Runge-Kutta (RK4) numerical solution for Differential Equations

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Runge-Kutta Method Order 4 Formula

$$y(x+h) = y(x) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where

$$F_1 = hf(x, y)$$

$$F_2 = hf\left(x + \frac{h}{2}, y + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(x + \frac{h}{2}, y + \frac{F_2}{2}\right)$$

$$F_4 = hf(x + h, y + F_3)$$

Where does this formula come from?

We learned earlier that Taylor's Series gives us a reasonably good approximation to a function, especially if we are near enough to some known starting point, and we take enough terms.

However, one of the drawbacks with Taylor's method is that you need to differentiate your function once for each new term you want to calculate. This can be troublesome for complicated functions, and doesn't work well in computerised modelling.

Carl Runge (pronounced "roonga") and Wilhelm Kutta (pronounced "koota") aimed to provide a method of approximating a function without having to differentiate the original equation.

Their approach was to simulate as many steps of the Taylor's Series method but using evaluation of the original function only.

Runge-Kutta Method of Order 2

We begin with two function evaluations of the form:

$$F_1 = hf(x, y)$$

$$F_2 = hf(x + \alpha h, y + \beta F_1)$$

The α and β are unknown quantities. The idea was to take a linear combination of the F_1 and F_2 terms to obtain an approximation for the y value at $x = x_0 + h$, and to find appropriate values of α and β.

By comparing the values obtains using Taylor's Series method and the above terms (I will spare you the details here), they obtained the following, which is **Runge-Kutta Method of Order 2**:

$$\boxed{y(x + h) = y(x) + \frac{1}{2}(F_1 + F_2)}$$

where

$$F_1 = hf(x, y)$$

$$F_2 = hf(x + h, y + F_1)$$

Runge-Kutta Method of Order 3

As usual in this work, the more terms we take, the better the solution. In practice, the Order 2 solution is rarely used because it is not very accurate.

A better result is given by the Order 3 method:

$$\boxed{y(x + h) = y(x) + \frac{1}{9}(2F_1 + 3F_2 + 4F_3)}$$

where

$$F_1 = hf(x, y)$$

$$F_2 = hf\left(x + \frac{h}{2}, y + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(x + \frac{3h}{4}, y + \frac{3F_2}{4}\right)$$

This was obtained in a similar way to the earlier formula, by comparing Taylor's Series results.

The most commonly used Runge-Kutta formula in use is the Order 4 formula (RK4), as it gives the best trade-off between computational requirements and accuracy.

Let's look at an example to see how it works.

Example

Use Runge-Kutta Method of Order 4 to solve the following, using a step size of $h = 0.1$ for $0 \le x \le 1$.

$$\frac{dy}{dx} = \frac{5x^2 - y}{e^{x+y}}$$

$$y(0) = 1$$

Step 1

Note: The following looks tedious, and it is. We'll use a computer (not calculator) to do most of the work for us. The following is here so you can see how the formula is applied.

We start with $x = 0$ and $y = 1$. We'll find the F values first:

$$F_1 = hf(x,y) = 0.1\frac{5(0)^2 - 1}{e^{0+1}} = -0.03678794411$$

For F_2, we need to know:

$$x + \frac{h}{2} = 0 + \frac{0.1}{2} = 0.05, \text{ and}$$

$$y + \frac{F_1}{2} = 1 + \frac{-0.03678794411}{2} = 0.98160602794$$

We substitute these into the F_2 expression:

$$F_2 = hf\left(x + \frac{h}{2}, y + \frac{F_1}{2}\right) = 0.1\left(\frac{5(0.05)^2 - 0.98160602794}{e^{0.05+0.98160602794}}\right) = -0.03454223937$$

For F_3, we need to know:

$$y + \frac{F_2}{2} = 1 + \frac{-0.03454223937}{2} = 0.98272888031$$

So

$$F_3 = hf\left(x + \frac{h}{2}, y + \frac{F_2}{2}\right) = 0.1\left(\frac{5(0.05)^2 - 0.98272888031}{e^{0.05+0.98272888031}}\right) = -0.03454345267$$

For F_4, we need to know:

$$y + F_3 = 1 - 0.03454345267 = 0.96545654732$$

So

$$F_4 = hf(x+h, y+F_3) = 0.1\left(\frac{5(0.1)^2 - 0.96545654732}{e^{0.1+0.96545654732}}\right) = -0.03154393258$$

Step 2

Next, we take those 4 values and substitute them into the Runge-Kutta RK4 formula:

$$y(x+h) = y(x) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$= 1 + \frac{1}{6}(-0.03678794411 - 2 \times 0.03454223937 - 2 \times 0.03454345267 - 0.03154393258)$$

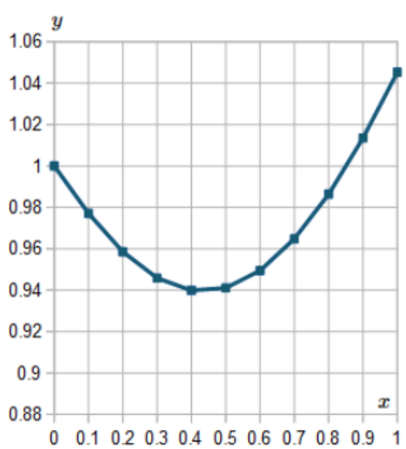
$$= 0.9655827899$$

Using this new y-value, we would start again, finding the new F_1, F_2, F_3 and F_4, and substitute into the Runge-Kutta formula.

We continue with this process, and construct the following table of Runge-Kutta values. (I used a spreadsheet to obtain the table. Using calculator is very tedious, and error-prone.)

x	y	$F_1 = h\frac{dy}{dx}$	$x + \frac{h}{2}$	$y + \frac{F_1}{2}$	F_2	$y + \frac{F_2}{2}$	F_3	$x + h$	$y + F_3$	F_4
0	1	−0.0367879441	0.05	0.9816060279	−0.0345422394	0.9827288803	−0.0345434527	0.1	0.9654565473	−0.0315439326
0.1	0.9655827899	−0.0315443	0.15	0.9498106398	−0.0278769283	0.9516443257	−0.0278867954	0.2	0.9376959945	−0.023647342
0.2	0.937796275	−0.023648185	0.25	0.9259721824	−0.0189267761	0.9283328869	−0.0189548088	0.3	0.9188414662	−0.0138576597
0.3	0.9189181059	−0.0138588628	0.35	0.9119886745	−0.0084782396	0.9146789861	−0.0085314167	0.4	0.9103866892	−0.0029773028
0.4	0.9104421929	−0.0029786344	0.45	0.9089528756	0.0026604329	0.9117724093	0.002580704	0.5	0.9130228969	0.0082022376
0.5	0.913059839	0.0082010354	0.55	0.9171603567	0.013727301	0.9199234895	0.0136258867	0.6	0.9266857257	0.018973147
0.6	0.9267065986	0.0189722976	0.65	0.9361927474	0.0240794197	0.9387463085	0.0239658709	0.7	0.9506724696	0.0287752146
0.7	0.9506796142	0.0287748718	0.75	0.9650670501	0.0332448616	0.967302045	0.0331305132	0.8	0.9838101274	0.0372312889
0.8	0.9838057659	0.0372315245	0.85	1.0024215282	0.0409408747	1.0042762033	0.0408359751	0.9	1.024641741	0.0441484563
0.9	1.024628046	0.0441492608	0.95	1.0467026764	0.0470593807	1.0481577363	0.0469712279	1	1.0715992739	0.0494916177
1	1.0715783953									

Here is the graph of the solutions we found, from $x = 0$ to $x = 1$.



Exercise

Solve the following using RK4 (Runge-Kutta Method of Order 4) for $0 \le x \le 2$. Use a step size of $h = 0.2$:

$$\frac{dy}{dx} = (x + y)\sin xy$$

$$y(0) = 5$$

Step 1

We have $x_0 = 0$ and $y_0 = 5$.

$F_1 = hf(x, y) = 0.2((0 + 5)\sin(0)(5)) = 0$

For F_2, we need to know:

$$x + \frac{h}{2} = 0 + \frac{0.2}{2} = 0.1, \text{ and}$$

$$y + \frac{F_1}{2} = 5 + \frac{0}{2} = 5$$

We substitute these into the F_2 expression:

$$F_2 = hf\left(x + \frac{h}{2}, y + \frac{F_1}{2}\right) = 0.2((0.1 + 5)\sin(0.1)(5)) = 0.48901404937$$

For F_3, we need to know:

$$y + \frac{F_2}{2} = 5 + \frac{0.48901404937}{2} = 5.24450702469$$

So

$$F_3 = hf\left(x + \frac{h}{2}, y + \frac{F_2}{2}\right)$$

$$= 0.2((0.1 + 5.24450702469) \times \sin(0.1)(5.24450702469))$$

$$= 0.53523913352$$

For F_4, we need to know:

$$y + F_3 = 5 + 0.53523913352 = 5.53523913352$$

So

$$F_4 = hf(x + h, y + F_3)$$

$$= 0.2((0.2 + 5.53523913352) \times \sin(0.2)(5.53523913352))$$

$$= 1.02589900571$$

Step 2

Next, we take those 4 values and substitute them into the Runge-Kutta RK4 formula:

$$y(x+h) = y(x) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$= 5 + \frac{1}{6}(0 + 2 \times 0.48901404937 + 2 \times 0.53523913352 + 1.02589900571)$$

$$= 5.5124008953$$

As before, we need to take this y_1 value and use the new $x_1 = 0.2$ value to find the next value, y_2, and so on up to $x = 2$.

Now complete the solution and upload in BLC as assignment.