8.2.5 Maxima and Minima of a Tabulated Function:

Newton's forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots,$$

$$p = \frac{x - x_0}{h} \qquad \ldots ①$$

Differentiating ① with respect to p

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots \qquad \ldots ②$$

For finding maxima/minima of a function $y = f(x)$, $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = 0$

$$\frac{dp}{dx} = \frac{1}{h} \neq 0, \ \therefore \frac{dy}{dp} = 0 \qquad \ldots ③$$

Neglecting 4th and higher order differences in equation ② and substituting in ③, we get a quadratic equation of the form $A + Bp + Cp^2 = 0$,where A, B, C are constants. Solving for p and substituting in $x = x_0 + ph$, we get points of maxima/minima for the function $y = f(x)$.

- Newton's forward method is apt for finding extreme values of a tabulated data, wherever their location may be, by index p assuming values $|p| \geq 1$, if the extreme value is not in vicinity of the point (x_0, y_0). Yet we may also use Newton's backward or Stirling's central differences formulae to locate extreme values, if desired.

Example 7 From the following data, find maximum and minimum values of y.

$\boldsymbol{x}$	0	2	4	6
$\boldsymbol{f(x)}$	2	0	-50	-196

Solution: Constructing forward difference table for the function $y = f(x)$

x	y	Δ	Δ^2	Δ^3
0	2			
		-2		
2	0		-48	
		-50		-48
4	-50		-96	
		-146		
6	-196			

Newton's forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots, \; p = \frac{x - x_0}{h} \qquad \ldots ①$$

Taking $x_0 = 0$, $y_0 = 2$, $\Delta y_0 = -2$, $\Delta^2 y_0 = -48$, $\Delta^3 y_0 = -48$

Substituting these values in ①, we get

$$y \equiv 2 + p(-2) + \frac{p(p-1)}{2}(-48) + \frac{p(p-1)(p-2)}{6}(-48)$$

$$\Rightarrow y \equiv 2 - 2p - 24(p^2 - p) - 8(p^3 - 3p^2 + 2p)$$

$$\Rightarrow y \equiv -8p^3 + 6p + 2 \qquad \ldots ②$$

Differentiating ② with respect to p, we get

$$\frac{dy}{dp} = -24p^2 + 6$$

For y to be maximum, $\frac{dy}{dp} = 0$

$$\Rightarrow -24p^2 + 6 = 0$$

$$\Rightarrow p = 0.5, -0.5$$

Substituting in ② , maximum and minimum values of y are 4 and 0 respectively.

Example 8 From the following table, find x for which y is maximum.

x	3	4	5	6	7	8
$f(x)$	0.205	0.240	0.259	0.262	0.250	0.224

Also find maximum value of y.

Solution: Constructing forward difference table for the function $y = f(x)$, upto third differences

x	$y = f(x)$	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		−0.016	
		0.019		0
5	0.259		−0.016	
		0.003		0.001
6	0.262		−0.015	
		−0.012		0.001
7	0.250		−0.014	
		−0.026		
8	0.224			

Newton's forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots,\ \ p = \frac{x - x_0}{h} \qquad \ldots①$$

Taking $x_0 = 3$, $y_0 = 0.205,\ \Delta y_0 = 0.035,\ \Delta^2 y_0 = -0.016,\ \Delta^3 y_0 = 0$

Substituting these values in ①, we get

$$y \equiv (0.205) + p(0.035) + \frac{p(p-1)}{2}(-0.016) + 0 \qquad \ldots②$$

Differentiating with respect to p, we get

$$\frac{dy}{dp} = 0.035 + \frac{2p-1}{2}(-0.016) = 0.035 - (0.008)(2p-1)$$

For y to be maximum, $\frac{dy}{dp} = 0$

$\Rightarrow 0.035 - (0.008)(2p-1) = 0$

$\Rightarrow p = 2.6875$

Also $p = \frac{x - x_0}{h}$ or $x = x_0 + ph$

$\Rightarrow x = 3 + 2.6875(1) = 5.6875$

$\therefore y$ is maximum when $x = 5.6875$ or $p = 2.6875$

Substituting in ② , maximum value of y is given by

$$y \equiv (0.205) + (2.6875)(0.035) + \frac{(2.6875)(2.6875-1)}{2}(-0.016) = 0.2628$$