Chapter 02

Error Discussion

After reading this chapter, you should be able to:

1. *Know about the exact, relative and percentage error.*
2. *Relate the absolute relative approximate error to the number of significant digits at least correct in your answers.*
3. *Know that there are two inherent sources of error in numerical methods – round-off and truncation error.*
4. *Know the difference between round-off and truncation error.*
5. *Know the concept of significant digits.*

2.1 Introduction:

Error in solving an engineering or science problem can arise due to several factors. First, the error may be in the modeling technique. A mathematical model may be based on using assumptions that are not acceptable. For example, there are two kinds of numbers such as exact and approximate numbers, such the exact numbers are $1, 2, 5, \dots., \frac{2}{5}, \frac{3}{2}, \dots\dots, \pi, e, \dots.. etc$ and the approximate numbers are representations of exact numbers to a certain degree of accuracy. Thus, 3.1416 is an approximate number of π and 3.14159265 is another approximate number of π. Second, errors may arise from mistakes in programs themselves or in the measurement of physical quantities. But, in applications of numerical methods itself, hear we focus on these type of errors,

1. Absolute error
2. Relative error
3. Percentage error
4. Round off error
5. Truncation error.

2.2 Absolute Error:

The absolute error of a quantity is the absolute value of the difference between the exact value and the approximate value. It is denoted by E_A. If the exact value is X and approximate value is x, the absolute error is define by, $E_A = |X - x|$.

2.3 Relative Error:

The relative error of a quantity is the ratio of its absolute error to its exact value. It is denoted by E_R, that is $E_R = \frac{E_A}{X}$.

2.4 Percentage Error:

The percentage error of a quantity is 100 times of its relative error. It is denoted by E_p, that is $E_P = 100E_R$

Problem-01: If an approximate value of π is 3.1428571 and let exact value is 3.1415926. Find the absolute, relative and percentage errors.

Solution: We have, exact value is $X = 3.1415926$ and approximate value is $x = 3.1428571$
The absolute error is,

$$E_A = |X - x| = |3.1415926 - 3.1428571| = |-0.0012645| = 0.0012645$$

The relative error is, $E_R = \frac{E_A}{X} = \frac{0.0012645}{3.1415926} = 0.000402$

The percentage error is, $E_P = 100E_R = 100 \times 0.000402 = 0.0402\%$

2.5 Round off Error:

When a calculator or digital computer is used to perform numerical calculations, an unavoidable error, called round-off error, must be considered. A round-off error, also called rounding error, is the difference between the calculated approximation of a number and its exact mathematical value due to rounding. For example, a number like $\frac{1}{3}$ may be represented as 0.333333 on a PC. Then the round off error in this case is $\frac{1}{3} - 0.333333 = 0.00000033\overline{3}$. Then there are other numbers that cannot be represented exactly. For example, π and $\sqrt{2}$ are numbers that need to be approximated in computer calculations.

Problem-02: Find the absolute, relative and percentage errors of the number 8.6 if both of its digits are correct (i.e, this number rounded to one decimal place).

Solution: The given number is $P = 8.6$
Since both digits are correct so $N = 1$
The absolute error is, $E_A = \frac{1}{2}(10^{-1}) = 0.05$ (i.e, exact value $X = 8.6$)
The relative error is, $E_R = \frac{E_A}{X} = \frac{0.05}{8.6} = 0.0058$
The percentage error is, $E_P = 100E_R = 100 \times 0.0058 = 0.58\%$

Formula: *If the number P is rounded to N decimal place then* $E_A = \frac{1}{2}(10^{-N})$

2.6 Truncation Error:

In numerical analysis and scientific computing, truncation error is the error made by truncating an infinite sum and approximating it by a finite sum. For instance, if we approximate the sine function by the first two non-zero term of its Taylor series, as in $\sin(x) \approx x - \frac{1}{6}x^3$ for small x, the resulting error is a truncation error.

Problem -03: Find the truncation error of e^x for first three terms.

Solution: Truncation error is defined as the error caused by truncating a mathematical procedure. For example, the Maclaurin series for e^x is given as, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots$

This series has an infinite number of terms but when using this series to calculate e^x, only a finite number of terms can be used. For example, if one uses three terms to calculate e^x, then $e^x \approx 1 + x + \frac{x^2}{2!}$.

The truncation error for such an approximation is

$$\text{Truncation error} = e^x - \left(1 + x + \frac{x^2}{2!}\right) = \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2.7 Difference Between the Round Off Error and the Truncation Error:

Round off error is the error caused by approximate representation of numbers. When we talk about round off error, it is the error between the number and its representation. For example 200/3 would be represented as 66.6667 in a six significant digit computer that rounds off the last digit. The last digit has been rounded up from 6 to a 7. The difference between 200/3 and 66.6667, that is, 200/3-66.6667 is the round off error.

Truncation error is error caused by truncating a mathematical procedure. In problem 03 we find the truncation error of e^x for first terms is, $\frac{x^3}{3!}+\frac{x^4}{4!}+$..........

So let us get this straight – round off error is caused by representing numbers approximately and truncation error is caused by approximating mathematical procedures.

2.8 Significant Digits:

Significant digits are certain digits that have significance or meaning and give more precise details about the value of the number. If I offered you $2,000, the 2 in 2000 is significant because it tells you exactly how many thousands. Again let's say two people ran a race. Runner 1 took 30.01 seconds, and runner 2 took 30.02 seconds. Who would win the race? Obviously, runner 1 because he took less time. All those numbers are significant because we need them all to tell us exactly who won the race.

So we have some rules to identify the significant numbers,

Rule 1: Every non-zero digit is significant

- 456 has 3 significant digits
- 68.29 has 4 significant digits

Rule 2: Zeros between non-zero digits are always significant

- 5,609 has 4 significant digits.
- 700.0879 has 7 significant digits.

Rule 3: Zeros before non-zero digits are never significant

- 0.067 has 2 significant digits
- 0.000008 has 1 significant digit

Rule 4: Zeros behind non-zero digits are sometimes significant

- 12.2300 has six significant figures: 1, 2, 2, 3, 0 and 0.
- 0.000122300 still has only six significant figures (the zeros before the 1 are not significant).
- 120.00 has five significant figures since it has three trailing zeros.

Note: If the digit immediately to the right of the last significant figure is greater than 5 or is a 5 followed by other non-zero digits, add 1 to the last significant figure. For example, 1.2459 as the result of a calculation or measurement that only allows for 3 significant figures should be written 1.25

2.9 More Examples:

Problem -04: Evaluate the sum $S=\sqrt{2}+\sqrt{3}+\sqrt{5}$ to 4 significant digits and also find its absolute, relative and percentage errors.

Solution: For the 4 significant digits we may write $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{2}=2.236$

So, sum $S=1.414+1.732+2.236=5.382$

Since the values of $\sqrt{2},\sqrt{3}$ and $\sqrt{5}$ are rounded to three decimal places, so $N=3$.

Now the absolute error is, $E_A = \frac{1}{2}(10^{-3}) + \frac{1}{2}(10^{-3}) + \frac{1}{2}(10^{-3})$

$= 0.0005 + 0.0005 + 0.0005 = 0.0015$

The absolute error shows that the sum is correct to 3 significant digits only, so we take exact value $X = 5.38$

The relative error is, $E_R = \frac{E_A}{X} = \frac{0.0015}{5.38} = 0.00028$

The percentage error is, $E_P = 100E_R = 100 \times 0.00028 = 0.028\%$

Problem -05: Evaluate the sum $S = \sqrt{11} + \sqrt{21} + \sqrt{31}$ to 5 significant digits and also find its E_A, E_R, E_P.

Solution: For the 5 significant digits we may write, $\sqrt{11} = 3.3166$, $\sqrt{21} = 4.5826$, $\sqrt{31} = 5.5678$

So, sum $S = 3.3166 + 4.5826 + 5.5678 = 13.467$

Since the values of $\sqrt{11}, \sqrt{21}$ and $\sqrt{31}$ are rounded to three decimal places, so $N = 3$.

Now the absolute error is, $E_A = \frac{1}{2}(10^{-3}) + \frac{1}{2}(10^{-3}) + \frac{1}{2}(10^{-3})$

$= 0.0005 + 0.0005 + 0.0005 = 0.0015$

The absolute error shows that the sum is correct to 4 significant digits only, so we take exact value $X = 13.47$

The relative error is, $E_R = \frac{E_A}{X} = \frac{0.0015}{13.47} = 0.00011136$

The percentage error is, $E_P = 100E_R = 100 \times 0.00011136 = 0.011136\%$

Problem-06: If the number 5.365489 is correct to 5 significant digits, then find the values of E_A, E_R, E_P.

Solution: The given number is 5.365489, but this number is correct to 5 significant digits so $P = 5.3655$.
Since P rounded to 4 decimal places so $N = 4$.

The absolute error is, $E_A = \frac{1}{2}(10^{-4}) = 0.00005$ (i.e, exact value $X = 5.3655$)

The relative error is, $E_R = \frac{E_A}{X} = \frac{0.00005}{5.3655} = 0.0000093188$

The percentage error is, $E_P = 100E_R = 100 \times 0.0000093188 = 0.00093188\%$

For Practices

1) Define percentage error, absolute error with example.
2) Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{11}$ to 6 significant digits and find its percentage error, absolute error, relative error?
3) Find the absolute, relative and percentage errors of the number 8.2154356 if **4** significant digits are correct.
4) Evaluate the sum **S** = $\sqrt{\mathbf{11}} + \sqrt{\mathbf{21}} + \sqrt{\mathbf{31}}$ to **5** significant digits and find its E_A, E_R, E_P ?
5) Find the absolute, relative and percentage errors of the number 0.3576 if **2** significant digits are correct.