82778$$

Comparing:

Result on Simpson's 3/8 rule and Weddle rule are closer to one another

and also to the true value. That means both methods work well.

Practice work

Problem 03: Compute the definite integral $\int_{0.2}^{1.4} \left(\sin x - \ln x + e^x \right) dx$ by using various rules using 6 equidistant sub-intervals correct up to three decimal places.


Thank You