Chapter–4 (A)

Numerical Differentiation

ভূমিকা :

পূর্ববর্তী অধ্যায়ে কোন ফাংশনের তালিকামান (tabulated values) হতে উক্ত ফাংশন নির্ধারণ বা তালিকায় অনুপস্থিত অন্য কোন অজানা রাশির জন্য উক্ত ফাংশনের মান নির্ণয় করা নিয়ে আলোচনা করা হয়েছে। এ অধ্যায়ে কোন ফাংশনের অন্তরকসমূহ বা ফাংশনের তালিকামান-এ অনুপস্থিত কোন বিন্দুতে অন্তরক নির্ণয় নিয়ে আলোচনা করা হবে।

যদি কোন তালিকামান সমদূরত্বের (equal spaced) হয়, তবে তালিকার প্রথম দিকের কোন বিন্দুতে অন্তরক নির্ণয় করতে Newton's forward interpolation formula, তালিকার শেষ দিকের কোন বিন্দুতে অন্তরক নির্ণয় করতে Newton's backward interpolation formula এবং তালিকার মধ্যভাগের কোন বিন্দুতে অন্তরক নির্ণয় করতে যে কোন central difference interplation formula ব্যবহার করতে হয়। আর তালিকামান অসমদূরত্বের (unequally spaced) বা সমদূরত্বের যে কোন ক্ষেত্রে কোন বিন্দুতে অন্তরক নির্ণয় করতে Newton's general interpolation formula বা Lagrange's formula ব্যবহার করা হয়।

সাধারণত: যে কোন ফাংশনের অন্তরক নির্ণয় করা হলেও প্রকৃতপক্ষে যদি কোন ফাংশনের বিস্তার সম্পর্কে ধারণা পাওয়া না যায় বা কোন ফাংশনের কেবলমাত্র কিছু তালিকামান দেওয়া থাকে তবে উক্ত তালিকায় অনুপস্থিত কোন বিন্দুতে অন্তরক নির্ণয় করার ক্ষেত্রে numerical differentiation গুরুত্বপূর্ণ ভূমিকা পালন করে। যদিও এ পদ্ধতিতে প্রাপ্ত ফলাফল প্রায় সর্বদাই আসন্নমান (approximate result) হিসেবে পাওয়া যায়।

4.1 ***Using (i) Newton's forward difference formula and (ii) Newton's backward difference formula, derive expressions for the first and second derivatives of a function.***

Hence also prove that *[(i) নিউটনের অগ্রজ পার্থক্য সূত্র এবং (ii) নিউটনের পশ্চাৎ পার্থক্যসূত্র ব্যবহার করে কোন ফাংশনের প্রথম ও দ্বিতীয় অন্তরকলনের রাশিমালা বাহির কর। অত:পর প্রমাণ কর যে,]*

$$y'_0 = \frac{1}{h}\left[\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots\dots\dots\right]y_0$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{2}\Delta^4 - \frac{5}{6}\Delta^5 + \dots\dots\dots\right]y_0$$

and $$y' = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \dots\dots\dots\right]y_n$$

$$y''_n = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \dots\dots\dots\right]y_n$$

Solution : (i) We have Newton's forward difference formula is

$$y(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!}$$

$$+ u(u-1)(u-2)(u-3)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\ldots\ldots$$

$$= y_0 + u\Delta y_0 + (u^2-u)\frac{\Delta^2 y_0}{2!} + (u^3 - 3u^2 + 2u)\frac{\Delta^3 y_0}{3!}$$

$$+ (u^4 - 6u^3 + 11u^2 - 6u)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $u = \dfrac{x - x_0}{h}$ $\qquad \therefore \dfrac{du}{dx} = \dfrac{1}{h}$

Differentiating (1) with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left[(y_0 + u\Delta y_0 + (u^2-u)\frac{\Delta^2 y_0}{2!} + (u^3 - 3u^2 + 2u)\frac{\Delta^3 y_0}{3!}\right.$$

$$\left. + (u^4 - 6u^3 + 11u^2 - 6u)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\right]$$

$$= \frac{d}{du}\left[y_0 + u\Delta y_0 + (u^2-u)\frac{\Delta_2 y_0}{2!} + (u^3 - 3u^2 + 2u)\frac{\Delta^3 y_0}{3!}\right.$$

$$\left. + (u^4 - 6u^3 + 11u^2 - 6u)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\right]\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + (2u-1)\frac{\Delta^2 y_0}{2!} + (3u^2 - 6u + 2)\frac{\Delta^3 y_0}{3!}\right.$$

$$\left. + (4u^3 - 18u^2 + 22u - u)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\right]\ldots\ldots\ldots(2)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h}\frac{d}{du}\left[\Delta y_0 + (2u-1)\frac{\Delta^2 y_0}{2!} + (3u^2 - 6u + 2)\frac{\Delta^3 y_0}{3!}\right.$$

$$\left. + (4u^3 - 18u^2 + 22u - 6)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\right]\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[2.\frac{\Delta^2 y_0}{2!} + (6u-6)\frac{\Delta^3 y_0}{3!} + (12u^2 - 36u + 22)\frac{\Delta^4 y_0}{4!} + \ldots\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + (12u^2 - 36u + 22)\frac{\Delta^4 y_0}{4!} + \ldots\ldots\right]\ldots(3)$$

Hence (2) and (3) are the 1st and 2nd derivatives respectively of a function with regards to Newtons forward difference formula.

(ii) We have Newton's backward difference formula is

$$y(x) = y_n + u\nabla y_n + u(u+1)\frac{\nabla^2 y_n}{2!} + u(u+1)(u+2)\frac{\nabla^3 y_n}{3!}$$

$$+ u(u+1)(u+2)(u+3)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\ldots\ldots$$

$$= y_n + u\nabla y_n + (u^2+u)\frac{\nabla^2 y_n}{2!} + (u^3+3u^2+2u)\frac{\nabla^3 y_n}{3!}$$

$$+ (u^4+6u^3+11u^2+6u)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $u = \frac{x - x_n}{h}$ $\therefore \frac{du}{dx} = \frac{1}{h}$.

Differentiating (4) with respect to x we get

$$\frac{dy}{dx} = \frac{d}{du}\left[y_n + u\nabla y_n + (u^2+u)\frac{\nabla^2 y_n}{2!} + (u^3+3u^2+2u)\frac{\nabla^3 y_n}{3!} + (u^4+6u^3+11u^2+6u)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\right]\frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + (2u+1)\frac{\nabla^2 y_n}{2!} + (3u^2+6u+2)\frac{\nabla^3 y_n}{3!} + (4u^3+18u^2+22u+6)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\right]\ldots\ldots(5)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h}\frac{d}{du}\left[\nabla y_n + (2u+1)\frac{\nabla^2 y_n}{2!} + (3u^2+6u+2)\frac{\nabla^3 y_n}{3!} + (4u^3+18u^2+22u+6)\frac{\Delta^4 y_n}{4!} + \ldots\ldots\right]\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[2.\frac{\nabla^2 y_n}{2!} + (6u+6)\frac{\nabla^3 y_n}{3!} + (12u^2+36u+22)\frac{\nabla^4 y_n}{4!} + \ldots\right]$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + (12u^2+36u+22)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\right]\ldots(6)$$

Hence (5) and (6) are the 1st and 2nd derivatives respectively of a function with regards to Newtons backward difference formula.

2nd pard : At the point $x = x_0$ (in a particular case) we get from forward formula $u = 0$

Hence on substituting this value of u we get.

$$+\frac{u^4-2u^3-u^2+2u}{48}(\Delta^4 y_{-2}+\Delta^4 y_{-1})+\ldots\ldots\Big]\frac{du}{dx}$$

$$=\frac{1}{h}\Big[\Delta y_0+\frac{2u-1}{4}(\Delta^2 y_{-1}+\Delta^2 y_0)+\frac{6u^2-6u+1}{12}\Delta^3 y_{-1}$$

$$+\frac{2u^3-3u^2-u+1}{24}(\Delta^4 y_{-2}+\Delta^4 y_{-1})+\ldots\ldots\Big]$$

$$\therefore\frac{d^2y}{dx^2}=\frac{1}{h^2}\Big[\frac{1}{2}(\Delta^2 y_{-1}+\Delta^2 y_0)+\frac{2u-1}{2}\Delta^3 y_{-1}$$

$$+\frac{6u^2-6u-1}{24}(\Delta^4 y_{-2}+\Delta^4 y_{-1})+\ldots\ldots\Big]$$

$$\frac{d^3y}{dx^3}=\frac{1}{h^3}\Big[\Delta^3 y_{-1}+\frac{2u-1}{4}(\Delta^4 y_{-2}+\Delta^4 y_{-1})+\ldots\ldots\Big]$$

4.3 ***Derive the expressions of first derivative of Newtons general interpolation formula.*** *[নিউটনের সাধারণ আন্ত:পাতন সূত্রের প্রথম অন্তরকলন রাশিমালা বের কর ।]*

Solution : We have the Newton's general interpolation formula is

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)+\ldots..$$

Differentiating w. r. to x, we get

$$f'(x)=f(x_0,x_1)+\{(x-x_0)+(x-x_1)\}f(x_0\ x_1,x_2)$$
$$+\{(x-x_0)(x-x_1)+(x-x_0)(x-x_2)+(x-x_1)(x-x_2)\}$$
$$f(x_0,x_1,x_2,x_3)+\{(x-x_0)(x-x_1)(x-x_2)+(x-x_0)(x-x_1)(x-x_3)$$
$$+(x-x_0)(x-x_2)(x-x_3)+(x-x_1)(x-x_2)(x-x_3)\}$$
$$f(x_0,x_1,x_2,x_3,x_4)+\ldots\ldots\ldots\ldots$$

4.1–1 ***Find*** $\frac{dy}{dx}$ ***at x = 1 from the following table ;*** *[নিচের টেবিল হতে* **x = 1** *বিন্দুতে* $\frac{dy}{dx}$ *নির্ণয় কর ।]*

x	1	2	3	4	5	6
y	198669	295520	389418	479425	564652	644217

Solution : Since the derivative is required at $x = 1$ which lies at the begining of the table. So Newton's forward formula is more suitable in this case.

We form a forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	198669	**96851**	**−2953**	**−938**	**39**	**8**
2	295520	93898	−3891	−899	47	
3	389418	90007	−4790	−852		
4	479425	85217	−5642			
5	564652	79575				
6	644217					

Newton's forward formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!} + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$

So at $x = 1, u = 0$

Differentiating (1) w. r. to x and putting $u = 0$ we get

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\right]$$

$$\therefore \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \left[96851 - \frac{1}{2}(-2953) + \frac{1}{3}(-938) - \frac{1}{4}(39) + \frac{1}{5}(8)\right]$$

$$= 98006.65$$

$$\approx 98007$$

Hence at $x = 1, \frac{dy}{dx} = 98007.$

4.1–2 ***The tabulated values of the function $y = \sqrt[3]{x}$ are given below. Find the first and second derivatives of the function at $x = 50$.*** [নিচের $y = \sqrt[3]{x}$ ফাংশনের তালিকামান দেওয়া হল। $x = 50$ বিন্দুতে প্রথম ও দ্বিতীয় অন্তরকলন বের কর।]

x	50	51	52	53	54	55	56
y	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution : The difference table of the data is,

x	$y = \sqrt[3]{x}$	Δy	$\Delta^2 y$
50	3.6840	0.0244	– 0.0003
51	3.7084	0.0241	– 0.0003
52	3.7325	0.0238	–0.0003
53	3.7563	0.0235	– 0.0003
54	3.7798	0.0232	– 0.0003
55	3.8030	0.0229	
56	3.8259		

Newton-Gregory forward interpolation formula is ,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \ldots\ldots\ldots\ldots$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \ldots\ldots\ldots\ldots\right] \text{ and}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \ldots\ldots\ldots\right]$$

$$\text{At } x = 50 \text{ we get } u = \frac{x - x_0}{h} = 0$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=50} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0\right]$$

$$= 0.0244 - \frac{1}{2}\cdot(-0.0003) = 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h}\left[\Delta^2 y_0\right] = -\ 0.0003$$

4.1–3 ***Find the 1st, 2nd and 3rd derivatives at the point $x = 1.00$ of the function $y = \sqrt{x}\ r(x)$ tabulated below : [$y = \sqrt{x}$ ফাংশনের নিম্নের তালিকামান হতে $x = 1.00$ বিন্দুতে ১ম, ২য় এবং ৩য় অন্তরকলন বের কর।]***

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$y = \sqrt{x}$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

Solution : We form a difference table of the data :

x	$y=\sqrt{x}$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.00	1.00000	**0.02470**	**– 0.00059**	**0.00005**
1.05	1.02470	0.02411	– 0.00054	0.00004
1.10	1.04881	0.2357	– 0.0050	0.00002
1.15	1.07238	0.2307	– 0.00048	0.00003
1.20	1.09544	0.02259	– 0.00045	
1.25	1.11803	0.02214		
1.30	1.14017			

Newton's forward interpolation formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots.(1)$$

$$\text{where } u = \frac{x - x_0}{h}$$

Here $x = 1.00$, $x_0 = 1.00$, $h = 0.5 \Rightarrow u = \frac{1.00 - 1.00}{0.05} = 0$

Now differentiating (1) w. r. to x and putting $u = 0$ we get.

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3}\left[\Delta^3 y_0\right]$$

Putting the value of certain difference we get

$$\left(\frac{dy}{dx}\right)_{u=0} = \left(\frac{dy}{dx}\right)_{x=1.00} = \frac{1}{0.05}$$

$$\left[0.02470 - \frac{1}{2}(-0.00059) + \frac{1}{3}(0.00005)\right]$$

$$= 0.500233$$

$$\left(\frac{d^2y}{dx^2}\right)_{u=0} = \left(\frac{d^2y}{dx^2}\right)_{x=1.00}$$

$$= \frac{1}{(0.5)^2}\left[-0.00059 - 0.00005)\right]$$

$$= -0.256$$

$$\left(\frac{d^3y}{dx^3}\right)_{u=0} = \left(\frac{d^3y}{dx^3}\right)_{x=1.00} = \frac{1}{(0.05)^3}\ [0.00005] = 0.4$$

which are the required derivatives at the mentioned point.

Note : The function tabulated above is

$$y = \sqrt{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-1/2} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{x=1.00} = 0.5$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}x^{-3/2} \quad \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1.00} = -0.250$$

$$\frac{d^3y}{dx^3} = \frac{3}{8}x^{-5/2} \quad \Rightarrow \left(\frac{d^3y}{dx^3}\right)_{x=1.00} = 0.375$$

4.1–4 ***Find the first, second and third derivatives of the function tabulated below, at the point x = 1.5.***

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

Solution : Form a forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.5	3.375	**3.625**	**3.000**	**0.750**
2.0	7.000	6.625	3.750	0.750
2.5	13.625	10.375	4.500	0.750
3.0	24.000	14.875	5.250	
3.5	38.875	20.125		
4.0	59.000			

We have Newton's forward formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots$$

$$\text{where } u = \frac{x - x_0}{h} \quad \therefore \frac{du}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \dots\dots\dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \dots\dots\dots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \dots\dots\dots\dots\right]$$

Here $x_0 = 1.5$ and $h = 0.5$ $\therefore u = \frac{x - x_0}{h} = 0$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.000) + \frac{1}{3}(0.750)\right] = 4.750$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = \frac{1}{0.25}[3.000 - (0.750)] = 9.000$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=1.5} = \frac{1}{0.125}[0.750] = 6.000$$

Thus at x = 1.5,

$$\frac{dy}{dx} = 4.75, \quad \frac{d^2y}{dx^2} = 9 \text{ and } \frac{d^3y}{dx^3} = 6$$

4.1-5 ***Find the derivative of f(x) at x = 0.4 from the following table :***

x	0.1	0.2	0.3	0.4
f(x)	1.10517	1.22140	1.34986	1.49182

Solution : Since the derivative is required at x = 0.4, which is at the end of the table. Therefore we shall use Newton's backward formula. The difference table is given below :

x	f(x)	∇f(x)	∇^2f(x)	∇^3f(x)
0.1	1.10517			
0.2	1.22140	0.11623		
0.3	1.34986	0.12846	0.01223	
0.4	1.49182	**0.14196**	**0.01350**	**0.00127**

Newton's backward formula is

$$f(x) = f(x_n) + u\nabla(f(x_n)) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots\ldots$$

$$\text{where } u = \frac{x - x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

$$\therefore f'(x) = \frac{1}{h}\left[\nabla f(x_n) + \frac{2u+1}{2!}\nabla^2 f(x_n) + \frac{3u^2 + 6u + 2}{3!}\nabla^3 f(x_n) + \ldots\ldots\right]$$

Here $x = 0.4$, $x_n = 0.4$, $h = 0.1$ $\quad \therefore u = 0$

$$\therefore f'(0.4) = \frac{1}{0.1}\left[\nabla f(0.4) + \frac{1}{2}\nabla^2 f(0.4) + \frac{1}{3}\nabla^3 f(0.4) + \ldots\ldots\right]$$

$$= 10\left[0.14196 + \frac{1}{2}(0.1350) + \frac{1}{3}(0.00127)\right]$$

$$= 1.4913$$

Thus the derivative of f(x) at x = 0.4 is 1.4913

4.2-3 ***Find the value of* (0.04) *from the following table :***

x	0.01	0.02	0.03	0.04	0.05	0.06
f(x)	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Solution : Here we want to find the derivative at the point x = 0.04 which lies near the middle of the table. So we should use any central difference formula.

Here h = 0.01, taking x_0 = 0.04 as the origin the central difference table is,

$u = \frac{x - 0.04}{0.01}$	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−3	0.01	0.1023	0.0024	0	0.0001	−0.0001
−2	0.02	0.1047	0.0024	0.0001	0	−0.0001
−1	0.03	0.1071	0.0025	0.0001	−0.0001	
0	0.04	0.1096	0.0026	0		
1	0.05	0.1122	0.0026			
2	0.06	0.1148				

The Gauss's forward formula is

$$y = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_{-1}}{2!} + u(u^2-1)\frac{\Delta^3 y_{-1}}{3!}$$

$$+ u(u^2-1)(u-2)\frac{\Delta^4 y_{-2}}{4!} + \ldots\ldots\ldots\ldots$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du} \qquad \left[\because u = \frac{x - x_0}{h}\right]$$

$$= \frac{1}{h}\left[\Delta y_0 + (2u-1)\frac{\Delta^2 y_{-1}}{2!} + (3u^2-1)\frac{\Delta^2 y_{-1}}{3!} + (4u^3 - 6u^2 - 2u + 2)\frac{\Delta^4 y_{-2}}{4!}\right]$$

At x = 0.04 , u = 0.

$$\therefore \left(\frac{dy}{dx}\right)_{x=0.04} = \frac{1}{0.01}\left[0.0026 - \frac{0.0001}{2} + \frac{0.001}{6} + \frac{2}{24}\cdot(-0.001)\right]$$

$$= 0.2558$$

Hence the required result is f ′(0.04) = 0.2558

4.2-4 ***Find the first and second derivatives from the following table at x = 1.0***

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y	0.644218	0.717356	0.783327	0.841471	0.891207	0.932039	0.963558

Solution : To find the derivatives at x = 1.0 (the middle of the table) we form a central difference table with the given data :

$u = \frac{x - x_0}{h}$	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
– 3	0.7	0.644218	0.073138	– 0.007167	– 0.000660	0.000079
– 2	0.8	0.717356	0.065971	– 0.007827	– 0.000581	0.000085
– 1	0.9	0.783327	0.058144	– 0.008408	– 0.000496	0.000087
0	1.0	0.841471	0.048736	– 0.008904	– 0.000409	
2	1.1	0.891207	0.040832	– 0.009313		
2	1.2	0.932039	0.031519			
3	1.3	0.963558				

We have the Stirling's formula is

$$y = y_0 + u\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2 - 1)}{3!}\cdot\frac{\Delta^3 y_{-1}}{2}$$

$$+ \frac{u^2(u^2 - 1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots(1)$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

Taking $x_0 = 1.0$ as the origin we get u = 0

Differentiating (1) w. r. to x and putting u = 0, we get

$$\left(\frac{dy}{dx}\right)_{u=0} = \left(\frac{dy}{dx}\right)_{x=1.0} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right]$$

$$= \frac{1}{0.1}\left[\frac{1}{2}(0.58144 + 0.048736) - \frac{1}{12}(-0.000581 - 0.000496)\right]$$

$= 0.535296666$

≈ 0.535297 approx.

4.2-5 ***Find first and second derivatives of the function given below at the point x = 1.2***

x	1	2	3	4	5
y	0	1	5	6	8

Solution : Since the derivatives are required at x = 1.2, which lies near the beginning of the table. So we shall use Newton's forward formula.

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0	**1**	**3**	**– 6**	**10**
2	1	4	– 3	4	
3	5	1	1		
4	6	2			
5	8				

We have the Newton's forward formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\ldots$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}.$$

$$\frac{du}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left[y_0 + u\Delta y_0 + \frac{u^2 - u}{2!}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!}\Delta^3 y_0 + \frac{u^4 - 6u^3 + 11u^2 - 6u}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6u}{24}\Delta^4 y\right]$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2 - 18u + 11}{12}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\right]$$

Here $x = 1.2$, $x_0 = 1$, $h = 1$ $\therefore u = \frac{x - x_0}{h} = \frac{1}{5}$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{1}\left[1 + \frac{1}{2}\left(\frac{2}{5} - 1\right).3 + \frac{1}{6}\left(\frac{3}{25} - \frac{6}{5} + 2\right)(-6) + \frac{1}{24}\left(\frac{4}{124} - \frac{18}{25} + \frac{22}{5} - 6\right)(10)\right]$$

$$= \frac{-133}{75} \approx -1.773$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=1.2} = \frac{1}{1^2}\left[3 + \left(\frac{1}{5} - 1\right)(-6) + \frac{1}{12}\left(\frac{6}{25} - \frac{18}{5} + 11\right)(10)\right]$$

$$= \frac{85}{6} \approx 14.167$$

4.2–6 ***Find the first and second derivatives of the function tabulated below at the point x = 1.1:***

x	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	0	0.1280	0.5440	1.2960	2.4320	4.0000

Solution : Since the derivatives are required at x = 1.1, which is near the beginning of the table. So Newtons forward formula is suitable for this case.

The forward difference table is

x	f(x)	Δf(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1.0	0.0000	0.1280	0.2880	0.0480
1.2	0.1280	0.4160	0.3360	0.0480
1.4	0.5440	0.7520	0.3840	0.0480
1.6	1.2960	1.1360	0.4320	
1.8	2.4320	1.5680		
2.0	4.0000			

Newton's forward formula is

$$y = y_0 + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

Now,
$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta f(x_0) + \frac{2u-1}{2}\Delta^2 f(x_0) + \frac{3u^2 - 6u + 2}{6}\Delta^3 f(x_0) + \ldots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f(x_0) + (u-1)\Delta^3 f(x_0) + \ldots\ldots\right]$$

Here $x = 1.1$, $x_0 = 1.0$, $h = 0.2$

$$\therefore u = \frac{1.1 - 1.0}{0.2} = \frac{1}{2}$$

Thus we get,

$$\left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.1280 + 0 + \frac{1}{6}\left(\frac{3}{4} - \frac{6}{2} + 2\right)(0.0480)\right]$$

$$= 0.63$$

and
$$\left(\frac{d^2y}{dx^2}\right)_{x=1.1} = \frac{1}{0.02}\left[0.2880 + \left(\frac{1}{2} - 1\right)(0.0480)\right]$$

$$= 6.60$$

Thus at the point x = 1.1, the first and second derivatives of the function are 0.63 and 6.60 respectively.

$$f'(2.5) = -\frac{1}{18}[(2.5-1)(2.5-3)+(2.5-1)(2.5-6)+(2.5-3)(2.5-6)]$$

$$+\frac{3}{10}[2.5(2.5-3)+2.5(2.5-6)+(2.5-3)(2.5-6)]$$

$$-\frac{31}{18}[2.5(2.5-1)+2.5(2.5-6)+(2.5-1)(2.5-6)]$$

$$+\frac{223}{90}[2.5(2.5-1)+2.5(2.5-3)+(2.5-1)(2.5-3)]$$

$$= 19.75$$

4.5–1 ***From the following table, find x correct to two decimal places, for which y is maximum and find this value of y.*** *[নিচের তালিকা হতে y এর সর্বোচ্চ মানের জন্য x-এর দুই দশমিক স্থান পর্যন্ত মান নির্ণয় কর, এবং y এর ঐ মানটিও বের কর।]*

x	1.2	1.3	1.4	1.5	1.6
f(x)	0.9320	0.9636	0.9855	0.9975	0.9996

Solution : Form a difference table of the given data :

x	y	$\Delta(y)$	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320	0.0316	– 0.0097	– 0.0002
1.3	0.9636	0.0219	– 0.0099	0
1.4	0.9855	0.0120	– 0.0099	
1.5	0.9975	0.0021		
1.6	0.9996			

From Newtons Gregory forward formula. We have

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \ldots\ldots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + ...]$$

We know, for maximum value of y, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + ...\right] = 0$$

$$\Rightarrow 0.0316 + \frac{2u-1}{2}(-0.0097) + \text{negligible terms} = 0$$

$$\Rightarrow u = \frac{1}{2}\left[\frac{2\times(0.0316)}{(0.0097)} + 1\right]$$

$$= 3.757731959$$

$$\Rightarrow \frac{x-1.2}{0.1} = 3.757731959$$

$$\Rightarrow x = 1.575773196$$

$$\therefore x = 1.58$$

Now, $\left(\frac{d^2y}{dx^2}\right)_{x=1.58} = \frac{1}{(0.1)^2}[-0.0097]$ = negative quantity.

So at x = 1.58, the value of y is maximum. And the value of y at x = 1.58 is

$$y(1.58) = 0.932 + (3.7577)(0.0316) + \frac{(3.7577)(3.7577-1)}{2}(-0.0097)$$

$$= 1.00048$$

$$\approx 1$$

i. e. the value of y is 1.

Note : প্রশ্নে দুই দশমিক স্থান পর্যন্ত নিতে বলা হয়েছে। কিন্তু table এ 3rd difference এর মান তিন দশমিক স্থান পর্যন্ত শূন্য; তাই 3rd difference এবং উহার উচ্চঘাত পরিহার করা হয়েছে।

4.5–2 ***Find the maximum and minimum values of the function tabulated below by using Bessel's formula :***

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16.00	56.25

Solution : Taking x = 2 as the origin, the central difference table is

The stirlings formula is

$$\theta = y_0 + u\left(\frac{\Delta\theta_{-1} + \Delta\theta_0}{2}\right) + \frac{u^2}{2}\Delta^2\theta_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3\theta_{-2} + \Delta^3\theta_{-1}}{2}\right)$$

$$+ \frac{u^2(u^2-1)}{4!}\Delta^4\theta_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}\left(\frac{\Delta^5\theta_{-3} + \Delta^5\theta_{-2}}{2}\right) + \ldots$$

where $u = \frac{t - t_0}{h} \Rightarrow \frac{du}{dt} = \frac{1}{h}$

$$\therefore \frac{d\theta}{dt} = \frac{d\theta}{du} \cdot \frac{du}{dt} = \frac{1}{h} \cdot \frac{d\theta}{du}$$

$$= \frac{1}{h}\left[\frac{\Delta\theta_{-1} + \Delta\theta_0}{2} + u\Delta^2\theta_{-1} + \frac{3u^2 - 1}{12}(\Delta^3\theta_{-2} + \Delta^3\theta_{-1})\right.$$

$$\left. + \frac{4u^3 - 2u}{24}\Delta^4 y_{-2} + \frac{5u^4 - 15u^2 + 4}{240}(\Delta^5\theta_{-3} + \Delta^5\theta_{-2}) + \ldots\ldots\right]$$

But at $t = 0.6$, $u = 0$

$$\therefore \left(\frac{d\theta}{dt}\right)_{t=0.06} = \frac{1}{0.02}\left[\frac{0.74 + 0.085}{2} - \frac{1}{12}(0 - 0.15)\right.$$

$$\left. + \frac{1}{60}(-0.14 + 0.034)\right]$$

$$= \frac{1}{0.02}[0.795 + 0.00125 + 0.000333]$$

$$= 4.054167$$

Thus the required angular velocity is 4.054 rad/sec approximately.

4.5-4 ***The population of a town is given below in difference time. Find the rate of grwoth of the population in 1921 and 1961.*** *[নিচে বিভিন্ন সময়ের কোন শহরের জনসংখ্যা দেওয়া হল। **1921** ও **1961** সালের জনসংখ্যা বৃদ্ধির হার নির্ণয় কর।]*

Year	1921	1931	1941	1951	1961
Poupulation (in thousand)	19.96	38.65	58.81	77.21	94.61

Solution : If we consider x as the year and y as the population, then the rate of growth of the population becomes $\frac{dy}{dx}$.

In order to find $\frac{dy}{dx}$, we form a difference table.

x	y	Δy / ∇y	$\Delta^2 y$ / $\nabla^2 y$	$\Delta^3 y$ / $\nabla^3 y$	$\Delta^4 y$ / $\nabla^4 y$
1921	19.96				
		18.69			
1931	38.65		1.47		
		20.16		– 3.23	
1941	58.81		– 1.76		3.99
		18.40		0.76	
1951	77.21		– 1.0		
		17.40			
1961	94.61				

(i) To find $\frac{dy}{dx}$ at x = 1921 we shall use Newtons forward formula.

The formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\ldots (1)$$

$$\text{where } u = \frac{x - x_0}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

At x = 1921, u = 0 ; [here h = 10, x_0 = 1921]

Now differentiating (1) w. r. to x and putting u = 0 we get

$$\left(\frac{dy}{dx}\right)_{u=0} = \left(\frac{dy}{dx}\right)_{x=1921}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{10}\left[18.69 - \frac{1}{2}(1.47) + \frac{1}{3}(-3.23) - \frac{1}{4}(3.99)\right]$$

$$= 1.588$$

(ii) To find $\frac{dy}{dx}$ at x = 1961, we shall use Newton's beackward formula (to get more accurate result).

The formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \quad \ldots\ldots\ldots\ldots(2)$$

$$\text{where } u = \frac{x - x_n}{h} \Rightarrow \frac{du}{dx} = \frac{1}{h}$$

At $x = 1961$, $u = 0$; [here $h = 10$, $x_n = 1961$]

Now differentiating (2) w. r. to x and putting $u = 0$ we get.

$$\left(\frac{dy}{dx}\right)_{u=0} = \left(\frac{dy}{dx}\right)_{x=1961}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{2}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\ldots\ldots\right]$$

$$= \frac{1}{10}\left[17.40 + \frac{1}{2}(-1.0) + \frac{1}{3}(0.76) + \frac{1}{4}(3.99)\right]$$

$$= 1.81508$$

Hence the rate of growth of the populations are 1.588 and 1.81508 in 1921 and 1961 respectively.

4.6–1 ***For more practice, solve the followings :***

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 3.0$ of the function tabulated below :

x	3.0	3.2	3.4	3.6	3.8	4.0
y	– 14.000	– 10.032	– 5.296	0.256	6.672	14.000

Answer : $\left(\frac{dy}{dx}\right)_{x=3.0} = 18$, $\left(\frac{d^2y}{dx^2}\right)_{x=3.0} = 18$

Note : The function tabulated above is

$$y = x^3 - 9x - 14$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=3} = 3.3^2 - 9 = 18, \quad \left(\frac{d^2y}{dx^2}\right)_{x=3} = 18$$

(ii) Find the 1st and 2nd derivatives of $\ln x$ at $x = 500$, after taking the values of $\ln 500$, $\ln 510$, $\ln 520$, $\ln 530$, $\ln 540$, $\ln 550$ correct to six decimal places.

Answer : 0.002000 and – 0.0000040.

(iii) Find $f'(1)$ for $f(x) = \frac{1}{(1+x)^2}$ using the following data :

x	1.0	1.1	1.2	1.3	1.4
f(x)	0.2500	0.2268	0.2066	0.1890	0.1736

Answer ; – 0.25

(iv) Find f′(15) and f″(15) of $f(x) = \sqrt{x}$ from the following data :

x	15	17	19	21	23
f(x)	3.8730	4.1231	4.3589	4.5826	4.7958

Answer : f′(15) = 0.1365 and f″(15) = 0.0161

(v) Find 1st and 2nd derivatives at x = 1.6 from the following table :

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Answer : 2.732 and – 1.475

(vi) Find f′(0.5) from the following table :

x	0.35	0.40	0.45	0.50	0.55	0.60	0.65
f(x)	1.521	1.506	1.488	1.467	1.444	1.418	1.389

Answer : f′(0.5)= – 0.44

(vii) Find y′(1) from the following table :

x	0.8	0.9	1.0	1.1	1.2
y	0.717356	0.783327	0.841471	0.891207	0.932039

Answer : y ′(1) = 0.54030

(viii) Find f′(x) and f″(x) at x = 1, given that

x	– 2	– 1	0	1	2	3
f(x)	104	17	0	– 1	8	69

Answer : f′(1) = 1, f′(1) = 6

(ix) Find f′(x) at x = 93 from the following table :

x	60	75	90	105	120
f(x)	28.2	38.2	43.2	40.9	33.7

Answer : f′(x) = – 0.036271

(x) From the following table, find f′(51) :

x	50	60	70	80	90
f(x)	19.96	36.65	58.81	77.21	94.61

Answer : f′(51) =1.031584

(xi) From the following table, find f ′(10) :

x	3	5	11	27	34
f(x)	– 13	23	899	17315	35605

Answer : 233

(xii) A curves passes through (1,0), (2, 1), (4, 5) (8, 21), (10, 25). Find its slope at x = 4. **Ansewer :** 2.883

(xiii) Find f‴(5) from the given values by using divided difference.

x	2	4	9	13	16	21	29
f(x)	57	1345	66340	402052	1118209	4287844	21242820

(xiv) A slider (গড়িয়ে চলতে পারে এমন বস্তু) moves along a fixed straight line. Its distance 'x' cm in various times 't' sec are given below.

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Find (a) the velocity of the slider and its acceleration at $t = 0.3$

Answer : 5.34cm/sec and – 45.6 cm/sec²

(xv) The following table gives the angular displacement θ (readians) at different intervals of time t (sec). Calculate the angular velicity and accelertion at time $t = 0.6$.

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Answer : velocity 3.81665 rad/see; Acceleration 6.75 rasec

(xvi) The following data gives the corresponding values for pressure p and volume v. Find the rate of change of pressure with respect to the volume $v = 2$

v	2	4	6	8	10
P	105	42.7	25.3	16.7	13

Answer : – 52.4

(xvii) Find the value of x for which y is minimum and find y in that place of x from the following tabel :

x	0.60	0.65	0.70	0.75
y	0.6221	0.6155	0.6138	0.6170

Answer : $x = 0.699235$ and $y = 0.61374$

(xviii) Find the value of x for which y is minimum of maximum :

x	0.60	0.65	0.70	0.80
y	0.36	0.4225	0.49	0.64

Answer : 0

(xix) Given a table of values of x and y :

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$ **Answer :** $\frac{dy}{dx} = 3.3205$ $\frac{d^2y}{dx^2} = 3.318$

(b) Find $\frac{dy}{dx}$ at $x = 2$ and 2.2

Answer : $\left(\frac{dy}{dx}\right)_2 = 7.3896, \left(\frac{dy}{dx}\right)_{2.2} = 9.0228$

(c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$

Answer : $\left(\frac{dy}{dx}\right)_{1.6} = 4.9530, \left(\frac{d^2y}{dx^2}\right)_{1.6} = 4.9525$

(xx) Derive the expression of first derivative of Gauss's formula.