SECTION 1.1 Functions

Just-In-Time REVIEW

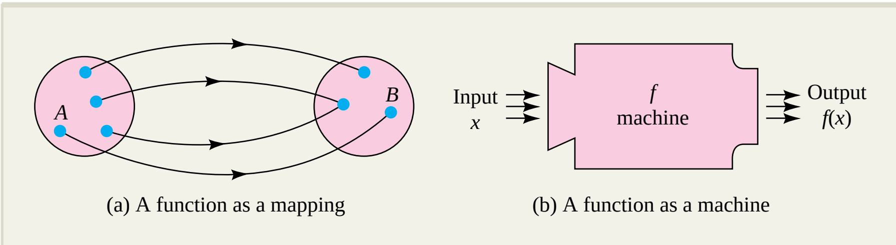
Appendices A1 and A2 contain a brief review of algebraic properties needed in calculus.

In many practical situations, the value of one quantity may depend on the value of a second. For example, the consumer demand for beef may depend on the current market price; the amount of air pollution in a metropolitan area may depend on the number of cars on the road; or the value of a rare coin may depend on its age. Such relationships can often be represented mathematically as **functions.**

Loosely speaking, a function consists of two sets and a rule that associates elements in one set with elements in the other. For instance, suppose you want to determine the effect of price on the number of units of a particular commodity that will be sold at that price. To study this relationship, you need to know the set of admissible prices, the set of possible sales levels, and a rule for associating each price with a particular sales level. Here is the definition of function we shall use.

Function ■ A **function** is a rule that assigns to each object in a set A exactly one object in a set B. The set A is called the **domain** of the function, and the set of assigned objects in B is called the **range.**

For most functions in this book, the domain and range will be collections of real numbers and the function itself will be denoted by a letter such as f. The value that the function f assigns to the number x in the domain is then denoted by $f(x)$ (read as "f of x"), which is often given by a formula, such as $f(x) = x^2 + 4$.



(a) A function as a mapping (b) A function as a machine

FIGURE 1.1 Interpretations of the function $f(x)$.

It may help to think of such a function as a "mapping" from numbers in A to numbers in B (Figure 1.1a), or as a "machine" that takes a given number from A and converts it into a number in B through a process indicated by the functional rule (Figure 1.1b). For instance, the function $f(x) = x^2 + 4$ can be thought of as an "f machine" that accepts an input x, then squares it and adds 4 to produce an output $y = x^2 + 4$.

No matter how you choose to think of a functional relationship, it is important to remember that *a function assigns one and only one number in the range (output) to each number in the domain (input)*. Here is an example.

EXPLORE!

Store $f(x) = x^2 + 4$ into your graphing utility. Evaluate at $x = -3, -1, 0, 1,$ and 3. Make a table of values. Repeat using $g(x) = x^2 - 1$. Explain how the values of $f(x)$ and $g(x)$ differ for each x value.

EXAMPLE 1.1.1

Find $f(3)$ if $f(x) = x^2 + 4$.

Solution

$$f(3) = 3^2 + 4 = 13$$

Observe the convenience and simplicity of the functional notation. In Example 1.1.1, the compact formula $f(x) = x^2 + 4$ completely defines the function, and you can indicate that 13 is the number the function assigns to 3 by simply writing $f(3) = 13$.

It is often convenient to represent a functional relationship by an equation $y = f(x)$, and in this context, x and y are called **variables.** In particular, since the numerical value of y is determined by that of x, we refer to y as the **dependent variable** and to x as the **independent variable.** Note that there is nothing sacred about the symbols x and y. For example, the function $y = x^2 + 4$ can just as easily be represented by $s = t^2 + 4$ or by $w = u^2 + 4$.

Functional notation can also be used to describe tabular data. For instance, Table 1.1 lists the average tuition and fees for private 4-year colleges at 5-year intervals from 1973 to 2003.

TABLE 1.1 Average Tuition and Fees for 4-Year Private Colleges

Academic Year Ending in	Period *n*	Tuition and Fees
1973	1	\$1,898
1978	2	\$2,700
1983	3	\$4,639
1988	4	\$7,048
1993	5	\$10,448
1998	6	\$13,785
2003	7	\$18,273

SOURCE: *Annual Survey of Colleges*, The College Board, New York.

We can describe this data as a function f defined by the rule

$$f(n) = \begin{bmatrix} \text{average tuition and fees at the} \\ \text{beginning of the } n\text{th 5-year period} \end{bmatrix}$$

Thus, $f(1) = 1{,}898$, $f(2) = 2{,}700$, . . . , $f(7) = 18{,}273$. Note that the domain of f is the set of integers $A = \{1, 2, \ldots, 7\}$.

The use of functional notation is illustrated further in Examples 1.1.2 and 1.1.3. In Example 1.1.2, notice that letters other than f and x are used to denote the function and its independent variable.

Just-In-Time REVIEW

Recall that $x^{a/b} = \sqrt[b]{x^a}$ whenever a and b are positive integers. Example 1.1.2 uses the case when $a = 1$ and $b = 2$; $x^{1/2}$ is another way of expressing $\sqrt{x}$.

EXAMPLE 1.1.2

If $g(t) = (t - 2)^{1/2}$, find (if possible) $g(27)$, $g(5)$, $g(2)$, and $g(1)$.

Solution

Rewrite the function as $g(t) = \sqrt{t - 2}$. (If you need to brush up on fractional powers, consult the discussion of exponential notation in Appendix A1. Then

$$g(27) = \sqrt{27 - 2} = \sqrt{25} = 5$$

$$g(5) = \sqrt{5 - 2} = \sqrt{3} \approx 1.7321$$

and

$$g(2) = \sqrt{2 - 2} = \sqrt{0} = 0$$

EXPLORE!

Store $g(x) = \sqrt{x-2}$ in the function editor of your graphing utility as $Y1 = \sqrt{(x-2)}$. Now on your **HOME SCREEN** create Y1(27), Y1(5), and Y1(2), or, alternatively, Y1({27, 5, 2}), where the braces are used to enclose a list of values. What happens when you construct Y1(1)?

EXPLORE!

Create a simple piecewise-defined function using the boolean algebra features of your graphing utility. Write Y1 = 2(X < 1) + (−1)(X ≥ 1) in the function editor. Examine the graph of this function, using the **ZOOM** Decimal Window. What values does Y1 assume at X = −2, 0, 1, and 3?

EXPLORE!

Store **f**(**x**) = 1/(**x** − 3) in your graphing utility as Y1, and display its graph using a **ZOOM** Decimal Window. **TRACE** values of the function from X = 2.5 to 3.5. What do you notice at X = 3? Next store $g(x) = \sqrt{(x-2)}$ into Y1, and graph using a **ZOOM** Decimal Window. **TRACE** values from X = 0 to 3, in 0.1 increments. When do the Y values start to appear, and what does this tell you about the domain of **g**(**x**)?

However, $g(1)$ is undefined since

$$g(1) = \sqrt{1-2} = \sqrt{-1}$$

and negative numbers do not have real square roots.

Functions are often defined using more than one formula, where each individual formula describes the function on a subset of the domain. A function defined in this way is sometimes called a **piecewise-defined function.** Here is an example of such a function.

EXAMPLE 1.1.3

Find $f\left(-\frac{1}{2}\right)$, $f(1)$, and $f(2)$ if

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 1 \\ 3x^2 + 1 & \text{if } x \ge 1 \end{cases}$$

Solution

Since $x = -\frac{1}{2}$ satisfies $x < 1$, use the top part of the formula to find

$$f\left(-\frac{1}{2}\right) = \frac{1}{-1/2 - 1} = \frac{1}{-3/2} = -\frac{2}{3}$$

However, $x = 1$ and $x = 2$ satisfy $x \ge 1$, so $f(1)$ and $f(2)$ are both found by using the bottom part of the formula:

$$f(1) = 3(1)^2 + 1 = 4 \quad \text{and} \quad f(2) = 3(2)^2 + 1 = 13$$

Domain Convention ■ Unless otherwise specified, if a formula (or several formulas, as in Example 1.1.3) is used to define a function f, then we assume the domain of f to be the set of all numbers for which $f(x)$ is defined (as a real number). We refer to this as the **natural domain** of f.

Determining the natural domain of a function often amounts to excluding all numbers x that result in dividing by 0 or in taking the square root of a negative number. This procedure is illustrated in Example 1.1.4.

EXAMPLE 1.1.4

Find the domain and range of each of these functions.

a. $f(x) = \dfrac{1}{x-3}$ **b.** $g(t) = \sqrt{t-2}$

Solution

a. Since division by any number other than 0 is possible, the domain of f is the set of all numbers x such that $x - 3 \ne 0$; that is, $x \ne 3$. The range of f is the set of all numbers y except 0, since for any $y \ne 0$, there is an x such that $y = \dfrac{1}{x-3}$; in particular, $x = 3 + \dfrac{1}{y}$.

b. Since negative numbers do not have real square roots, $g(t)$ can be evaluated only when $t - 2 \geq 0$, so the domain of g is the set of all numbers t such that $t \geq 2$. The range of g is the set of all nonnegative numbers, for if $y \geq 0$ is any such number, there is a t such that $y = \sqrt{t-2}$; namely, $t = y^2 + 2$.

***Just-In-Time* REVIEW**

Recall that $\sqrt{a}$ is defined to be the *positive* number whose square is a.

Functions Used in Economics

There are several functions associated with the marketing of a particular commodity:

The **demand function** $D(x)$ for the commodity is the price $p = D(x)$ that must be charged for each unit of the commodity if x units are to be sold (demanded).

The **supply function** $S(x)$ for the commodity is the unit price $p = S(x)$ at which producers are willing to supply x units to the market.

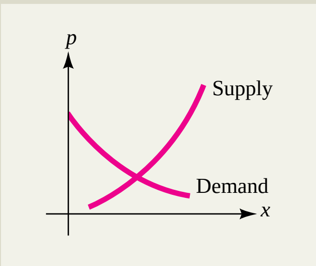
The **revenue** $R(x)$ obtained from selling x units of the commodity is given by the product

$$R(x) = (\text{number of items sold})(\text{price per item}) = xp(x)$$

The **cost function** $C(x)$ is the cost of producing x units of the commodity.

The **profit function** $P(x)$ is the profit obtained from selling x units of the commodity and is given by the difference

$$\begin{aligned} P(x) &= \text{revenue} - \text{cost} \\ &= R(x) - C(x) = xp(x) - C(x) \end{aligned}$$



Generally speaking, the higher the unit price, the fewer the number of units demanded, and vice versa. Conversely, an increase in unit price leads to an increase in the number of units supplied. Thus, demand functions are typically decreasing ("falling" from left to right), while supply functions are increasing ("rising"), as illustrated in the margin. Here is an example that uses several of these special economic functions.

EXAMPLE 1.1.5

Market research indicates that consumers will buy x thousand units of a particular kind of coffee maker when the unit price is

$$p(x) = -0.27x + 51$$

dollars. The cost of producing the x thousand units is

$$C(x) = 2.23x^2 + 3.5x + 85$$

thousand dollars.

a. What are the revenue and profit functions, $R(x)$ and $P(x)$, for this production process?

b. For what values of x is production of the coffee makers profitable?

Solution

a. The revenue is

$$R(x) = xp(x) = -0.27x^2 + 51x$$

thousand dollars, and the profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.27x^2 + 51x - (2.23x^2 + 3.5x + 85) \\ &= -2.5x^2 + 47.5x - 85 \end{aligned}$$

thousand dollars.

***Just-In-Time* REVIEW**

The product of two numbers is positive if they have the same sign and is negative if they have different signs. That is, $ab > 0$ if $a > 0$ and $b > 0$ and also if $a < 0$ and $b < 0$. On the other hand, $ab < 0$ if $a < 0$ and $b > 0$ or if $a > 0$ and $b < 0$.

b. Production is profitable when $P(x) > 0$. We find that

$$\begin{aligned} P(x) &= -2.5x^2 + 47.5x - 85 \\ &= -2.5(x^2 - 19x + 34) \\ &= -2.5(x - 2)(x - 17) \end{aligned}$$

Since the coefficient -2.5 is negative, it follows that $P(x) > 0$ only if the terms $(x - 2)$ and $(x - 17)$ have different signs; that is, when $x - 2 > 0$ and $x - 17 < 0$. Thus, production is profitable for $2 < x < 17$.

Example 1.1.6 illustrates how functional notation is used in a practical situation. Notice that to make the algebraic formula easier to interpret, letters suggesting the relevant practical quantities are used for the function and its independent variable. (In this example, the letter C stands for "cost" and q stands for "quantity" manufactured.)

EXPLORE!

Refer to Example 1.1.6, and store the cost function $C(q)$ into Y1 as

$X^3 - 30X^2 + 500X + 200$

Construct a **TABLE** of values for $C(q)$ using your calculator, setting TblStart at X = 5 with an increment ΔTbl = 1 unit. On the table of values observe the cost of manufacturing the 10th unit.

EXAMPLE 1.1.6

Suppose the total cost in dollars of manufacturing q units of a certain commodity is given by the function $C(q) = q^3 - 30q^2 + 500q + 200$.

a. Compute the cost of manufacturing 10 units of the commodity.

b. Compute the cost of manufacturing the 10th unit of the commodity.

Solution

a. The cost of manufacturing 10 units is the value of the total cost function when $q = 10$. That is,

$$\begin{aligned} \text{Cost of 10 units} &= C(10) \\ &= (10)^3 - 30(10)^2 + 500(10) + 200 \\ &= \$3{,}200 \end{aligned}$$

b. The cost of manufacturing the 10th unit is the difference between the cost of manufacturing 10 units and the cost of manufacturing 9 units. That is,

$$\text{Cost of 10th unit} = C(10) - C(9) = 3{,}200 - 2{,}999 = \$201$$

Composition of Functions

There are many situations in which a quantity is given as a function of one variable that, in turn, can be written as a function of a second variable. By combining the functions in an appropriate way, you can express the original quantity as a function of the second variable. This process is called **composition of functions** or **functional composition.**

For instance, suppose environmentalists estimate that when p thousand people live in a certain city, the average daily level of carbon monoxide in the air will be $c(p)$ parts per million, and that separate demographic studies indicate the population in t years will be $p(t)$ thousand. What level of pollution should be expected in t years? You would answer this question by substituting $p(t)$ into the pollution formula $c(p)$ to express c as a composite function of t.

We shall return to the pollution problem in Example 1.1.11 with specific formulas for $c(p)$ and $p(t)$, but first you need to see a few examples of how composite functions are formed and evaluated. Here is a definition of functional composition.

Composition of Functions ■ Given functions $f(u)$ and $g(x)$, the composition $f(g(x))$ is the function of x formed by substituting $u = g(x)$ for u in the formula for $f(u)$.

Note that the composite function $f(g(x))$ "makes sense" only if the domain of f contains the range of g. In Figure 1.2, the definition of composite function is illustrated as an "assembly line" in which "raw" input x is first converted into a transitional product $g(x)$ that acts as input the f machine uses to produce $f(g(x))$.

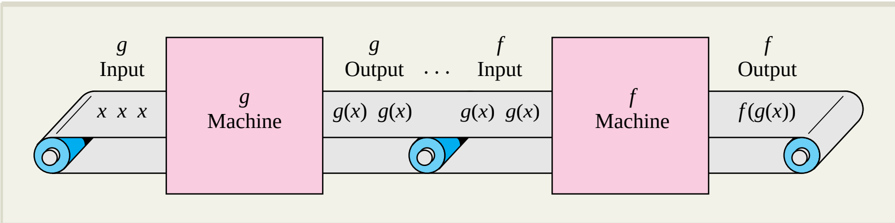


FIGURE 1.2 The composition $f(g(x))$ as an assembly line.

EXAMPLE 1.1.7

Find the composite function $f(g(x))$, where $f(u) = u^2 + 3u + 1$ and $g(x) = x + 1$.

Solution

Replace u by $x + 1$ in the formula for $f(u)$ to get

$$\begin{aligned} f(g(x)) &= (x + 1)^2 + 3(x + 1) + 1 \\ &= (x^2 + 2x + 1) + (3x + 3) + 1 \\ &= x^2 + 5x + 5 \end{aligned}$$

NOTE By reversing the roles of f and g in the definition of composite function, you can define the composition $g(f(x))$. In general, $f(g(x))$ and $g(f(x))$ will *not* be the same. For instance, with the functions in Example 1.1.7, you first write

$$g(w) = w + 1 \qquad \text{and} \qquad f(x) = x^2 + 3x + 1$$

and then replace w by $x^2 + 3x + 1$ to get

$$\begin{aligned} g(f(x)) &= (x^2 + 3x + 1) + 1 \\ &= x^2 + 3x + 2 \end{aligned}$$

which is equal to $f(g(x)) = x^2 + 5x + 5$ only when $x = -\frac{3}{2}$ (you should verify this). ■

EXPLORE!

Store the functions $f(x) = x^2$ and $g(x) = x + 3$ into Y1 and Y2, respectively, of the function editor. Deselect (turn off) Y1 and Y2. Set Y3 = Y1(Y2) and Y4 = Y2(Y1). Show graphically (using **ZOOM** Standard) and analytically (by table values) that $f(g(x))$ represented by Y3 and $g(f(x))$ represented by Y4 are not the same functions. What are the explicit equations for both of these composites?

Example 1.1.7 could have been worded more compactly as follows: Find the composite function $f(x + 1)$ where $f(x) = x^2 + 3x + 1$. The use of this compact notation is illustrated further in Example 1.1.8.

EXPLORE!

Refer to Example 1.1.8. Store $f(x) = 3x^2 + 1/x + 5$ into Y1. Write Y2 = Y1(X − 1). Construct a table of values for Y1 and Y2 for 0, 1, . . . , 6. What do you notice about the values for Y1 and Y2?

EXAMPLE 1.1.8

Find $f(x - 1)$ if $f(x) = 3x^2 + \dfrac{1}{x} + 5$.

Solution

At first glance, this problem may look confusing because the letter x appears both as the independent variable in the formula defining f and as part of the expression $x - 1$. Because of this, you may find it helpful to begin by writing the formula for f in more neutral terms, say as

$$f(\square) = 3(\square)^2 + \frac{1}{\square} + 5$$

To find $f(x - 1)$, you simply insert the expression $x - 1$ inside each box, getting

$$f(x-1) = 3(x-1)^2 + \frac{1}{x-1} + 5$$

Occasionally, you will have to "take apart" a given composite function $g(h(x))$ and identify the "outer function" $g(u)$ and "inner function" $h(x)$ from which it was formed. The procedure is demonstrated in Example 1.1.9.

EXAMPLE 1.1.9

If $f(x) = \dfrac{5}{x-2} + 4(x-2)^3$, find functions $g(u)$ and $h(x)$ such that $f(x) = g(h(x))$.

Solution

The form of the given function is

$$f(x) = \frac{5}{\square} + 4(\square)^3$$

where each box contains the expression $x - 2$. Thus, $f(x) = g(h(x))$, where

$$\underbrace{g(u) = \frac{5}{u} + 4u^3}_{\text{outer function}} \qquad \text{and} \qquad \underbrace{h(x) = x - 2}_{\text{inner function}}$$

Actually, in Example 1.1.9, there are infinitely many pairs of functions $g(u)$ and $h(x)$ that combine to give $g(h(x)) = f(x)$. [For example, $g(u) = \dfrac{5}{u+1} + 4(u+1)^3$ and $h(x) = x - 3$.] The particular pair selected in the solution to this example is the most natural one and reflects most clearly the structure of the original function $f(x)$.

EXAMPLE 1.1.10

A **difference quotient** is an expression of the general form

$$\frac{f(x+h) - f(x)}{h}$$

where f is a given function of x and h is a number. Difference quotients will be used in Chapter 2 to define the *derivative*, one of the fundamental concepts of calculus. Find the difference quotient for $f(x) = x^2 - 3x$.

Solution

You find that

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\
&= \frac{[x^2 + 2xh + h^2 - 3x - 3h] - [x^2 - 3x]}{h} && \text{expand the numerator} \\
&= \frac{2xh + h^2 - 3h}{h} && \text{combine terms in the numerator} \\
&= 2x + h - 3 && \text{divide by } h
\end{aligned}$$

Example 1.1.11 illustrates how a composite function may arise in an applied problem.

EXAMPLE 1.1.11

An environmental study of a certain community suggests that the average daily level of carbon monoxide in the air will be $c(p) = 0.5p + 1$ parts per million when the population is p thousand. It is estimated that t years from now the population of the community will be $p(t) = 10 + 0.1t^2$ thousand.

a. Express the level of carbon monoxide in the air as a function of time.

b. When will the carbon monoxide level reach 6.8 parts per million?

Solution

a. Since the level of carbon monoxide is related to the variable p by the equation

$$c(p) = 0.5p + 1$$

and the variable p is related to the variable t by the equation

$$p(t) = 10 + 0.1t^2$$

it follows that the composite function

$$c(p(t)) = c(10 + 0.1t^2) = 0.5(10 + 0.1t^2) + 1 = 6 + 0.05t^2$$

expresses the level of carbon monoxide in the air as a function of the variable t.

b. Set $c(p(t))$ equal to 6.8 and solve for t to get

$$\begin{aligned}
6 + 0.05t^2 &= 6.8 \\
0.05t^2 &= 0.8 \\
t^2 &= \frac{0.8}{0.05} = 16 \\
t &= \sqrt{16} = 4 && \text{discard } t = -4
\end{aligned}$$

That is, 4 years from now the level of carbon monoxide will be 6.8 parts per million.

EXERCISES ■ 1.1

In Exercises 1 through 14, compute the indicated values of the given function.

1. $f(x) = 3x + 5$; $f(0)$, $f(-1)$, $f(2)$

2. $f(x) = -7x + 1$; $f(0)$, $f(1)$, $f(-2)$

3. $f(x) = 3x^2 + 5x - 2$; $f(0)$, $f(-2)$, $f(1)$

4. $h(t) = (2t + 1)^3$; $h(-1)$, $h(0)$, $h(1)$

5. $g(x) = x + \dfrac{1}{x}$; $g(-1)$, $g(1)$, $g(2)$

6. $f(x) = \dfrac{x}{x^2 + 1}$; $f(2)$, $f(0)$, $f(-1)$

7. $h(t) = \sqrt{t^2 + 2t + 4}$; $h(2)$, $h(0)$, $h(-4)$

8. $g(u) = (u + 1)^{3/2}$; $g(0)$, $g(-1)$, $g(8)$

9. $f(t) = (2t - 1)^{-3/2}$; $f(1)$, $f(5)$, $f(13)$

10. $f(t) = \dfrac{1}{\sqrt{3 - 2t}}$; $f(1)$, $f(-3)$, $f(0)$

11. $f(x) = x - |x - 2|$; $f(1)$, $f(2)$, $f(3)$

12. $g(x) = 4 + |x|$; $g(-2)$, $g(0)$, $g(2)$

13. $h(x) = \begin{cases} -2x + 4 & \text{if } x \le 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$; $h(3)$, $h(1)$, $h(0)$, $h(-3)$

14. $f(t) = \begin{cases} 3 & \text{if } t < -5 \\ t + 1 & \text{if } -5 \le t \le 5 \\ \sqrt{t} & \text{if } t > 5 \end{cases}$; $f(-6)$, $f(-5)$, $f(16)$

In Exercises 15 through 18, determine whether or not the given function has the set of all real numbers as its domain.

15. $g(x) = \dfrac{x}{1 + x^2}$

16. $f(x) = \dfrac{x + 1}{x^2 - 1}$

17. $f(t) = \sqrt{1 - t}$

18. $h(t) = \sqrt{t^2 + 1}$

In Exercises 19 through 24, determine the domain of the given function.

19. $g(x) = \dfrac{x^2 + 5}{x + 2}$

20. $f(x) = x^3 - 3x^2 + 2x + 5$

21. $f(x) = \sqrt{2x + 6}$

22. $f(t) = \dfrac{t + 1}{t^2 - t - 2}$

23. $f(t) = \dfrac{t + 2}{\sqrt{9 - t^2}}$

24. $h(s) = \sqrt{s^2 - 4}$

In Exercises 25 through 32, find the composite function $f(g(x))$.

25. $f(u) = 3u^2 + 2u - 6$, $g(x) = x + 2$

26. $f(u) = u^2 + 4$, $g(x) = x - 1$

27. $f(u) = (u - 1)^3 + 2u^2$, $g(x) = x + 1$

28. $f(u) = (2u + 10)^2$, $g(x) = x - 5$

29. $f(u) = \dfrac{1}{u^2}$, $g(x) = x - 1$

30. $f(u) = \dfrac{1}{u}$, $g(x) = x^2 + x - 2$

31. $f(u) = \sqrt{u + 1}$, $g(x) = x^2 - 1$

32. $f(u) = u^2$, $g(x) = \dfrac{1}{x - 1}$

In Exercises 33 through 38, find the difference quotient of f; namely, $\dfrac{f(x + h) - f(x)}{h}$.

33. $f(x) = 4 - 5x$

34. $f(x) = 2x + 3$

35. $f(x) = 4x - x^2$

36. $f(x) = x^2$

37. $f(x) = \dfrac{x}{x + 1}$

38. $f(x) = \dfrac{1}{x}$

In Exercises 39 through 42, first obtain the composite functions $f(g(x))$ and $g(f(x))$, and then find all numbers x (if any) such that $f(g(x)) = g(f(x))$.

39. $f(x) = \sqrt{x}$, $g(x) = 1 - 3x$

40. $f(x) = x^2 + 1$, $g(x) = 1 - x$

41. $f(x) = \dfrac{2x+3}{x-1}$, $g(x) = \dfrac{x+3}{x-2}$

42. $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{4-x}{2+x}$

In Exercises 43 through 50, find the indicated composite function.

43. $f(x-2)$ where $f(x) = 2x^2 - 3x + 1$

44. $f(x+1)$ where $f(x) = x^2 + 5$

45. $f(x-1)$ where $f(x) = (x+1)^5 - 3x^2$

46. $f(x+3)$ where $f(x) = (2x-6)^2$

47. $f(x^2 + 3x - 1)$ where $f(x) = \sqrt{x}$

48. $f\left(\dfrac{1}{x}\right)$ where $f(x) = 3x + \dfrac{2}{x}$

49. $f(x+1)$ where $f(x) = \dfrac{x-1}{x}$

50. $f(x^2 - 2x + 9)$ where $f(x) = 2x - 20$

In Exercises 51 through 56, find functions h(x) and g(u) such that f(x) = g(h(x)).

51. $f(x) = (x-1)^2 + 2(x-1) + 3$

52. $f(x) = (x^5 - 3x^2 + 12)^3$

53. $f(x) = \dfrac{1}{x^2+1}$

54. $f(x) = \sqrt{3x-5}$

55. $f(x) = \sqrt[3]{2-x} + \dfrac{4}{2-x}$

56. $f(x) = \sqrt{x+4} - \dfrac{1}{(x+4)^3}$

CONSUMER DEMAND *In Exercises 57 through 60, the demand function p = D(x) and the total cost function C(x) for a particular commodity are given in terms of the level of production x. In each case, find:*

(a) The revenue R(x) and profit P(x).

(b) All values of x for which production of the commodity is profitable.

57. $D(x) = -0.02x + 29$
$C(x) = 1.43x^2 + 18.3x + 15.6$

58. $D(x) = -0.37x + 47$
$C(x) = 1.38x^2 + 15.15x + 115.5$

59. $D(x) = -0.5x + 39$
$C(x) = 1.5x^2 + 9.2x + 67$

60. $D(x) = -0.09x + 51$
$C(x) = 1.32x^2 + 11.7x + 101.4$

61. MANUFACTURING COST Suppose the total cost of manufacturing q units of a certain commodity is $C(q)$ thousand dollars, where

$$C(q) = 0.01q^2 + 0.9q + 2$$

a. Compute the cost of manufacturing 10 units.
b. Compute the cost of manufacturing the 10th unit.

62. MANUFACTURING COST Suppose the total cost in dollars of manufacturing q units of a certain commodity is given by the function

$$C(q) = q^3 - 30q^2 + 400q + 500$$

a. Compute the cost of manufacturing 20 units.
b. Compute the cost of manufacturing the 20th unit.

63. DISTRIBUTION COST Suppose that the number of worker-hours required to distribute new telephone books to x% of the households in a certain rural community is given by the function

$$W(x) = \frac{600x}{300 - x}$$

a. What is the domain of the function W?
b. For what values of x does $W(x)$ have a practical interpretation in this context?
c. How many worker-hours were required to distribute new telephone books to the first 50% of the households?
d. How many worker-hours were required to distribute new telephone books to the entire community?
e. What percentage of the households in the community had received new telephone books by the time 150 worker-hours had been expended?

64. WORKER EFFICIENCY An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled

$$f(x) = -x^3 + 6x^2 + 15x$$

television sets x hours later.

a. How many sets will such a worker have assembled by 10:00 A.M.? [*Hint:* At 10:00 A.M., $x = 2$.]
b. How many sets will such a worker assemble between 9:00 and 10:00 A.M.?

SECTION 1.2 The Graph of a Function

Graphs have visual impact. They also reveal information that may not be evident from verbal or algebraic descriptions. Two graphs depicting practical relationships are shown in Figure 1.3.

The graph in Figure 1.3a describes the variation in total industrial production in a certain country over a 4-year period of time. Notice that the highest point on the graph occurs near the end of the third year, indicating that production was greatest at that time.

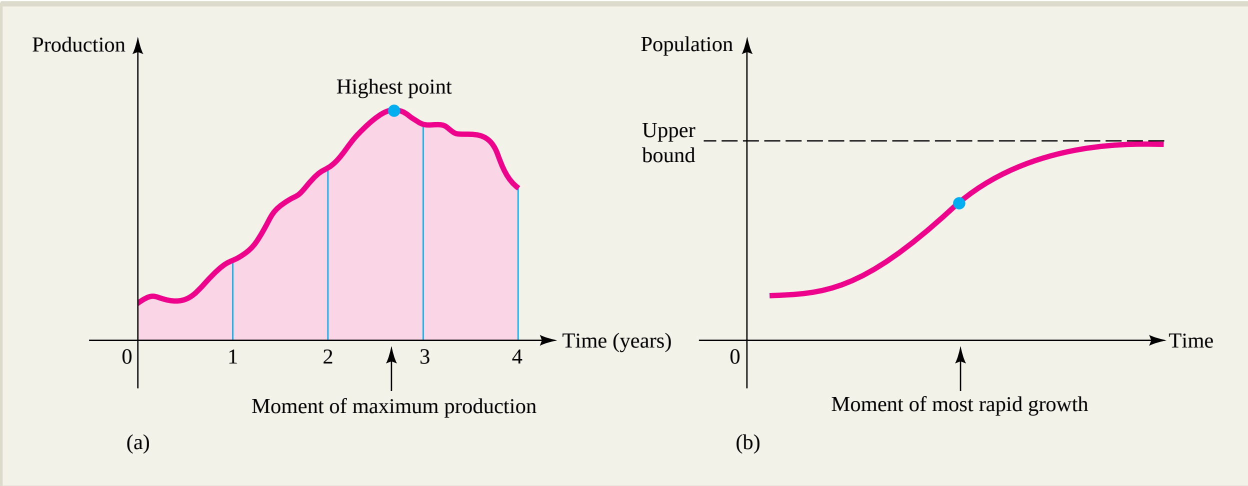


FIGURE 1.3 (a) A production function. (b) Bounded population growth.

The graph in Figure 1.3b represents population growth when environmental factors impose an upper bound on the possible size of the population. It indicates that the *rate* of population growth increases at first and then decreases as the size of the population gets closer and closer to the upper bound.

Rectangular Coordinate System

To represent graphs in the plane, we shall use a **rectangular (Cartesian) coordinate system,** which is an extension of the representation introduced for number lines in Section 1.1. To construct such a system, we begin by choosing two perpendicular number lines that intersect at the origin of each line. For convenience, one line is taken to be horizontal and is called the ***x* axis,** with positive direction to the right. The other line, called the ***y* axis,** is vertical with positive direction upward. Scaling on the two coordinate axes is often the same, but this is not necessary. The coordinate axes separate the plane into four parts called **quadrants,** which are numbered counterclockwise I through IV, as shown in Figure 1.4.

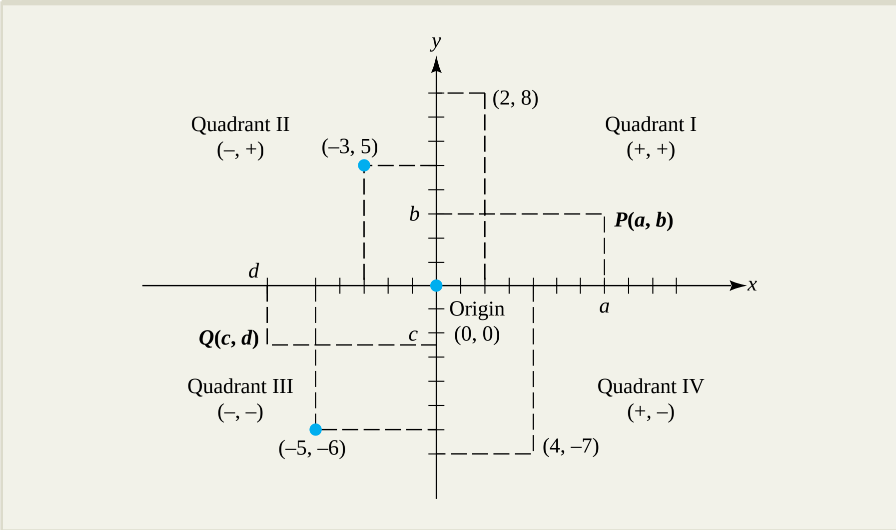


FIGURE 1.4 A rectangular coordinate system.

Any point P in the plane can be associated with a unique ordered pair of numbers (a, b) called the **coordinates** of P. Specifically, a is called the ***x* coordinate** (or **abscissa**) and b is called the ***y* coordinate** (or **ordinate**). To find a and b, draw the vertical and horizontal lines through P. The vertical line intersects the x axis at a, and the horizontal line intersects the y axis at b. Conversely, if c and d are given, the vertical line through c and horizontal line through d intersect at the unique point Q with coordinates (c, d).

Several points are plotted in Figure 1.4. In particular, note that the point (2, 8) is 2 units to the right of the vertical axis and 8 units above the horizontal axis, while $(-3, 5)$ is 3 units to the left of the vertical axis and 5 units above the horizontal axis. Each point P has unique coordinates (a, b), and conversely each ordered pair of numbers (c, d) uniquely determines a point in the plane.

The Distance Formula

There is a simple formula for finding the distance D between two points in a coordinate plane. Figure 1.5 shows the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Note that the difference $x_2 - x_1$ of the x coordinates and the difference $y_2 - y_1$ of the y coordinates represent the lengths of the sides of a right triangle, and the length of the hypotenuse is the required distance D between P and Q. Thus, the Pythagorean theorem gives us the **distance formula** $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. To summarize:

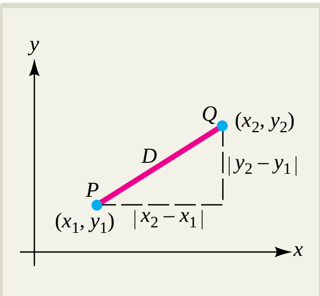


FIGURE 1.5 The distance formula.

The Distance Formula ■ The distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

NOTE The distance formula is valid for all points in the plane even though we have considered only the case in which Q is above and to the right of P. ■

EXAMPLE 1.2.1

Find the distance between the points $P(-2, 5)$ and $Q(4, -1)$.

Solution

In the distance formula, we have $x_1 = -2$, $y_1 = 5$, $x_2 = 4$, and $y_2 = -1$, so the distance between P and Q may be found as follows:

$$D = \sqrt{(4 - (-2))^2 + (-1 - 5)^2} = \sqrt{72} = 6\sqrt{2}$$

The Graph of a Function

To represent a function $y = f(x)$ geometrically as a graph, we plot values of the independent variable x on the (horizontal) x axis and values of the dependent variable y on the (vertical) y axis. The graph of the function is defined as follows.

The Graph of a Function ■ The graph of a function f consists of all points (x, y) where x is in the domain of f and $y = f(x)$; that is, all points of the form $(x, f(x))$.

In Chapter 3, you will study efficient techniques involving calculus that can be used to draw accurate graphs of functions. For many functions, however, you can make a fairly good sketch by plotting a few points, as illustrated in Example 1.2.2.

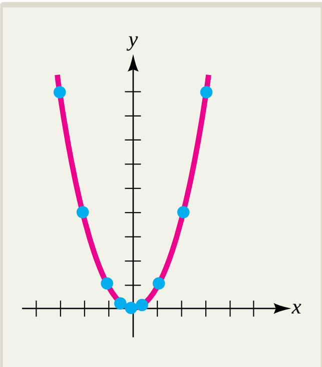


FIGURE 1.6 The graph of $y = x^2$.

EXAMPLE 1.2.2

Graph the function $f(x) = x^2$.

Solution

Begin by constructing the table

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
$y = x^2$	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9

Then plot the points (x, y) and connect them with the smooth curve shown in Figure 1.6.

NOTE Many different curves pass through the points in Example 1.2.2. Several of these curves are shown in Figure 1.7. There is no way to guarantee that the curve we pass through the plotted points is the actual graph of f. However, in general, the more points that are plotted, the more likely the graph is to be reasonably accurate. ■

EXPLORE!

Store $f(x) = x^2$ into Y1 of the equation editor, using a bold graphing style. Represent $g(x) = x^2 + 2$ by Y2 = Y1 + 2 and $h(x) = x^2 - 3$ by Y3 = Y1 − 3. Use **ZOOM** decimal graphing to show how the graphs of $g(x)$ and $h(x)$ relate to that of $f(x)$. Now deselect Y2 and Y3 and write Y4 = Y1(X + 2) and Y5 = Y1(X − 3). Explain how the graphs of Y1, Y4, and Y5 relate.

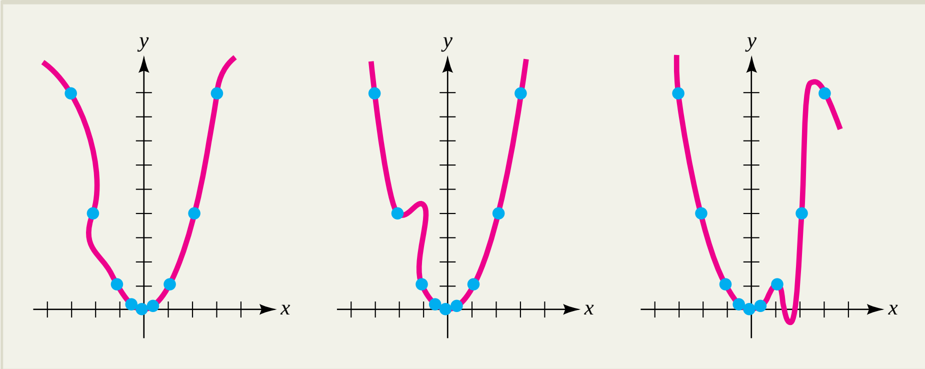


FIGURE 1.7 Other graphs through the points in Example 1.2.2.

Example 1.2.3 illustrates how to sketch the graph of a function defined by more than one formula.

EXPLORE!

Certain functions that are defined piecewise can be entered into a graphing calculator using indicator functions in sections. For example, the absolute value function,

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

can be represented by Y1 = X(X ≥ 0) + (−X)(X < 0). Now represent the function in Example 1.2.3, using indicator functions and graph it with an appropriate viewing window. [*Hint:* You will need to represent the interval, 0 < X < 1, by the boolean expression, (0 < X)(X < 1).]

EXAMPLE 1.2.3

Graph the function

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1 \\ \dfrac{2}{x} & \text{if } 1 \leq x < 4 \\ 3 & \text{if } x \geq 4 \end{cases}$$

Solution

When making a table of values for this function, remember to use the formula that is appropriate for each particular value of x. Using the formula $f(x) = 2x$ when $0 \leq x < 1$, the formula $f(x) = \dfrac{2}{x}$ when $1 \leq x < 4$, and the formula $f(x) = 3$ when $x \geq 4$, you can compile this table:

x	0	$\frac{1}{2}$	1	2	3	4	5	6
$f(x)$	0	1	2	1	$\frac{2}{3}$	3	3	3

Now plot the corresponding points $(x, f(x))$ and draw the graph as in Figure 1.8. Notice that the pieces for $0 \leq x < 1$ and $1 \leq x < 4$ are connected to one another at (1, 2) but that the piece for $x \geq 4$ is separated from the rest of the graph. [The "open dot" at $\left(4, \dfrac{1}{2}\right)$ indicates that the graph approaches this point but that the point is not actually on the graph.]

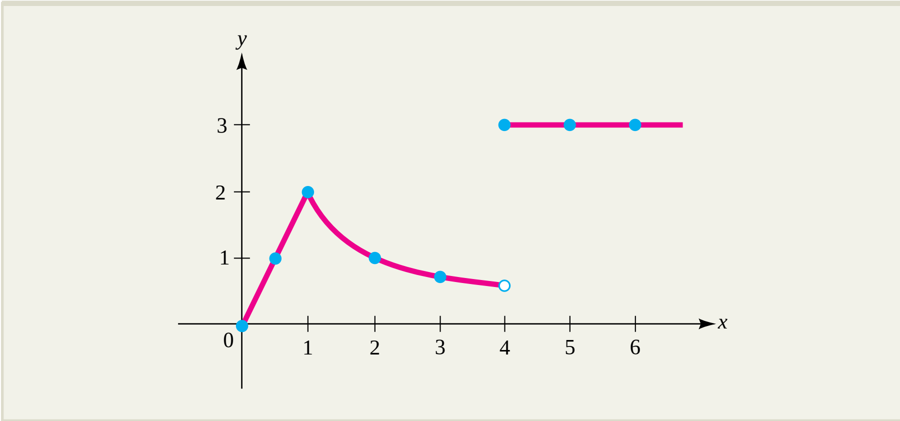


FIGURE 1.8 The graph of $f(x) = \begin{cases} 2x & 0 \le x < 1 \\ \dfrac{2}{x} & 1 \le x < 4 \\ 3 & x \ge 4 \end{cases}$

Intercepts

The points (if any) where a graph crosses the x axis are called ***x* intercepts,** and similarly, a ***y* intercept** is a point where the graph crosses the y axis. Intercepts are key features of a graph and can be determined using algebra or technology in conjunction with these criteria.

EXPLORE!

Using your graphing utility, locate the x intercepts of $f(x) = -x^2 + x + 2$. These intercepts can be located by first using the **ZOOM** button and then confirmed by using the root finding feature of the graphing utility. Do the same for $g(x) = x^2 + x - 4$. What radical form do these roots have?

How to Find the *x* and *y* Intercepts ■ To find any x intercept of a graph, set $y = 0$ and solve for x. To find any y intercept, set $x = 0$ and solve for y. For a function f, the only y intercept is $y_0 = f(0)$, but finding x intercepts may be difficult.

EXAMPLE 1.2.4

Graph the function $f(x) = -x^2 + x + 2$. Include all x and y intercepts.

Solution

The y intercept is $f(0) = 2$. To find the x intercepts, solve the equation $f(x) = 0$. Factoring, we find that

$$\begin{aligned} -x^2 + x + 2 &= 0 && \text{factor} \\ -(x+1)(x-2) &= 0 && uv = 0 \text{ if and only if } u = 0 \text{ or} \\ x = -1,\ x &= 2 && v = 0 \end{aligned}$$

Thus, the x intercepts are $(-1, 0)$ and $(2, 0)$.

Next, make a table of values and plot the corresponding points $(x, f(x))$.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	-10	-4	0	2	2	0	-4	-10

The graph of f is shown in Figure 1.9.

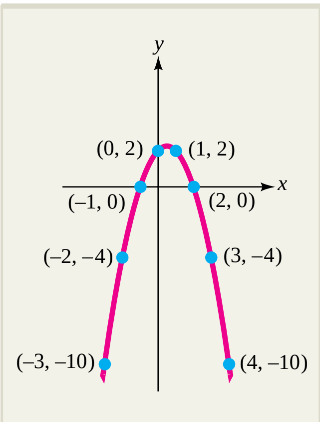


FIGURE 1.9 The graph of $f(x) = -x^2 + x + 2$.

NOTE The factoring in Example 1.2.4 is fairly straightforward, but in other problems, you may need to review the factoring procedure provided in Appendix A2. ■

Graphing Parabolas

The graphs in Figures 1.6 and 1.9 are called **parabolas.** In general, the graph of $y = Ax^2 + Bx + C$ is a parabola as long as $A \neq 0$. All parabolas have a "U shape," and the parabola $y = Ax^2 + Bx + C$ opens up if $A > 0$ and down if $A < 0$. The "peak" or "valley" of the parabola is called its **vertex** and occurs where $x = \dfrac{-B}{2A}$ (Figure 1.10; also see Exercise 72). These features of the parabola are easily obtained by the methods of calculus developed in Chapter 3. Note that to get a reasonable sketch of the parabola $y = Ax^2 + Bx + C$, you need only determine three key features:

1. The location of the vertex $\left(\text{where } x = \dfrac{-B}{2A}\right)$
2. Whether the parabola opens up $(A > 0)$ or down $(A < 0)$
3. Any intercepts

For instance, in Figure 1.9, the parabola $y = -x^2 + x + 2$ opens downward (since $A = -1$ is negative) and has its vertex (high point) where $x = \dfrac{-B}{2A} = \dfrac{-1}{2(-1)} = \dfrac{1}{2}$.

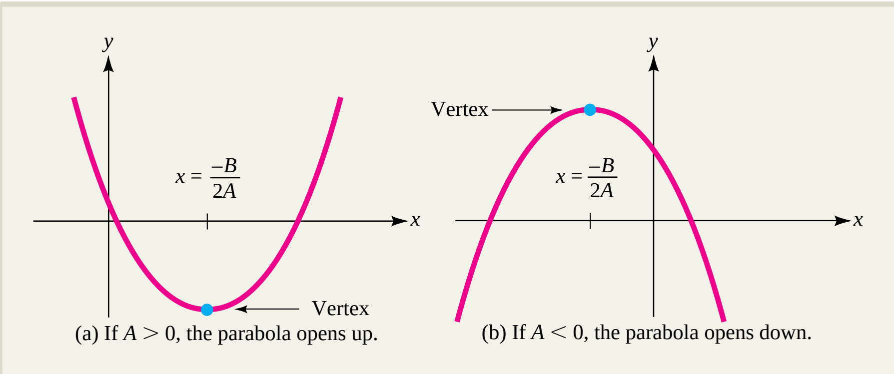


FIGURE 1.10 The graph of the parabola $y = Ax^2 + Bx + C$.

In Chapter 3, we will develop a procedure in which the graph of a function of practical interest is first obtained by calculus and then interpreted to obtain useful information about the function, such as its largest and smallest values. In Example 1.2.5 we preview this procedure by using what we know about the graph of a parabola to determine the maximum revenue obtained in a production process.

EXAMPLE 1.2.5

A manufacturer determines that when x hundred units of a particular commodity are produced, they can all be sold for a unit price given by the demand function $p = 60 - x$ dollars. At what level of production is revenue maximized? What is the maximum revenue?

Solution

The revenue derived from producing x hundred units and selling them all at $60 - x$ dollars is $R(x) = x(60 - x)$ hundred dollars. Note that $R(x) \geq 0$ only for $0 \leq x \leq 60$. The graph of the revenue function

$$R(x) = x(60 - x) = -x^2 + 60x$$

is a parabola that opens downward (since $A = -1 < 0$) and has its high point (vertex) where

$$x = \frac{-B}{2A} = \frac{-60}{2(-1)} = 30$$

as shown in Figure 1.11. Thus, revenue is maximized when $x = 30$ hundred units are produced, and the corresponding maximum revenue is

$$R(30) = 30(60 - 30) = 900$$

hundred dollars. The manufacturer should produce 3,000 units and at that level of production should expect maximum revenue of $90,000.

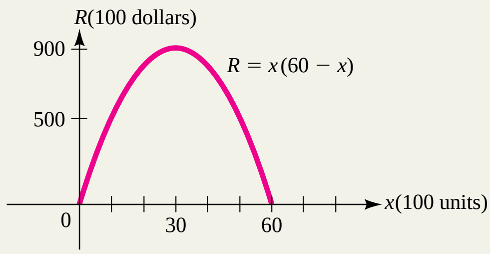


FIGURE 1.11 A revenue function.

Just-In-Time REVIEW

Completing the square is reviewed in Appendix A2 and illustrated in Examples A.2.12 and A.2.13.

Note that we can also find the largest value of $R(x) = -x^2 + 60x$ by completing the square:

$$\begin{aligned} R(x) &= -x^2 + 60x = -(x^2 - 60x) && \text{factor out } -1\text{, the coefficient of } x \\ &= -(x^2 - 60x + 900) + 900 && \text{complete the square inside parentheses by adding } (-60/2)^2 = 900 \\ & \qquad -900 + 900 && \\ &= -(x - 30)^2 + 900 \end{aligned}$$

Thus, $R(30) = 0 + 900 = 900$ and if c is any number other than 30, then

$$R(c) = -(c - 30)^2 + 900 < 900 \qquad \text{since } -(c - 30)^2 < 0$$

so the maximum revenue is $90,000 when $x = 30$ (3,000 units).

Intersections of Graphs

Sometimes it is necessary to determine when two functions are equal. For instance, an economist may wish to compute the market price at which the consumer demand for a commodity will be equal to supply. Or a political analyst may wish to predict how long it will take for the popularity of a certain challenger to reach that of the incumbent. We shall examine some of these applications in Section 1.4.

In geometric terms, the values of x for which two functions $f(x)$ and $g(x)$ are equal are the x coordinates of the points where their graphs intersect. In Figure 1.12, the graph of $y = f(x)$ intersects that of $y = g(x)$ at two points, labeled P and Q. To find the points of intersection algebraically, set $f(x)$ equal to $g(x)$ and solve for x. This procedure is illustrated in Example 1.2.6.

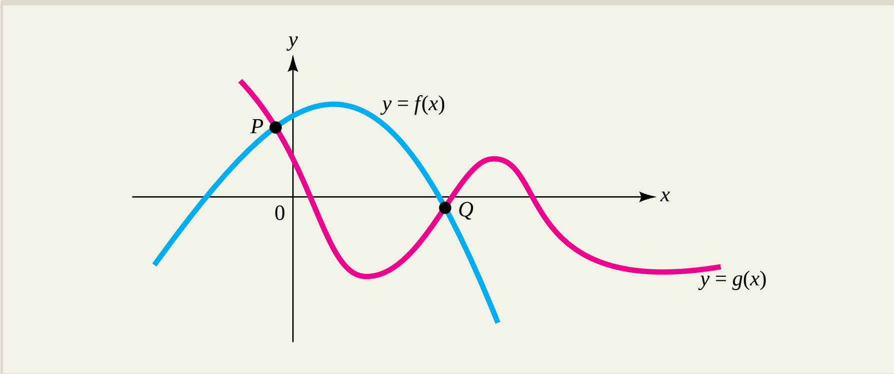


FIGURE 1.12 The graphs of $y = f(x)$ and $y = g(x)$ intersect at P and Q.

Just-In-Time REVIEW

The **quadratic formula** is used in Example 1.2.6. Recall that this result says that the equation $Ax^2 + Bx + C = 0$ has real solutions if and only if $B^2 - 4AC \geq 0$, in which case, the solutions are

$$r_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

and

$$r_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

A review of the quadratic formula may be found in Appendix A2.

EXPLORE!

Refer to Example 1.2.6. Use your graphing utility to find all points of intersection of the graphs of $f(x) = 3x + 2$ and $g(x) = x^2$. Also find the roots of $g(x) - f(x) = x^2 - 3x - 2$. What can you conclude?

EXAMPLE 1.2.6

Find all points of intersection of the graphs of $f(x) = 3x + 2$ and $g(x) = x^2$.

Solution

You must solve the equation $x^2 = 3x + 2$. Rewrite the equation as $x^2 - 3x - 2 = 0$ and apply the quadratic formula to obtain

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)} = \frac{3 \pm \sqrt{17}}{2}$$

The solutions are

$$x = \frac{3 + \sqrt{17}}{2} \approx 3.56 \qquad \text{and} \qquad x = \frac{3 - \sqrt{17}}{2} \approx -0.56$$

(The computations were done on a calculator, with results rounded off to two decimal places.)

Computing the corresponding y coordinates from the equation $y = x^2$, you find that the points of intersection are approximately (3.56, 12.67) and (−0.56, 0.31). [As a result of round-off errors, you will get slightly different values for the y coordinates if you substitute into the equation $y = 3x + 2$.] The graphs and the intersection points are shown in Figure 1.13.

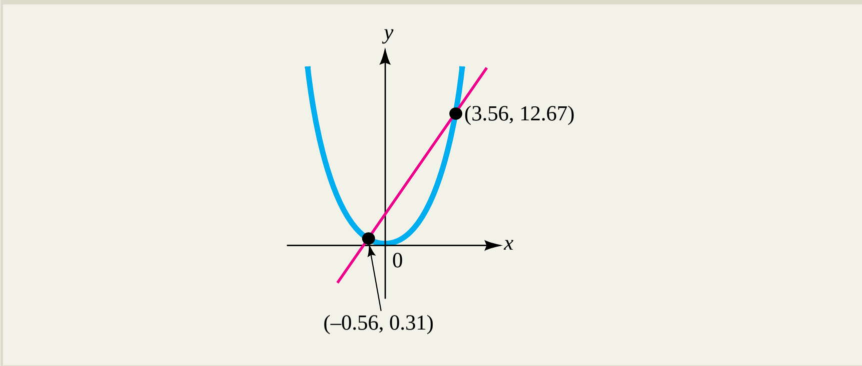


FIGURE 1.13 The intersection of the graphs of $f(x) = 3x + 2$ and $g(x) = x^2$.

Power Functions, Polynomials, and Rational Functions

A **power function** is a function of the form $f(x) = x^n$, where n is a real number. For example, $f(x) = x^2$, $f(x) = x^{-3}$, and $f(x) = x^{1/2}$ are all power functions. So are $f(x) = \dfrac{1}{x^2}$ and $f(x) = \sqrt[3]{x}$ since they can be rewritten as $f(x) = x^{-2}$ and $f(x) = x^{1/3}$, respectively.

EXPLORE!

Use your calculator to graph the third-degree polynomial $f(x) = x^3 - x^2 - 6x + 3$. Conjecture the values of the x intercepts and confirm them using the root finding feature of your calculator.

A **polynomial** is a function of the form

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are constants. If $a_n \neq 0$, the integer n is called the **degree** of the polynomial. For example, $f(x) = 3x^5 - 6x^2 + 7$ is a polynomial of degree 5. It can be shown that the graph of a polynomial of degree n is an unbroken curve that crosses the x axis no more than n times. To illustrate some of the possibilities, the graphs of three polynomials of degree 3 are shown in Figure 1.14.

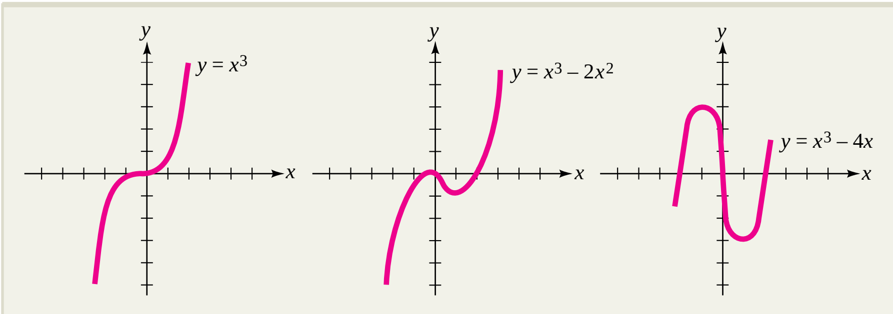


FIGURE 1.14 Three polynomials of degree 3.

A quotient $\dfrac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ is called a **rational function.** Such functions appear throughout this text in examples and exercises. Graphs of three rational functions are shown in Figure 1.15. You will learn how to sketch such graphs in Section 3.3 of Chapter 3.

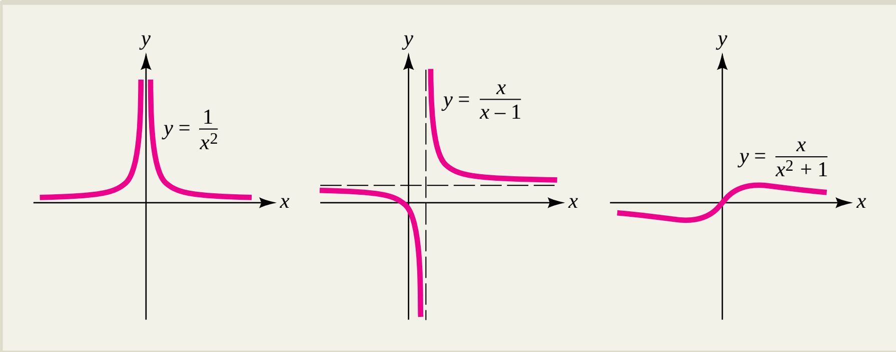


FIGURE 1.15 Graphs of three rational functions.

The Vertical Line Test

It is important to realize that not every curve is the graph of a function (Figure 1.16). For instance, suppose the circle $x^2 + y^2 = 5$ were the graph of some function $y = f(x)$. Then, since the points (1, 2) and (1, −2) both lie on the circle, we would have $f(1) = 2$ and $f(1) = -2$, contrary to the requirement that a function assigns one and *only* one value to each number in its domain. The **vertical line test** is a geometric rule for determining whether a curve is the graph of a function.

> **The Vertical Line Test** ■ A curve is the graph of a function if and only if no vertical line intersects the curve more than once.

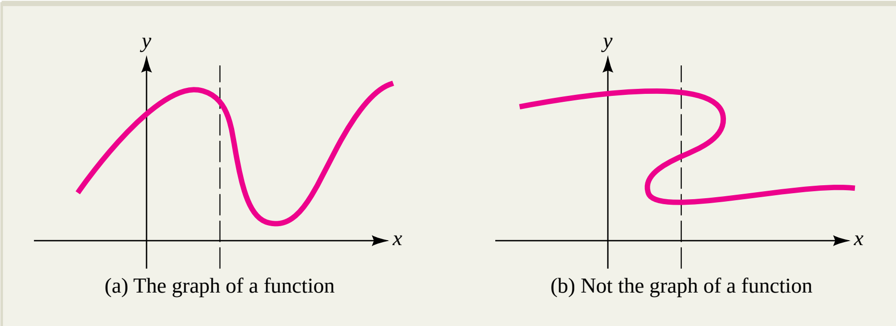


FIGURE 1.16 The vertical line test.

EXERCISES ■ 1.2

In Exercises 1 through 6, plot the given points in a rectangular coordinate plane.

1. (4, 3)
2. (−2, 7)
3. (5, −1)
4. (−1, −8)
5. (0, −2)
6. (3, 0)

In Exercises 7 through 10, find the distance between the given points P and Q.

7. $P(3, -1)$ and $Q(7, 1)$

8. $P(4, 5)$ and $Q(-2, -1)$

9. $P(7, -3)$ and $Q(5, 3)$

10. $P\left(0, \frac{1}{2}\right)$ and $Q\left(\frac{-1}{5}, \frac{3}{8}\right)$

In Exercises 11 and 12, classify each function as a polynomial, a power function, or a rational function. If the function is not one of these types, classify it as "different."

11. **a.** $f(x) = x^{1.4}$
b. $f(x) = -2x^3 - 3x^2 + 8$
c. $f(x) = (3x - 5)(4 - x)^2$
d. $f(x) = \dfrac{3x^2 - x + 1}{4x + 7}$

12. **a.** $f(x) = -2 + 3x^2 + 5x^4$
b. $f(x) = \sqrt{x} + 3x$
c. $f(x) = \dfrac{(x - 3)(x + 7)}{-5x^3 - 2x^2 + 3}$
d. $f(x) = \left(\dfrac{2x + 9}{x^2 - 3}\right)^3$

In Exercises 13 through 28, sketch the graph of the given function. Include all x and y intercepts.

13. $f(x) = x$

14. $f(x) = x^2$

15. $f(x) = \sqrt{x}$

16. $f(x) = \sqrt{1 - x}$

17. $f(x) = 2x - 1$

18. $f(x) = 2 - 3x$

19. $f(x) = x(2x + 5)$

20. $f(x) = (x - 1)(x + 2)$

21. $f(x) = -x^2 - 2x + 15$

22. $f(x) = x^2 + 2x - 8$

23. $f(x) = x^3$

24. $f(x) = -x^3 + 1$

25. $f(x) = \begin{cases} x - 1 & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

26. $f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$

27. $f(x) = \begin{cases} x^2 + x - 3 & \text{if } x < 1 \\ 1 - 2x & \text{if } x \ge 1 \end{cases}$

28. $f(x) = \begin{cases} 9 - x & \text{if } x \le 2 \\ x^2 + x - 2 & \text{if } x > 2 \end{cases}$

In Exercises 29 through 34, find the points of intersection (if any) of the given pair of curves and draw the graphs.

29. $y = 3x + 5$ and $y = -x + 3$

30. $y = 3x + 8$ and $y = 3x - 2$

31. $y = x^2$ and $y = 3x - 2$

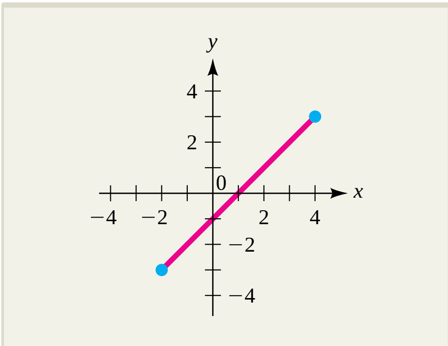
32. $y = x^2 - x$ and $y = x - 1$

33. $3y - 2x = 5$ and $y + 3x = 9$

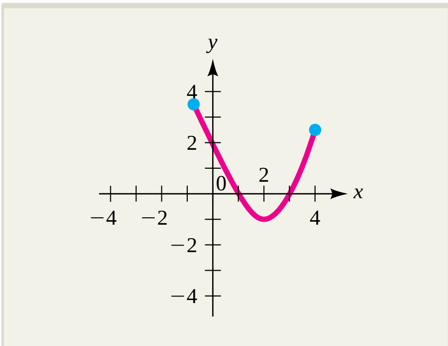
34. $2x - 3y = -8$ and $3x - 5y = -13$

In Exercises 35 through 38, the graph of a function f(x) is given. In each case find:
(a) The y intercept.
(b) All x intercepts.
(c) The largest value of f(x) and the value(s) of x for which it occurs.
(d) The smallest value of f(x) and the value(s) of x for which it occurs.

35.

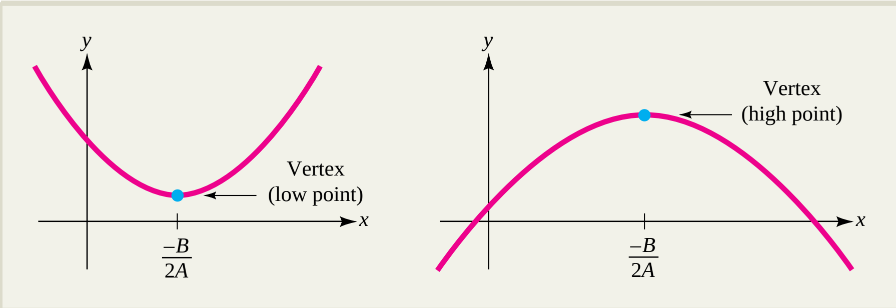


36.



72. Show that the vertex of the parabola $y = Ax^2 + Bx + C$ ($A \neq 0$) occurs at the point where $x = \dfrac{-B}{2A}$. [*Hint:* First verify that

$$Ax^2 + Bx + C = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right].$$ Then note that the largest or smallest value of $f(x) = Ax^2 + Bx + C$ must occur where $x = \dfrac{-B}{2A}$.]



EXERCISE 72

SECTION 1.3 Linear Functions

In many practical situations, the rate at which one quantity changes with respect to another is constant. Here is a simple example from economics.

EXPLORE!

Input the cost function Y1 = 50*x* + {200, 300, 400} into the equation editor, using braces to list various overhead costs. Set the WINDOW dimensions to [0, 5]1 by [−100, 700]100 to view the effect of varying the overhead values.

EXAMPLE 1.3.1

A manufacturer's total cost consists of a fixed overhead of \$200 plus production costs of \$50 per unit. Express the total cost as a function of the number of units produced and draw the graph.

Solution

Let x denote the number of units produced and $C(x)$ the corresponding total cost. Then,

$$\text{Total cost} = (\text{cost per unit})(\text{number of units}) + \text{overhead}$$

where
$$\text{Cost per unit} = 50$$
$$\text{Number of units} = x$$
$$\text{Overhead} = 200$$

Hence,

$$C(x) = 50x + 200$$

The graph of this cost function is sketched in Figure 1.17.

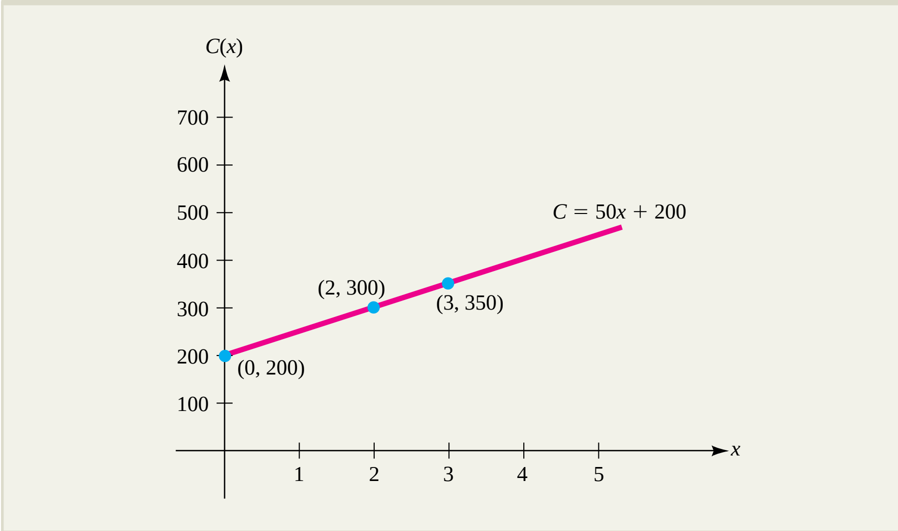


FIGURE 1.17 The cost function, $C(x) = 50x + 200$.

The total cost in Example 1.3.1 increases at a constant rate of \$50 per unit. As a result, its graph in Figure 1.17 is a straight line that increases in height by 50 units for each 1-unit increase in x.

In general, a function whose value changes at a constant rate with respect to its independent variable is said to be a **linear function.** Such a function has the form

$$f(x) = mx + b$$

where m and b are constants, and its graph is a straight line. For example, $f(x) = \frac{3}{2} + 2x$, $f(x) = -5x$, and $f(x) = 12$ are all linear functions. To summarize:

Linear Functions ■ A linear function is a function that changes at a constant rate with respect to its independent variable.

The graph of a linear function is a straight line.

The equation of a linear function can be written in the form

$$y = mx + b$$

where m and b are constants.

The Slope of a Line

A surveyor might say that a hill with a "rise" of 2 feet for every foot of "run" has a **slope** of

$$m = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$$

The steepness of a line can be measured by slope in much the same way. In particular, suppose (x_1, y_1) and (x_2, y_2) lie on a line as indicated in Figure 1.18. Between these points, x changes by the amount $x_2 - x_1$ and y by the amount $y_2 - y_1$. The slope is the ratio

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

It is sometimes convenient to use the symbol Δy instead of $y_2 - y_1$ to denote the change in y. The symbol Δy is read "delta y." Similarly, the symbol Δx is used to denote the change $x_2 - x_1$.

The Slope of a Line ■ The slope of the nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by the formula

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

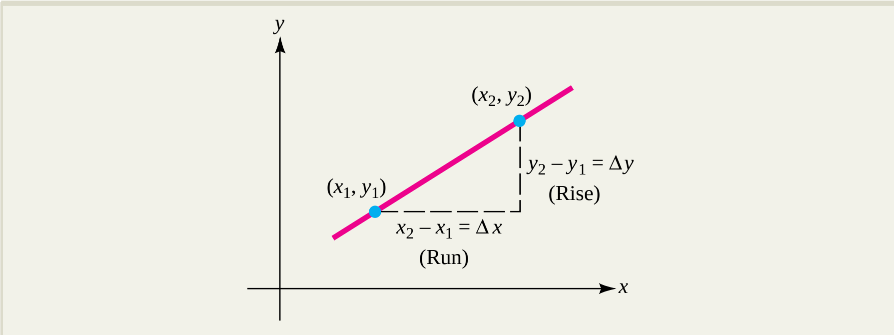


FIGURE 1.18 Slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$.

The use of this formula is illustrated in Example 1.3.2.

EXAMPLE 1.3.2

Find the slope of the line joining the points $(-2, 5)$ and $(3, -1)$.

Solution

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-1 - 5}{3 - (-2)} = \frac{-6}{5}$$

The line is shown in Figure 1.19.

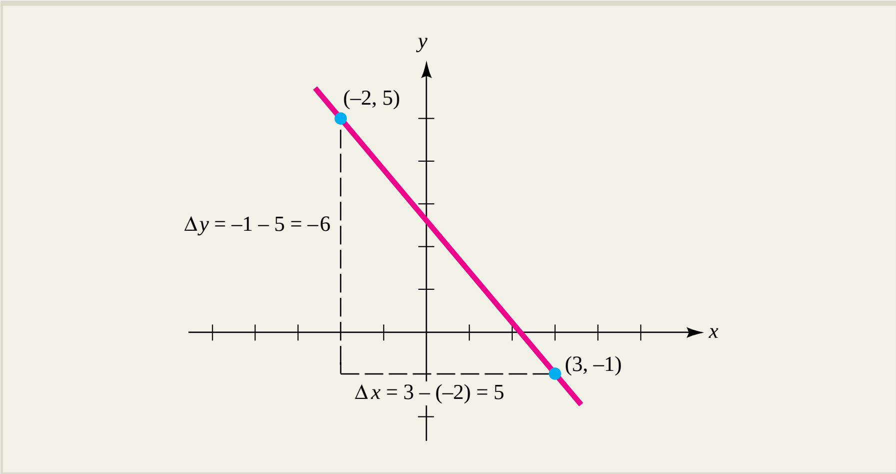


FIGURE 1.19 The line joining (−2, 5) and (3, −1).

The sign and magnitude of the slope of a line indicate the line's direction and steepness, respectively. The slope is positive if the height of the line increases as x increases and is negative if the height decreases as x increases. The absolute value of the slope is large if the slant of the line is severe and small if the slant of the line is gradual. The situation is illustrated in Figure 1.20.

EXPLORE!

Store the varying slope values {2, 1, 0.5, −0.5, −1, −2} into List 1, using the **STAT** menu and the **EDIT** option. Display a family of straight lines through the origin, similar to Figure 1.20, by placing Y1 = L1∗X into your calculator's equation editor. Graph using a **ZOOM** Decimal Window and **TRACE** the values for the different lines at X = 1.

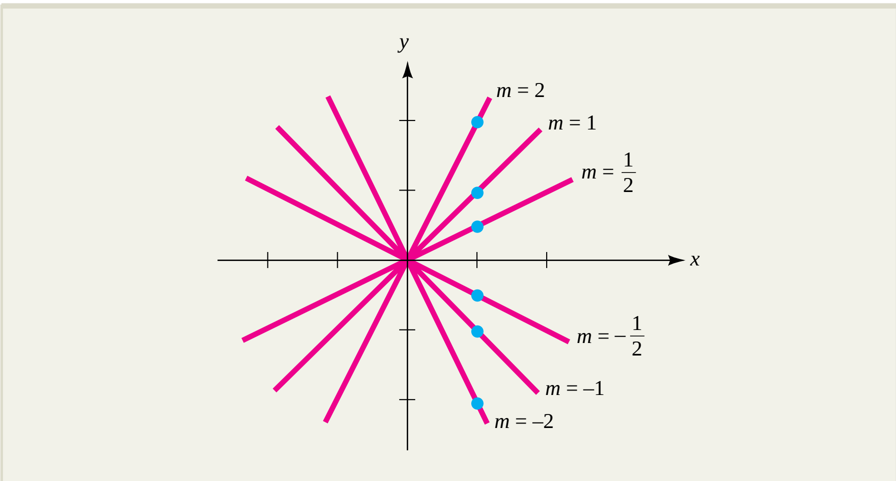


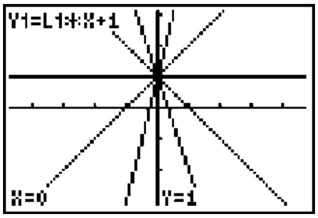
FIGURE 1.20 The direction and steepness of a line.

Horizontal and Vertical Lines

Horizontal and vertical lines (Figures 1.21a and 1.21b) have particularly simple equations. The y coordinates of all points on a horizontal line are the same. Hence, a horizontal line is the graph of a linear function of the form $y = b$, where b is a constant. The slope of a horizontal line is zero, since changes in x produce no changes in y.

The x coordinates of all points on a vertical line are equal. Hence, vertical lines are characterized by equations of the form $x = c$, where c is a constant. The slope of a vertical line is undefined because only the y coordinates of points on the line can change, so the denominator of the quotient $\dfrac{\text{change in } y}{\text{change in } x}$ is zero.

EXPLORE!



Determine what three slope values must be placed into List 1 so that Y1 = L1*X + 1 creates the screen pictured here.

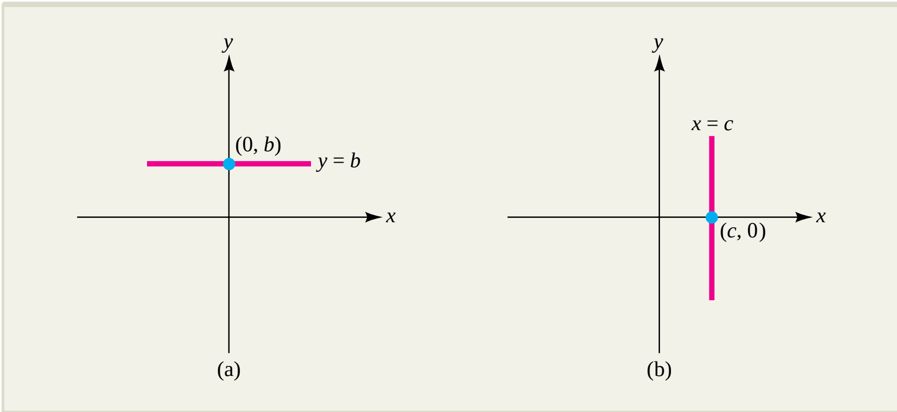


FIGURE 1.21 Horizontal and vertical lines.

The Slope-Intercept Form of the Equation of a Line

The constants m and b in the equation $y = mx + b$ of a nonvertical line have geometric interpretations. The coefficient m is the slope of the line. To see this, suppose that (x_1, y_1) and (x_2, y_2) are two points on the line $y = mx + b$. Then, $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, and so

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1}$$
$$= \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

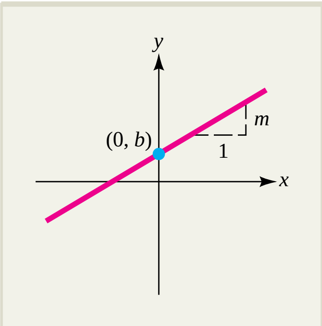


FIGURE 1.22 The slope and y intercept of the line $y = mx + b$.

The constant b in the equation $y = mx + b$ is the value of y corresponding to $x = 0$. Hence, b is the height at which the line $y = mx + b$ crosses the y axis, and the corresponding point $(0, b)$ is the y intercept of the line. The situation is illustrated in Figure 1.22.

Because the constants m and b in the equation $y = mx + b$ correspond to the slope and y intercept, respectively, this form of the equation of a line is known as the **slope-intercept form.**

> **The Slope-Intercept Form of the Equation of a Line** ■ The equation
>
> $$y = mx + b$$
>
> is the equation of the line whose slope is m and whose y intercept is $(0, b)$.

The slope-intercept form of the equation of a line is particularly useful when geometric information about a line (such as its slope or y intercept) is to be determined from the line's algebraic representation. Here is a typical example.

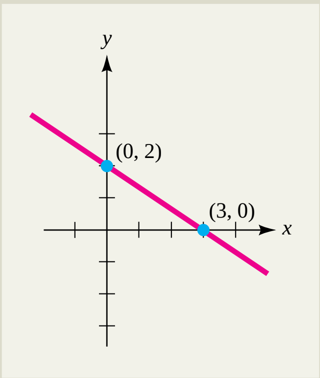


FIGURE 1.23 The line $3y + 2x = 6$.

EXAMPLE 1.3.3

Find the slope and y intercept of the line $3y + 2x = 6$ and draw the graph.

Solution

First put the equation $3y + 2x = 6$ in slope-intercept form $y = mx + b$. To do this, solve for y to get

$$3y = -2x + 6 \qquad \text{or} \qquad y = -\frac{2}{3}x + 2$$

It follows that the slope is $-\frac{2}{3}$ and the y intercept is $(0, 2)$.

To graph a linear function, plot two of its points and draw a straight line through them. In this case, you already know one point, the y intercept $(0, 2)$. A convenient choice for the x coordinate of the second point is $x = 3$, since the corresponding y coordinate is $y = -\frac{2}{3}(3) + 2 = 0$. Draw a line through the points $(0, 2)$ and $(3, 0)$ to obtain the graph shown in Figure 1.23.

The Point-Slope Form of the Equation of a Line

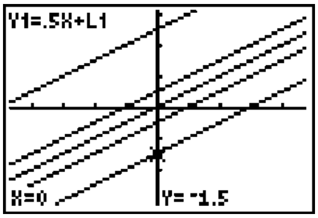
Geometric information about a line can be obtained readily from the slope-intercept formula $y = mx + b$. There is another form of the equation of a line, however, that is usually more efficient for problems in which the geometric properties of a line are known and the goal is to find the equation of the line.

The Point-Slope Form of the Equation of a Line ■ The equation

$$y - y_0 = m(x - x_0)$$

is an equation of the line that passes through the point (x_0, y_0) and that has slope equal to m.

EXPLORE!



Find the y intercept values needed in List L1 so that the function Y1 = 0.5X + L1 creates the screen shown here.

The point-slope form of the equation of a line is simply the formula for slope in disguise. To see this, suppose the point (x, y) lies on the line that passes through a given point (x_0, y_0) and has slope m. Using the points (x, y) and (x_0, y_0) to compute the slope, you get

$$\frac{y - y_0}{x - x_0} = m$$

which you can put in point-slope form

$$y - y_0 = m(x - x_0)$$

by simply multiplying both sides by $x - x_0$.

The use of the point-slope form of the equation of a line is illustrated in Examples 1.3.4 and 1.3.5.

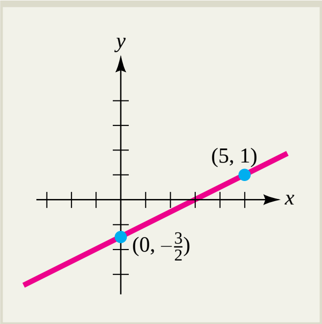


FIGURE 1.24 The line $y = \frac{1}{2}x - \frac{3}{2}$.

EXAMPLE 1.3.4

Find the equation of the line that passes through the point (5, 1) with slope $\frac{1}{2}$.

Solution

Use the formula $y - y_0 = m(x - x_0)$ with $(x_0, y_0) = (5, 1)$ and $m = \frac{1}{2}$ to get

$$y - 1 = \frac{1}{2}(x - 5)$$

which you can rewrite as

$$y = \frac{1}{2}x - \frac{3}{2}$$

The graph is shown in Figure 1.24.

For practice, solve the problem in Example 1.3.4 using the slope-intercept formula. Notice that the solution based on the point-slope formula is more efficient.

In Chapter 2, the point-slope formula will be used extensively for finding the equation of the tangent line to the graph of a function at a given point. Example 1.3.5 illustrates how the point-slope formula can be used to find the equation of a line through two given points.

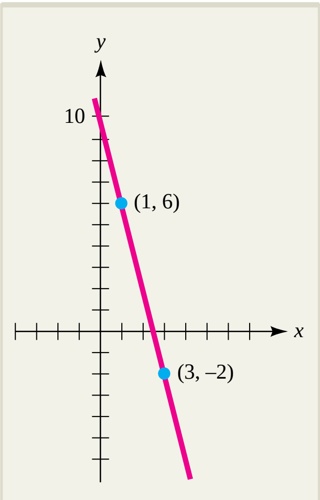


FIGURE 1.25 The line $y = -4x + 10$.

EXAMPLE 1.3.5

Find the equation of the line that passes through the points $(3, -2)$ and $(1, 6)$.

Solution

First compute the slope

$$m = \frac{6 - (-2)}{1 - 3} = \frac{8}{-2} = -4$$

Then use the point-slope formula with (1, 6) as the given point (x_0, y_0) to get

$$y - 6 = -4(x - 1) \qquad \text{or} \qquad y = -4x + 10$$

Convince yourself that the resulting equation would have been the same if you had chosen $(3, -2)$ to be the given point (x_0, y_0). The graph is shown in Figure 1.25.

NOTE The general form for the equation of a line is $Ax + By + C = 0$, where A, B, C are constants, with A and B not both equal to 0. If $B = 0$, the line is vertical, and when $B \neq 0$, the equation $Ax + By + C = 0$ can be rewritten as

$$y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$$

Comparing this equation with the slope-intercept form $y = mx + b$, we see that the slope of the line is given by $m = -A/B$ and the y intercept by $b = -C/B$. The line is horizontal (slope 0) when $A = 0$. ■

Practical Applications

If the rate of change of one quantity with respect to a second quantity is constant, the function relating the quantities must be linear. The constant rate of change is the slope of the corresponding line. Examples 1.3.6 and 1.3.7 illustrate techniques you can use to find the appropriate linear functions in such situations.

EXAMPLE 1.3.6

Since the beginning of the year, the price of a bottle of soda at a local discount supermarket has been rising at a constant rate of 2 cents per month. By November first, the price had reached \$1.56 per bottle. Express the price of the soda as a function of time and determine the price at the beginning of the year.

Solution

Let x denote the number of months that have elapsed since the first of the year and y the price of a bottle of soda (in cents). Since y changes at a constant rate with respect to x, the function relating y to x must be linear, and its graph is a straight line. Since the price y increases by 2 each time x increases by 1, the slope of the line must be 2. The fact that the price was 156 cents (\$1.56) on November first, 10 months after the first of the year, implies that the line passes through the point (10, 156). To write an equation defining y as a function of x, use the point-slope formula

$$y - y_0 = m(x - x_0)$$

with
$$m = 2,\ x_0 = 10,\ y_0 = 156$$

to get
$$y - 156 = 2(x - 10) \qquad \text{or} \qquad y = 2x + 136$$

The corresponding line is shown in Figure 1.26. Notice that the y intercept is (0, 136), which implies that the price of soda at the beginning of the year was \$1.36 per bottle.

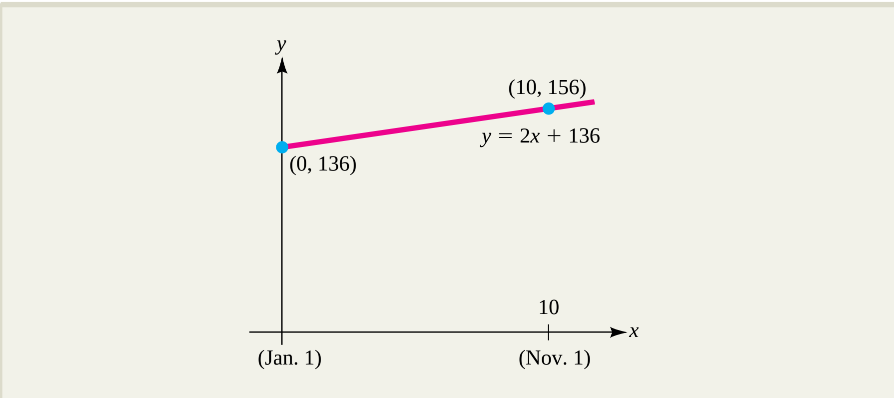


FIGURE 1.26 The rising price of soda: $y = 2x + 136$.

Sometimes it is hard to tell how two quantities, x and y, in a data set are related by simply examining the data. In such cases, it may be useful to graph the data to see if the points (x, y) follow a clear pattern (say, lie along a line). Here is an example of this procedure.

TABLE 1.4 Percentage of Civilian Unemployment 1991–2000

Year	Number of Years from 1991	Percentage of Unemployed
1991	0	6.8
1992	1	7.5
1993	2	6.9
1994	3	6.1
1995	4	5.6
1996	5	5.4
1997	6	4.9
1998	7	4.5
1999	8	4.2
2000	9	4.0

SOURCE: U.S. Bureau of Labor Statistics, Bulletin 2307; and *Employment and Earnings*, monthly.

EXAMPLE 1.3.7

Table 1.4 lists the percentage of the labor force that was unemployed during the decade 1991–2000. Plot a graph with time (years after 1991) on the x axis and percentage of unemployment on the y axis. Do the points follow a clear pattern? Based on these data, what would you expect the percentage of unemployment to be in the year 2005?

Solution

The graph is shown in Figure 1.27. Note that except for the initial point (0, 6.8), the pattern is roughly linear. There is not enough evidence to infer that unemployment is linearly related to time, but the pattern does suggest that we may be able to get useful information by finding a line that "best fits" the data in some meaningful way. One such procedure, called "least-squares approximation," requires the approximating line to be positioned so that the sum of squares of vertical distances from the data points to the line is minimized. The least-squares procedure, which will be developed in Section 7.4 of Chapter 7, can be carried out on your calculator. When this procedure is applied to the unemployment data in this example, it produces the "best-fitting line" $y = -0.389x + 7.338$, as displayed in Figure 1.27. We can then use this formula to attempt a prediction of the unemployment rate in the year 2005 (when $x = 14$):

$$y(14) = -0.389(14) + 7.338 = 1.892$$

Thus, least-squares extrapolation from the given data predicts roughly a 1.9% unemployment rate in 2005.

EXPLORE!

Place the data in Table 1.4 into L1 and L2 of the **STAT** data editor, where L1 is the number of years from 1991 and L2 is the percentage of unemployed. Following the Calculator Introduction for statistical graphing using the **STAT** and **STAT PLOT** keys, verify the scatterplot and best-fit line displayed in Figure 1.27.

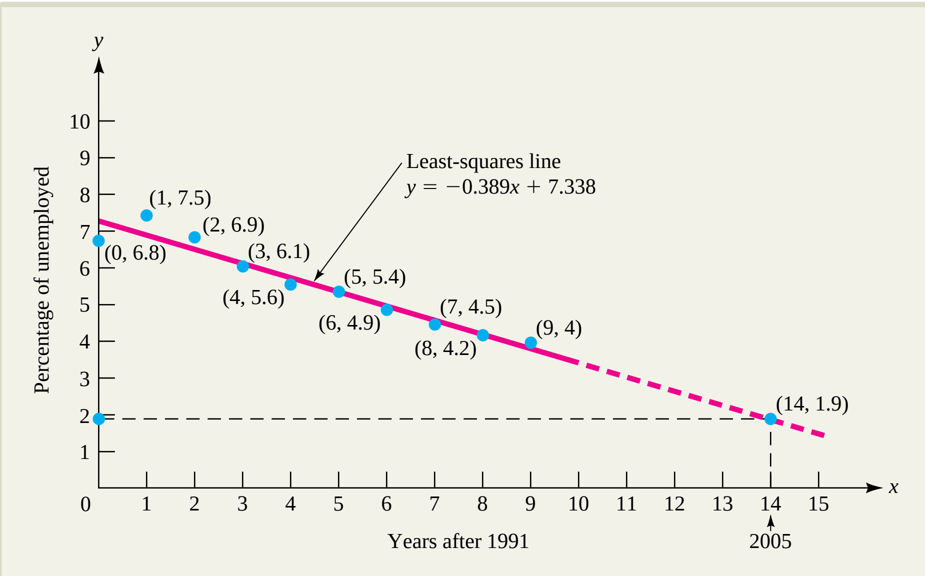


FIGURE 1.27 Percentage of unemployed in the United States for 1991–2000.

NOTE Care must be taken when making predictions by extrapolating from known data, especially when the data set is as small as the one in Example 1.3.7. In particular, the economy began to weaken after the year 2000, but the least-squares line in Figure 1.27 predicts a steadily decreasing unemployment rate. Is this reasonable? In Exercise 63, you are asked to explore this question by first using the Internet to obtain unemployment data for years subsequent to 2000 and then comparing this new data with the values predicted by the least-squares line. ■

Parallel and Perpendicular Lines

In applications, it is sometimes necessary or useful to know whether two given lines are parallel or perpendicular. A vertical line is parallel only to other vertical lines and is perpendicular to any horizontal line. Cases involving nonvertical lines can be handled by the following slope criteria.

Parallel and Perpendicular Lines ■ Let m_1 and m_2 be the slopes of the nonvertical lines L_1 and L_2. Then

L_1 and L_2 are **parallel** if and only if $m_1 = m_2$.

L_1 and L_2 are **perpendicular** if and only if $m_2 = \dfrac{-1}{m_1}$.

These criteria are demonstrated in Figure 1.28a. Geometric proofs are outlined in Exercises 64 and 65. We close this section with an example illustrating one way the criteria can be used.

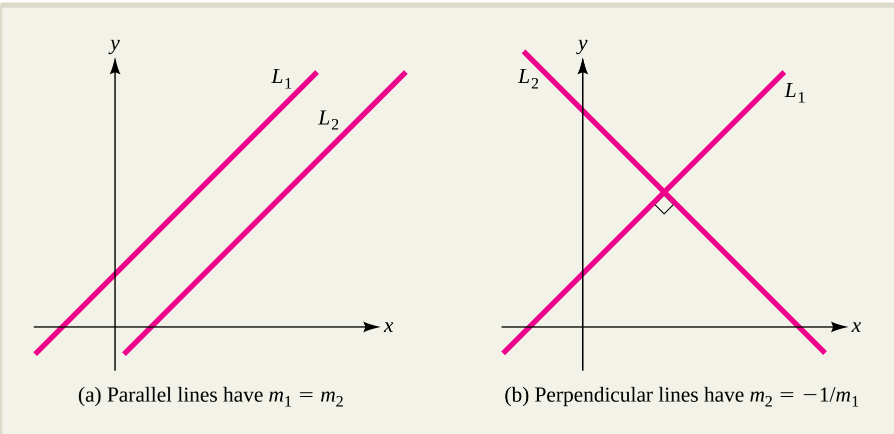


FIGURE 1.28 Slope criteria for parallel and perpendicular lines.

EXAMPLE 1.3.8

Let L be the line $4x + 3y = 3$.

a. Find the equation of a line L_1 parallel to L through $P(-1, 4)$.

b. Find the equation of a line L_2 perpendicular to L through $Q(2, -3)$.

EXPLORE!

Write Y1 = AX + 2 and Y2 = (−1/A)X + 5 in the equation editor of your graphing calculator. On the home screen, store different values into A and then graph both lines using a **ZOOM** Square Window. What do you notice for different values of A (A ≠ 0)? Can you solve for the point of intersection in terms of the value A?

Solution

By rewriting the equation $4x + 3y = 3$ in the slope-intercept form $y = -\frac{4}{3}x + 1$, we see that L has slope $m_L = -\frac{4}{3}$.

a. Any line parallel to L must also have slope $m = -\frac{4}{3}$. The required line L_1 contains $P(-1, 4)$, so

$$y - 4 = -\frac{4}{3}(x + 1)$$
$$y = -\frac{4}{3}x + \frac{8}{3}$$

b. A line perpendicular to L must have slope $m = -\frac{1}{m_L} = \frac{3}{4}$. Since the required line L_2 contains $Q(2, -3)$, we have

$$y + 3 = \frac{3}{4}(x - 2)$$
$$y = \frac{3}{4}x - \frac{9}{2}$$

The given line L and the required lines L_1 and L_2 are shown in Figure 1.29.

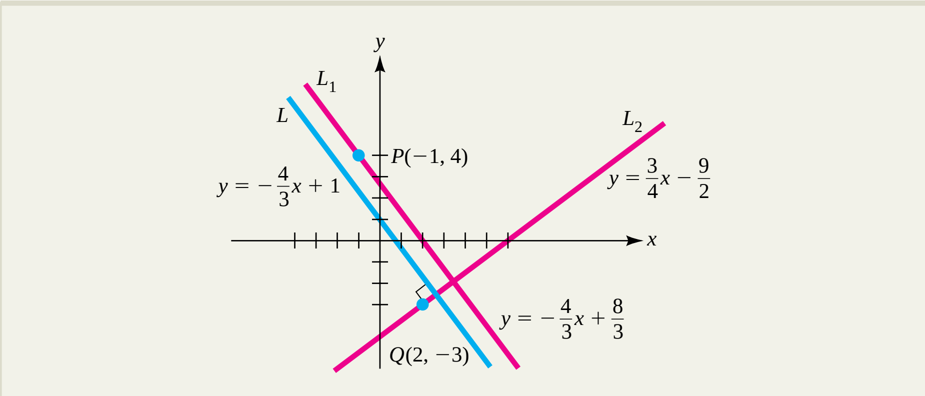


FIGURE 1.29 Lines parallel and perpendicular to a given line L.

EXERCISES 1.3

In Exercises 1 through 8, find the slope (if possible) of the line that passes through the given pair of points.

1. $(2, -3)$ and $(0, 4)$
2. $(-1, 2)$ and $(2, 5)$
3. $(2, 0)$ and $(0, 2)$
4. $(5, -1)$ and $(-2, -1)$
5. $(2, 6)$ and $(2, -4)$

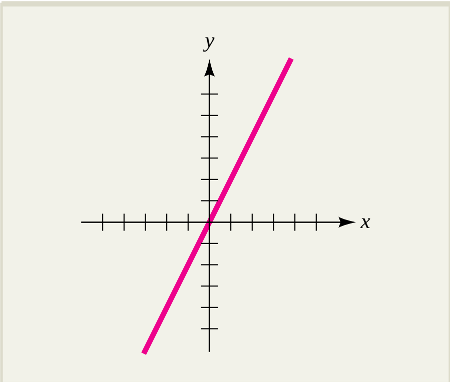
6. $\left(\frac{2}{3}, -\frac{1}{5}\right)$ and $\left(-\frac{1}{7}, \frac{1}{8}\right)$

7. $\left(\frac{1}{7}, 5\right)$ and $\left(-\frac{1}{11}, 5\right)$

8. $(-1.1, 3.5)$ and $(-1.1, -9)$

In Exercises 9 through 12, find the slope and intercepts of the line shown. Then find an equation for the line.

9.



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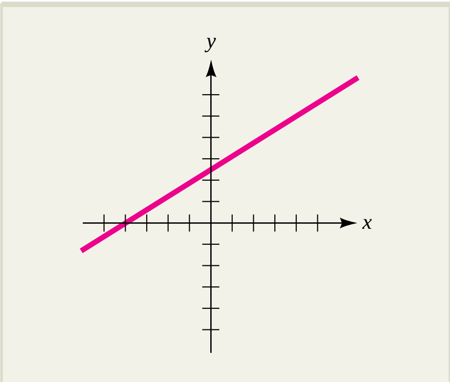

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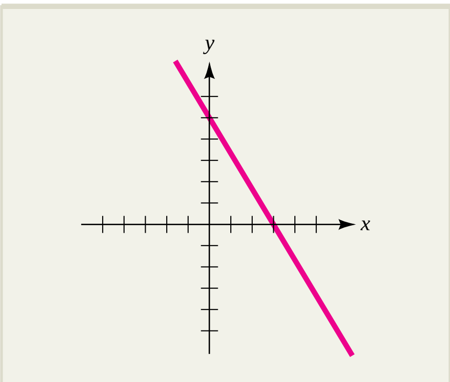

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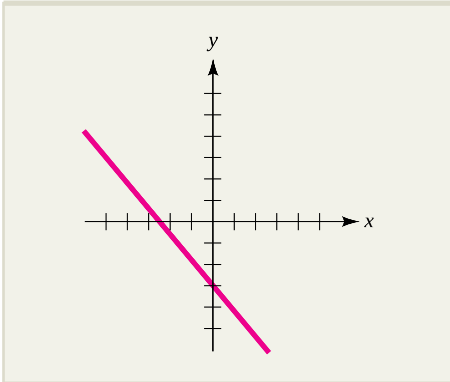


In Exercises 13 through 20, find the slope and intercepts of the line whose equation is given and sketch the graph of the line.

13. $x = 3$

14. $y = 5$

15. $y = 3x$

16. $y = 3x - 6$

17. $3x + 2y = 6$

18. $5y - 3x = 4$

19. $\frac{x}{2} + \frac{y}{5} = 1$

20. $\frac{x + 3}{-5} + \frac{y - 1}{2} = 1$

In Exercises 21 through 36, write an equation for the line with the given properties.

21. Through $(2, 0)$ with slope 1

22. Through $(-1, 2)$ with slope $\frac{2}{3}$

23. Through $(5, -2)$ with slope $-\frac{1}{2}$

24. Through $(0, 0)$ with slope 5

25. Through $(2, 5)$ and parallel to the x axis

26. Through $(2, 5)$ and parallel to the y axis

27. Through $(1, 0)$ and $(0, 1)$

28. Through $(2, 5)$ and $(1, -2)$

29. Through $\left(-\frac{1}{5}, 1\right)$ and $\left(\frac{2}{3}, \frac{1}{4}\right)$

30. Through $(-2, 3)$ and $(0, 5)$

31. Through $(1, 5)$ and $(3, 5)$

32. Through $(1, 5)$ and $(1, -4)$

33. Through $(4, 1)$ and parallel to the line $2x + y = 3$

34. Through $(-2, 3)$ and parallel to the line $x + 3y = 5$

35. Through $(3, 5)$ and perpendicular to the line $x + y = 4$

36. Through $\left(-\frac{1}{2}, 1\right)$ and perpendicular to the line $2x + 5y = 3$

37. MANUFACTURING COST A manufacturer's total cost consists of a fixed overhead of \$5,000 plus production costs of \$60 per unit. Express the total cost as a function of the number of units produced and draw the graph.

38. MANUFACTURING COST A manufacturer estimates that it costs \$75 to produce each unit of a particular commodity. The fixed overhead is \$4,500. Express the total cost of production as a function of the number of units produced and draw the graph.

39. CREDIT CARD DEBT A credit card company estimates that the average cardholder owed \$7,853 in the year 2000 and \$9,127 in 2005. Suppose average cardholder debt D grows at a constant rate.

a. Express D as a linear function of time t, where t is the number of years after 2000. Draw the graph.

b. Use the function in part (a) to predict the average cardholder debt in the year 2010.

c. Approximately when will the average cardholder debt be double the amount in the year 2000?

40. CAR RENTAL A car rental agency charges \$75 per day plus 70 cents per mile.

a. Express the cost of renting a car from this agency for 1 day as a function of the number of miles driven and draw the graph.

b. How much does it cost to rent a car for a l-day trip of 50 miles?

c. The agency also offers a rental for a flat fee of \$125 per day. How many miles must you drive on a l-day trip for this to be the better deal?

41. COURSE REGISTRATION Students at a state college may preregister for their fall classes by mail during the summer. Those who do not preregister must register in person in September. The registrar can process 35 students per hour during the September registration period. Suppose that after 4 hours in September, a total of 360 students (including those who preregistered) have been registered.

a. Express the number of students registered as a function of time and draw the graph.

b. How many students were registered after 3 hours?

c. How many students preregistered during the summer?

42. MEMBERSHIP FEES Membership in a swimming club costs \$250 for the 12-week summer season. If a member joins after the start of the season, the fee is prorated; that is, it is reduced linearly.

a. Express the membership fee as a function of the number of weeks that have elapsed by the time the membership is purchased and draw the graph.

b. Compute the cost of a membership that is purchased 5 weeks after the start of the season.

43. LINEAR DEPRECIATION A doctor owns \$1,500 worth of medical books which, for tax purposes, are assumed to depreciate linearly to zero over a 10-year period. That is, the value of the books decreases at a constant rate so that it is equal to zero at the end of 10 years. Express the value of the books as a function of time and draw the graph.

44. LINEAR DEPRECIATION A manufacturer buys \$20,000 worth of machinery that depreciates linearly so that its trade-in value after 10 years will be \$1,000.

a. Express the value of the machinery as a function of its age and draw the graph.

b. Compute the value of the machinery after 4 years.

c. When does the machinery become worthless? The manufacturer might not wait this long to dispose of the machinery. Discuss the issues the manufacturer may consider in deciding when to sell.

45. WATER CONSUMPTION Since the beginning of the month, a local reservoir has been losing

SECTION 1.5 Limits

As you will see in subsequent chapters, calculus is an enormously powerful branch of mathematics with a wide range of applications, including curve sketching, optimization of functions, analysis of rates of change, and computation of area and probability. What gives calculus its power and distinguishes it from algebra is the concept of limit, and the purpose of this section is to provide an introduction to this important concept. Our approach will be intuitive rather than formal. The ideas outlined here form the basis for a more rigorous development of the laws and procedures of calculus and lie at the heart of much of modern mathematics.

Intuitive Introduction to the Limit

Roughly speaking, the limit process involves examining the behavior of a function $f(x)$ as x approaches a number c that may or may not be in the domain of f. Limiting behavior occurs in a variety of practical situations. For instance, absolute zero, the temperature T_c at which all molecular activity ceases, can be approached but never actually attained in practice. Similarly, economists who speak of profit under ideal conditions or engineers profiling the ideal specifications of a new engine are really dealing with limiting behavior.

To illustrate the limit process, consider a manager who determines that when x% of her company's plant capacity is being used, the total cost of operation is C hundred thousand dollars, where

$$C(x) = \frac{8x^2 - 636x - 320}{x^2 - 68x - 960}$$

The company has a policy of rotating maintenance in an attempt to ensure that approximately 80% of capacity is always in use. What cost should the manager expect when the plant is operating at this ideal capacity?

It may seem that we can answer this question by simply evaluating $C(80)$, but attempting this evaluation results in the meaningless fraction $\frac{0}{0}$. However, it is still possible to evaluate $C(x)$ for values of x that approach 80 from the right ($x > 80$, when capacity is temporarily overutilized) and from the left ($x < 80$, when capacity is underutilized). A few such calculations are summarized in the following table.

	x approaches 80 from the left $\rightarrow$				$\leftarrow$ x approaches 80 from the right		
x	79.8	79.99	79.999	80	80.0001	80.001	80.04
$C(x)$	6.99782	6.99989	6.99999	$\times$	7.000001	7.00001	7.00043

The values of $C(x)$ displayed on the lower line of this table suggest that $C(x)$ approaches the number 7 as x gets closer and closer to 80. Thus, it is reasonable for the manager to expect a cost of \$700,000 when 80% of plant capacity is utilized.

The functional behavior in this example can be described by saying "$C(x)$ has the limiting value 7 as x approaches 80" or, equivalently, by writing

$$\lim_{x \to 80} C(x) = 7$$

More generally, the limit of $f(x)$ as x approaches the number c can be defined informally as follows.

The Limit of a Function ■ If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is the limit of $f(x)$ as x approaches c. The behavior is expressed by writing

$$\lim_{x \to c} f(x) = L$$

Geometrically, the limit statement $\lim_{x \to c} f(x) = L$ means that the height of the graph $y = f(x)$ approaches L as x approaches c, as shown in Figure 1.43. This interpretation is illustrated along with the tabular approach to computing limits in Example 1.5.1.

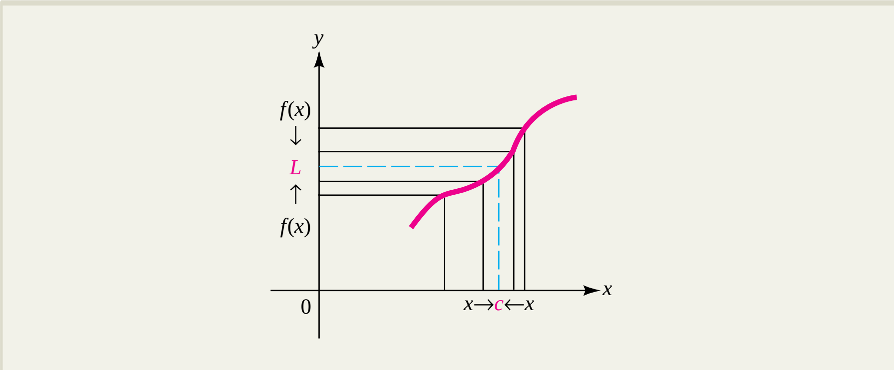


FIGURE 1.43 If $\lim_{x \to c} f(x) = L$, the height of the graph of f approaches L as x approaches c.

EXPLORE!

Graph $f(x) = \dfrac{\sqrt{x} - 1}{x - 1}$, using the modified decimal viewing window [0, 4.7]1 by [−1.1, 2.1]1. Trace values near $x = 1$. Also construct a table of values, using an initial value of 0.97 for x with an incremental change of 0.01. Describe what you observe. Now use an initial value of 0.997 for x with an incremental change of 0.001. Specifically what happens as x approaches 1 from either side? What would be the most appropriate value for $f(x)$ at $x = 1$ to fill the hole in the graph?

EXAMPLE 1.5.1

Use a table to estimate the limit

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$$

Solution

Let

$$f(x) = \frac{\sqrt{x} - 1}{x - 1}$$

and compute $f(x)$ for a succession of values of x approaching 1 from the left and from the right:

$x \to 1 \leftarrow x$

x	0.99	0.999	0.9999	1	1.00001	1.0001	1.001
$f(x)$	0.50126	0.50013	0.50001		0.499999	0.49999	0.49988

The numbers on the bottom line of the table suggest that $f(x)$ approaches 0.5 as x approaches 1; that is,

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = 0.5$$

The graph of $f(x)$ is shown in Figure 1.44. The limit computation says that the height of the graph of $y = f(x)$ approaches $L = 0.5$ as x approaches 1. This corresponds to the "hole" in the graph of $f(x)$ at (1, 0.5). We will compute this same limit using an algebraic procedure in Example 1.5.6.

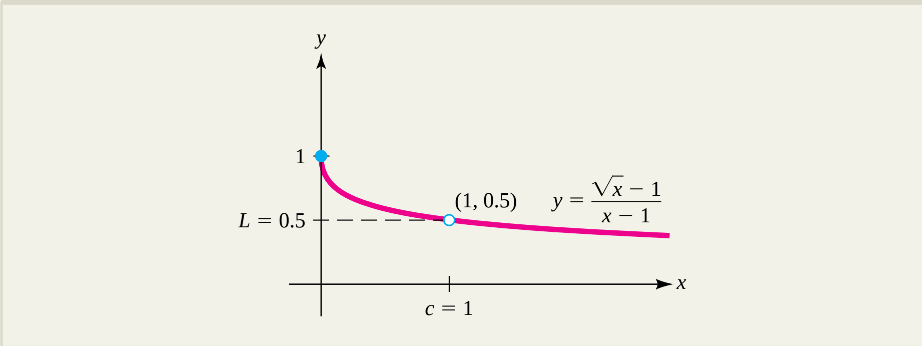


FIGURE 1.44 The function $f(x) = \dfrac{\sqrt{x}-1}{x-1}$ tends toward $L = 0.5$ as x approaches $c = 1$.

It is important to remember that limits describe the behavior of a function *near* a particular point, not necessarily *at* the point itself. This is illustrated in Figure 1.45. For all three functions graphed, the limit of $f(x)$ as x approaches 3 is equal to 4. Yet the functions behave quite differently at $x = 3$ itself. In Figure 1.45a, $f(3)$ is equal to the limit 4; in Figure 1.45b, $f(3)$ is different from 4; and in Figure 1.45c, $f(3)$ is not defined at all.

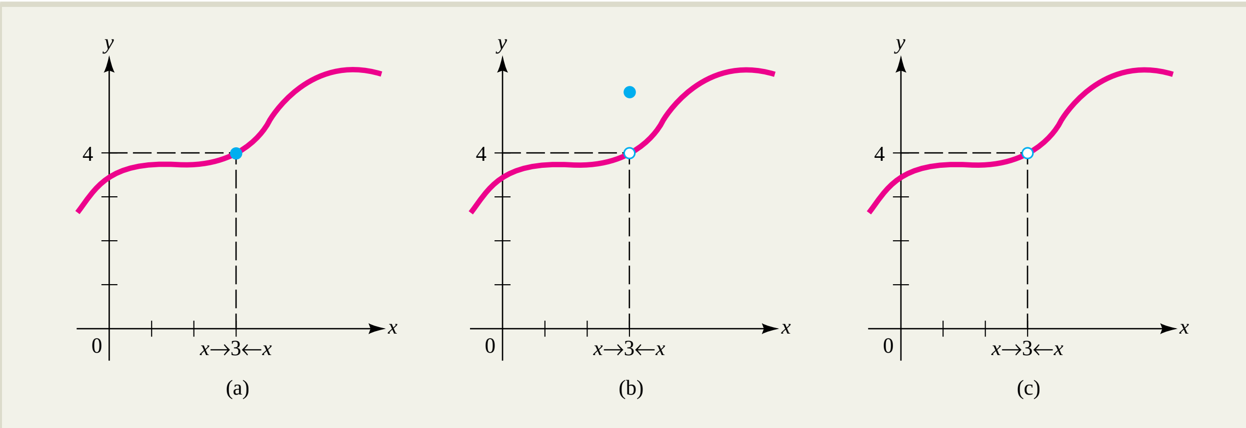


FIGURE 1.45 Three functions for which $\lim\limits_{x \to 3} f(x) = 4$.

Figure 1.46 shows the graph of two functions that do not have a limit as x approaches 2. The limit does not exist in Figure 1.46a because $f(x)$ tends toward 5 as x approaches 2 from the right and tends toward a different value, 3, as x approaches 2 from the left. The function in Figure 1.46b has no finite limit as x

approaches 2 because the values of $f(x)$ increase without bound as x tends toward 2 and hence tend to no finite number L. Such so-called *infinite limits* will be discussed later in this section.

EXPLORE!

Graph $f(x) = \dfrac{2}{(x-2)^2}$ using the window [0, 4]1 by [−5, 40]5. Trace the graph on both sides of $x = 2$ to view the behavior of $\boldsymbol{f(x)}$ about $x = 2$. Also display the table value of the function with the incremental change of x set to 0.01 and the initial value $x = 1.97$. What happens to the values of $f(x)$ as $\boldsymbol{x}$ approaches 2?

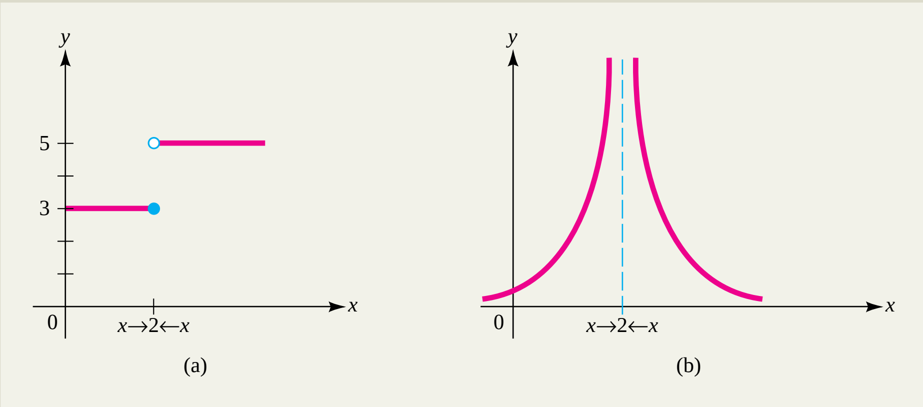


FIGURE 1.46 Two functions for which $\lim_{x\to 2} f(x)$ does not exist.

Properties of Limits

Limits obey certain algebraic rules that can be used to simplify computations. These rules, which should seem plausible on the basis of our informal definition of limit, are proved formally in more theoretical courses.

EXPLORE!

Graph the function

$$f(x) = \begin{cases} 3 & x \le 2 \\ 5 & x > 2 \end{cases}$$

using the dot graphing style and writing

Y1 = 3(X ≤ 2) + 5(X > 2)

in the equation editor of your graphing calculator. Use your **TRACE** key to determine the values of y when x is near 2. Does it make a difference from which side $x = 2$ is approached? Also evaluate $f(2)$.

Algebraic Properties of Limits ■ If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

$$\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

$$\lim_{x\to c} [f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

$$\lim_{x\to c} [kf(x)] = k \lim_{x\to c} f(x) \quad \text{for any constant } k$$

$$\lim_{x\to c} [f(x)g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)]$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{if } \lim_{x\to c} g(x) \ne 0$$

$$\lim_{x\to c} [f(x)]^p = [\lim_{x\to c} f(x)]^p \quad \text{if } [\lim_{x\to c} f(x)]^p \text{ exists}$$

That is, the limit of a sum, a difference, a multiple, a product, a quotient, or a power exists and is the sum, difference, multiple, product, quotient, or power of the individual limits, as long as all expressions involved are defined.

Here are two elementary limits that we will use along with the limit rules to compute limits involving more complex expressions.

Limits of Two Linear Functions ■ For any constant k,

$$\lim_{x \to c} k = k \quad \text{and} \quad \lim_{x \to c} x = c$$

That is, the limit of a constant is the constant itself, and the limit of $f(x) = x$ as x approaches c is c.

In geometric terms, the limit statement $\lim_{x \to c} k = k$ says that the height of the graph of the constant function $f(x) = k$ approaches k as x approaches c. Similarly, $\lim_{x \to c} x = c$ says that the height of the linear function $f(x) = x$ approaches c as x approaches c. These statements are illustrated in Figure 1.47.

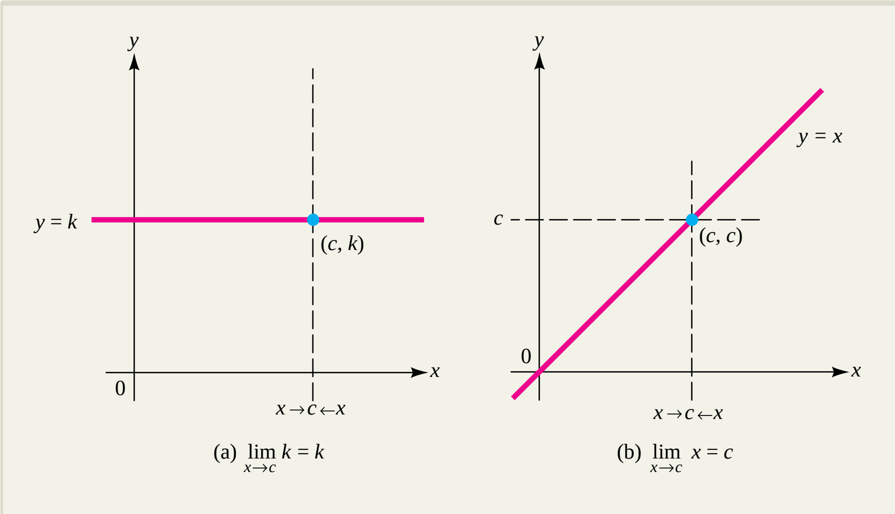


FIGURE 1.47 Limits of two linear functions.

Computation of Limits

Examples 1.5.2 through 1.5.6 illustrate how the properties of limits can be used to calculate limits of algebraic functions. In Example 1.5.2, you will see how to find the limit of a polynomial.

EXAMPLE 1.5.2

Find $\lim_{x \to -1} (3x^3 - 4x + 8)$.

Solution

Apply the properties of limits to obtain

$$\begin{aligned} \lim_{x \to -1} (3x^3 - 4x + 8) &= 3\left(\lim_{x \to -1} x\right)^3 - 4\left(\lim_{x \to -1} x\right) + \lim_{x \to -1} 8 \\ &= 3(-1)^3 - 4(-1) + 8 = 9 \end{aligned}$$

In Example 1.5.3, you will see how to find the limit of a rational function whose denominator does not approach zero.

EXPLORE!

Graph $f(x) = \dfrac{x^2 + x - 2}{x - 1}$ using the viewing window [0, 2]0.5 by [0, 5]0.5. Trace to $x = 1$ and notice there is no corresponding y value. Create a table with an initial value of 0.5 for x, increasing in increments of 0.1. Notice that an error is displayed for $x = 1$, confirming that $f(x)$ is undefined at $x = 1$. What would be the appropriate y value if this gap were filled? Change the initial value of x to 0.9 and the increment size to 0.01 to get a better approximation. Finally, zoom in on the graph about $x = 1$ to conjecture a limiting value for the function at $x = 1$.

EXAMPLE 1.5.3

Find $\displaystyle\lim_{x\to1} \frac{3x^3 - 8}{x - 2}$.

Solution

Since $\displaystyle\lim_{x\to1} (x - 2) \neq 0$, you can use the quotient rule for limits to get

$$\lim_{x\to1} \frac{3x^3 - 8}{x - 2} = \frac{\lim_{x\to1} (3x^3 - 8)}{\lim_{x\to1} (x - 2)} = \frac{3 \lim_{x\to1} x^3 - \lim_{x\to1} 8}{\lim_{x\to1} x - \lim_{x\to1} 2} = \frac{3 - 8}{1 - 2} = 5$$

In general, you can use the properties of limits to obtain these formulas, which can be used to evaluate many limits that occur in practical problems.

Limits of Polynomials and Rational Functions ■ If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to c} p(x) = p(c)$$

and

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

In Example 1.5.4, the denominator of the given rational function approaches zero, while the numerator does not. When this happens, you can conclude that the limit does not exist. The absolute value of such a quotient increases without bound and hence does not approach any finite number.

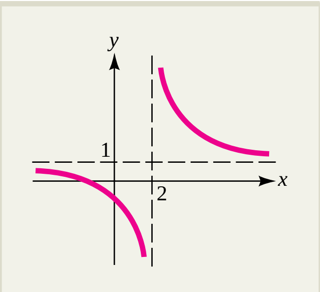


FIGURE 1.48 The graph of $f(x) = \dfrac{x + 1}{x - 2}$.

EXAMPLE 1.5.4

Find $\displaystyle\lim_{x\to2} \frac{x + 1}{x - 2}$.

Solution

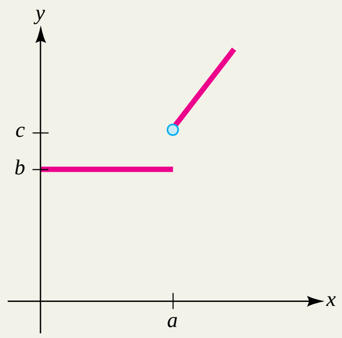
The quotient rule for limits does not apply in this case since the limit of the denominator is

$$\lim_{x\to2} (x - 2) = 0$$

Since the limit of the numerator is $\displaystyle\lim_{x\to2} (x + 1) = 3$, which is not equal to zero, you can conclude that the limit of the quotient does not exist.

The graph of the function $f(x) = \dfrac{x + 1}{x - 2}$ in Figure 1.48 gives you a better idea of what is actually happening in this example. Note that $f(x)$ increases without bound as x approaches 2 from the right and decreases without bound as x approaches 2 from the left.

4.



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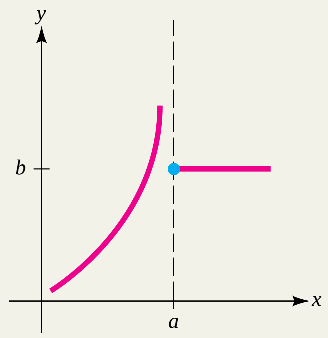

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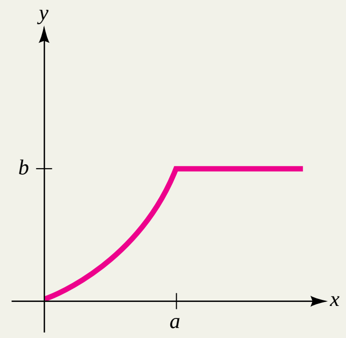


In Exercises 7 through 26, find the indicated limit if it exists.

7. $\lim_{x \to 2} (3x^2 - 5x + 2)$

8. $\lim_{x \to -1} (x^3 - 2x^2 + x - 3)$

9. $\lim_{x \to 0} (x^5 - 6x^4 + 7)$

10. $\lim_{x \to -1/2} (1 - 5x^3)$

11. $\lim_{x \to 3} (x - 1)^2(x + 1)$

12. $\lim_{x \to -1} (x^2 + 1)(1 - 2x)^2$

13. $\lim_{x \to 1/3} \dfrac{x + 1}{x + 2}$

14. $\lim_{x \to 1} \dfrac{2x + 3}{x + 1}$

15. $\lim_{x \to 5} \dfrac{x + 3}{5 - x}$

16. $\lim_{x \to 3} \dfrac{2x + 3}{x - 3}$

17. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

18. $\lim_{x \to 3} \dfrac{9 - x^2}{x - 3}$

19. $\lim_{x \to 5} \dfrac{x^2 - 3x - 10}{x - 5}$

20. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

21. $\lim_{x \to 4} \dfrac{(x + 1)(x - 4)}{(x - 1)(x - 4)}$

22. $\lim_{x \to 0} \dfrac{x(x^2 - 1)}{x^2}$

23. $\lim_{x \to -2} \dfrac{x^2 - x - 6}{x^2 + 3x + 2}$

24. $\lim_{x \to 1} \dfrac{x^2 + 4x - 5}{x^2 - 1}$

25. $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

26. $\lim_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

For Exercises 27 through 36, find $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

27. $f(x) = x^3 - 4x^2 - 4$

28. $f(x) = 1 - x + 2x^2 - 3x^3$

29. $f(x) = (1 - 2x)(x + 5)$

30. $f(x) = (1 + x^2)^3$

31. $f(x) = \dfrac{x^2 - 2x + 3}{2x^2 + 5x + 1}$

32. $f(x) = \dfrac{1 - 3x^3}{2x^3 - 6x + 2}$

33. $f(x) = \dfrac{2x + 1}{3x^2 + 2x - 7}$

34. $f(x) = \dfrac{x^2 + x - 5}{1 - 2x - x^3}$

35. $f(x) = \dfrac{3x^2 - 6x + 2}{2x - 9}$

36. $f(x) = \dfrac{1 - 2x^3}{x + 1}$

SECTION 1.6 One-Sided Limits and Continuity

The dictionary defines continuity as an "unbroken or uninterrupted succession." Continuous behavior is certainly an important part of our lives. For instance, the growth of a tree is continuous, as are the motion of a rocket and the volume of water flowing into a bathtub. In this section, we shall discuss what it means for a function to be continuous and shall examine a few important properties of such functions.

One-Sided Limits

Informally, a continuous function is one whose graph can be drawn without the "pen" leaving the paper (Figure 1.52a). Not all functions have this property, but those that do play a special role in calculus. A function is *not* continuous where its graph has a "hole or gap" (Figure 1.52b), but what do we really mean by "holes and gaps" in a graph? To describe such features mathematically, we require the concept of a **one-sided limit** of a function; that is, a limit in which the approach is either from the right or from the left, rather than from both sides as required for the "two-sided" limit introduced in Section 1.5.

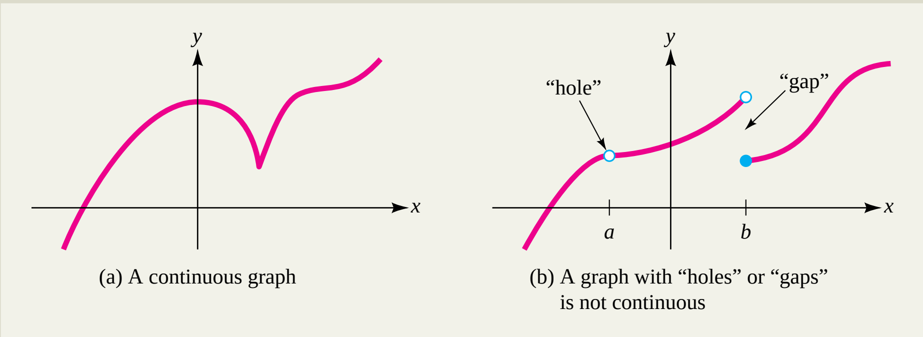


FIGURE 1.52 Continuity and discontinuity.

For instance, Figure 1.53 shows the graph of inventory I as a function of time t for a company that immediately restocks to level L_1 whenever the inventory falls to a certain minimum level L_2 (this is called *just-in-time inventory*). Suppose the first restocking time occurs at $t = t_1$. Then as t tends toward t_1 from the left, the limiting value of $I(t)$ is L_2, while if the approach is from the right, the limiting value is L_1.

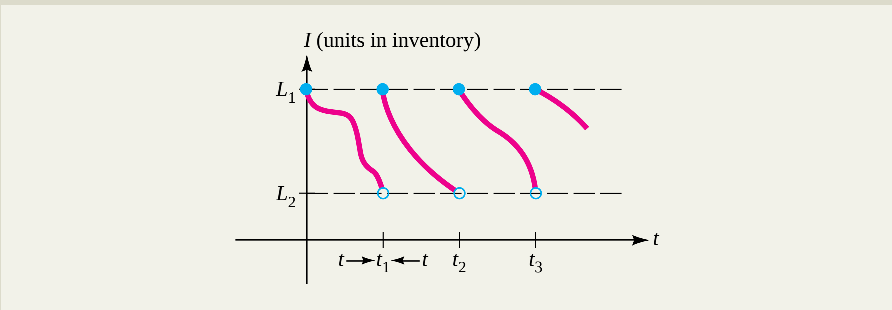


FIGURE 1.53 One-sided limits in a just-in-time inventory example.

Here is the notation we will use to describe one-sided limiting behavior.

One-Sided Limits ■ If $f(x)$ approaches L as x tends toward c from the left ($x < c$), we write $\lim_{x \to c^-} f(x) = L$. Likewise, if $f(x)$ approaches M as x tends toward c from the right ($c < x$), then $\lim_{x \to c^+} f(x) = M$.

If this notation is used in our inventory example, we would write

$$\lim_{t \to t_1^-} I(t) = L_2 \qquad \text{and} \qquad \lim_{t \to t_1^+} I(t) = L_1$$

Here are two more examples involving one-sided limits.

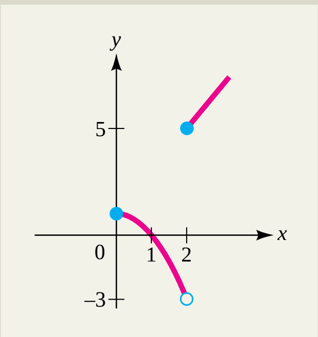


FIGURE 1.54 The graph of $f(x) = \begin{cases} 1 - x^2 & \text{if } 0 \le x < 2 \\ 2x + 1 & \text{if } x \ge 2 \end{cases}$

EXAMPLE 1.6.1

For the function

$$f(x) = \begin{cases} 1 - x^2 & \text{if } 0 \le x < 2 \\ 2x + 1 & \text{if } x \ge 2 \end{cases}$$

evaluate the one-sided limits $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

Solution

The graph of $f(x)$ is shown in Figure 1.54. Since $f(x) = 1 - x^2$ for $0 \le x < 2$, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 - x^2) = -3$$

Similarly, $f(x) = 2x + 1$ if $x \ge 2$, so

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x + 1) = 5$$

EXPLORE!

Refer to Example 1.6.2. Graph $f(x) = \dfrac{x - 2}{x - 4}$ using the window [0, 9.4]1 by [−4, 4]1 to verify the limit results as x approaches 4 from the left and the right. Now trace $f(x)$ for large positive or negative values of x. What do you observe?

EXAMPLE 1.6.2

Find $\lim \dfrac{x - 2}{x - 4}$ as x approaches 4 from the left and from the right.

Solution

First, note that for $2 < x < 4$ the quantity

$$f(x) = \frac{x - 2}{x - 4}$$

is negative, so as x approaches 4 from the left, $f(x)$ *decreases* without bound. We denote this fact by writing

$$\lim_{x \to 4^-} \frac{x - 2}{x - 4} = -\infty$$

Likewise, as x approaches 4 from the right (with $x > 4$), $f(x)$ increases without bound and we write

$$\lim_{x \to 4^+} \frac{x-2}{x-4} = +\infty$$

The graph of f is shown in Figure 1.55.

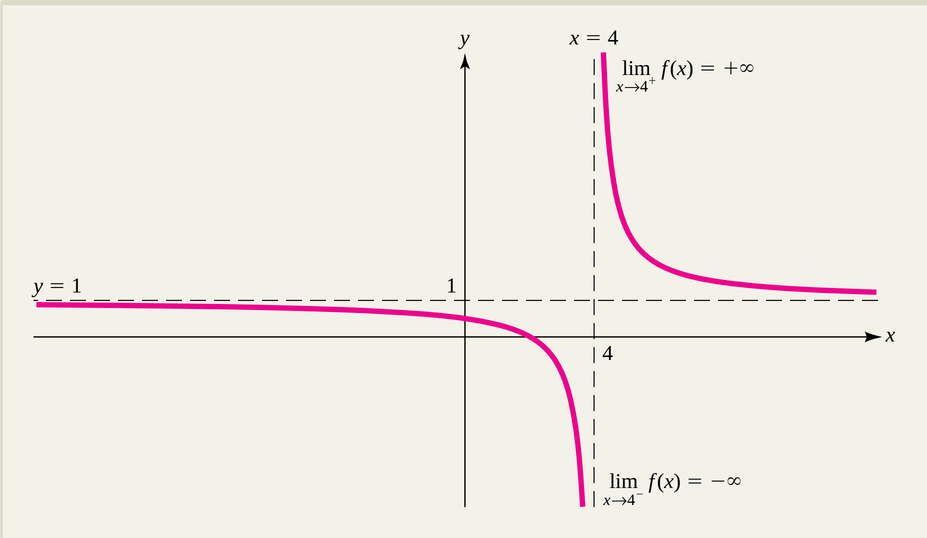


FIGURE 1.55 The graph of $f(x) = \dfrac{x-2}{x-4}$.

Notice that the two-sided limit $\lim\limits_{x \to 4} f(x)$ does *not* exist for the function in Example 1.6.2 since the functional values $f(x)$ do not approach a single value L as x tends toward 4 from each side. In general, we have the following useful criterion for the existence of a limit.

Existence of a Limit ■ The two-sided limit $\lim\limits_{x \to c} f(x)$ exists if and only if the two one-sided limits $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ both exist and are equal, and then

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

EXPLORE!

Re-create the piecewise linear function $f(x)$ defined in the Explore! Box on page 66. Verify graphically that $\lim\limits_{x \to 2^-} f(x) = 3$ and $\lim\limits_{x \to 2^+} f(x) = 5$.

EXAMPLE 1.6.3

Determine whether $\lim\limits_{x \to 1} f(x)$ exists, where

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ -x^2 + 4x - 1 & \text{if } x \geq 1 \end{cases}$$

Solution

Computing the one-sided limits at $x = 1$, we find

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = (1) + 1 = 2 \qquad \text{since } f(x) = x + 1 \text{ when } x < 1$$

and

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (-x^2 + 4x - 1) \qquad \text{since } f(x) = -x^2 + 4x - 1 \text{ when } x \geq 1\\ &= -(1)^2 + 4(1) - 1 = 2\end{aligned}$$

Since the two one-sided limits are equal, it follows that the two-sided limit of $f(x)$ at $x = 1$ exists, and we have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$$

The graph of $f(x)$ is shown in Figure 1.56.

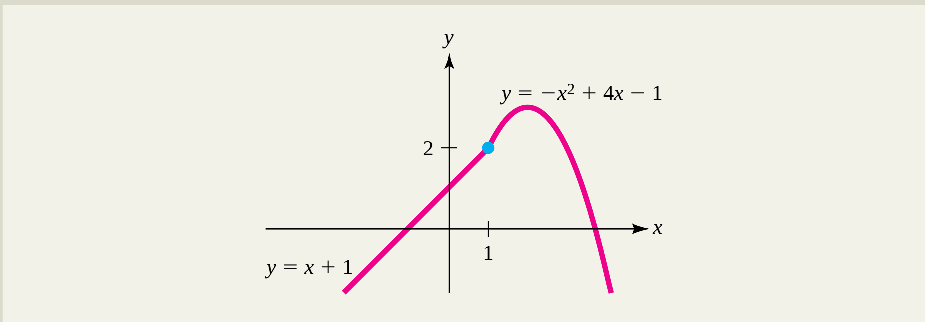


FIGURE 1.56 The graph of $f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ -x^2 + 4x - 1 & \text{if } x \geq 1 \end{cases}$.

Continuity

At the beginning of this section, we observed that a continuous function is one whose graph has no "holes or gaps." A "hole" at $x = c$ can arise in several ways, three of which are shown in Figure 1.57.

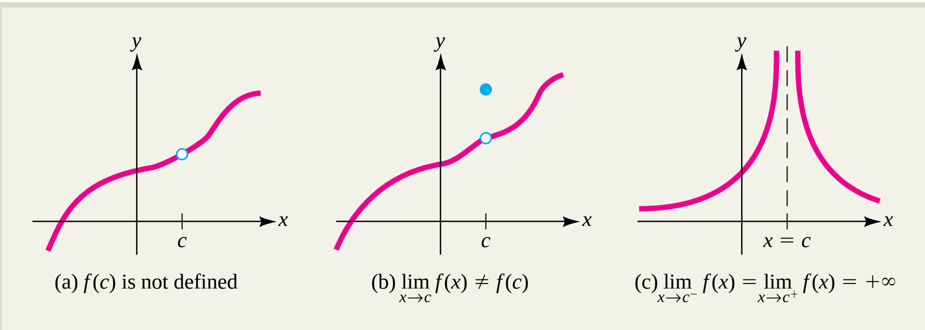
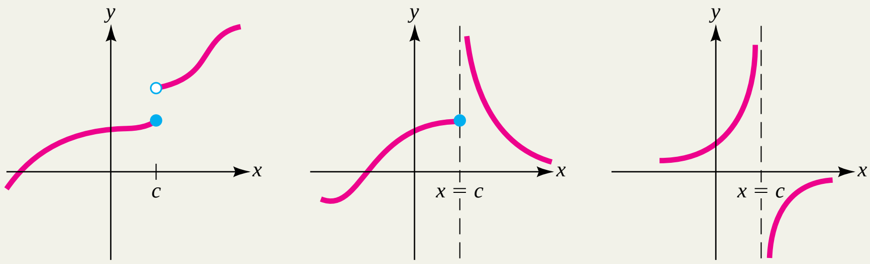


FIGURE 1.57 Three ways the graph of a function can have a "hole" at $x = c$.

The graph of $f(x)$ will have a "gap" at $x = c$ if the one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ are not equal. Three ways this can happen are shown in Figure 1.58.



(a) A finite gap: $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

(b) An infinite gap: $\lim_{x \to c^-} f(x)$ is finite but $\lim_{x \to c^+} f(x) = +\infty$

(c) An infinite gap: $\lim_{x \to c^-} f(x) = +\infty$ and $\lim_{x \to c^+} f(x) = -\infty$

FIGURE 1.58 Three ways for the graph of a function to have a "gap" at $x = c$.

So what properties will guarantee that $f(x)$ does not have a "hole or gap" at $x = c$? The answer is surprisingly simple. The function must be defined at $x = c$, it must have a finite, two-sided limit at $x = c$; and $\lim_{x \to c} f(x)$ must equal $f(c)$. To summarize:

Continuity ■ A function f is continuous at c if all three of these conditions are satisfied:

a. $f(c)$ is defined.

b. $\lim_{x \to c} f(x)$ exists.

c. $\lim_{x \to c} f(x) = f(c)$.

If $f(x)$ is not continuous at c, it is said to have a **discontinuity** there.

Continuity of Polynomials and Rational Functions

Recall that if $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c)$$

and

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

These limit formulas can be interpreted as saying that **a polynomial or a rational function is continuous wherever it is defined.** This is illustrated in Examples 1.6.4 through 1.6.7.

EXAMPLE 1.6.4

Show that the polynomial $p(x) = 3x^3 - x + 5$ is continuous at $x = 1$.

Solution

Verify that the three criteria for continuity are satisfied. Clearly $p(1)$ is defined; in fact, $p(1) = 7$. Moreover, $\lim_{x\to 1} p(x)$ exists and $\lim_{x\to 1} p(x) = 7$. Thus,

$$\lim_{x\to 1} p(x) = 7 = p(1)$$

as required for $p(x)$ to be continuous at $x = 1$.

EXPLORE!

Graph $f(x) = \dfrac{x+1}{x-2}$ using the enlarged decimal window $[-9.4, 9.4]1$ by $[-6.2, 6.2]1$. Is the function continuous? Is it continuous at $x = 2$? How about at $x = 3$? Also examine this function using a table with an initial value of x at 1.8, increasing in increments of 0.2.

EXAMPLE 1.6.5

Show that the rational function $f(x) = \dfrac{x+1}{x-2}$ is continuous at $x = 3$.

Solution

Note that $f(3) = \dfrac{3+1}{3-2} = 4$. Since $\lim_{x\to 3} (x-2) \neq 0$, you find that

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x+1}{x-2} = \frac{\lim_{x\to 3}(x+1)}{\lim_{x\to 3}(x-2)} = \frac{4}{1} = 4 = f(3)$$

as required for $f(x)$ to be continuous at $x = 3$.

EXPLORE!

Store $h(x)$ of Example 1.6.6(c) into the equation editor as Y1 $= (X + 1)(X < 1) + (2 - X)(X \geq 1)$. Use a decimal window with a dot graphing style. Is this function continuous at $x = 1$? Use the **TRACE** key to display the value of the function at $x = 1$ and to find the limiting values of y as x approaches 1 from the left side and from the right side.

EXAMPLE 1.6.6

Discuss the continuity of each of the following functions:

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x + 1}$ **c.** $h(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$

Solution

The functions in parts (a) and (b) are rational and are therefore continuous wherever they are defined (that is, wherever their denominators are not zero).

a. $f(x) = \dfrac{1}{x}$ is defined everywhere except $x = 0$, so it is continuous for all $x \neq 0$ (Figure 1.59a).

b. Since $x = -1$ is the only value of x for which $g(x)$ is undefined, $g(x)$ is continuous except at $x = -1$ (Figure 1.59b).

c. This function is defined in two pieces. First check for continuity at $x = 1$, the value of x that separates the two pieces. You find that $\lim_{x\to 1} h(x)$ does not exist, since $h(x)$ approaches 2 from the left and 1 from the right. Thus, $h(x)$ is not continuous at 1 (Figure 1.59c). However, since the polynomials $x + 1$ and $2 - x$ are each continuous for every value of x, it follows that $h(x)$ is continuous at every number x other than 1.

EXPLORE!

Graph $f(x) = \dfrac{x^3 - 8}{x - 2}$, using a standard window. Does this graph appear continuous? Now use a modified decimal window $[-4.7, 4.7]1$ by $[0, 14.4]1$ and describe what you observe. Which case in Example 1.6.6 does this resemble?

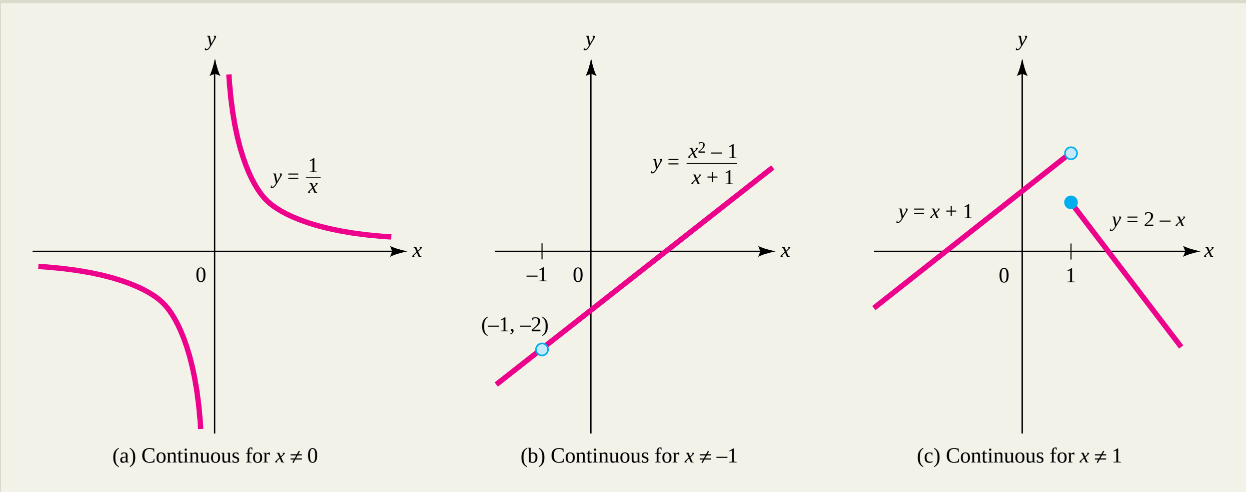


FIGURE 1.59 Functions for Example 1.6.6.

EXAMPLE 1.6.7

For what value of the constant A is the following function continuous for all real x?

$$f(x) = \begin{cases} Ax + 5 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } x \geq 1 \end{cases}$$

Solution

Since $Ax + 5$ and $x^2 - 3x + 4$ are both polynomials, it follows that $f(x)$ will be continuous everywhere except possibly at $x = 1$. Moreover, $f(x)$ approaches $A + 5$ as x approaches 1 from the left and approaches 2 as x approaches 1 from the right. Thus, for $\lim_{x \to 1} f(x)$ to exist, we must have $A + 5 = 2$ or $A = -3$, in which case

$$\lim_{x \to 1} f(x) = 2 = f(1)$$

This means that f is continuous for all x only when $A = -3$.

Continuity on an Interval

For many applications of calculus, it is useful to have definitions of continuity on open and closed intervals.

Continuity on an Interval ■ A function $f(x)$ is said to be continuous on an open interval $a < x < b$ if it is continuous at each point $x = c$ in that interval.

Moreover, f is continuous on the closed interval $a \leq x \leq b$ if it is continuous on the open interval $a < x < b$ and

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to b^-} f(x) = f(b)$$

In other words, continuity on an interval means that the graph of f is “one piece” throughout the interval.

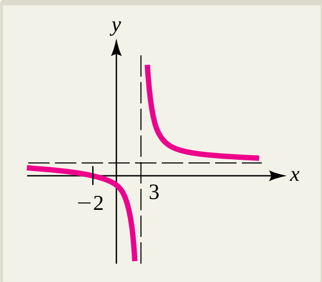


FIGURE 1.60 The graph of $f(x) = \dfrac{x+2}{x-3}$.

EXAMPLE 1.6.8

Discuss the continuity of the function

$$f(x) = \frac{x+2}{x-3}$$

on the open interval $-2 < x < 3$ and on the closed interval $-2 \le x \le 3$.

Solution

The rational function $f(x)$ is continuous for all x except $x = 3$. Therefore, it is continuous on the open interval $-2 < x < 3$ but not on the closed interval $-2 \le x \le 3$, since it is discontinuous at the endpoint 3 (where its denominator is zero). The graph of f is shown in Figure 1.60.

The Intermediate Value Property

An important feature of continuous functions is the **intermediate value property,** which says that if $f(x)$ is continuous on the interval $a \le x \le b$ and L is a number between $f(a)$ and $f(b)$, then $f(c) = L$ for some number c between a and b (see Figure 1.61). In other words, *a continuous function attains all values between any two of its values.* For instance, a girl who weighs 5 pounds at birth and 100 pounds at age 12 must have weighed exactly 50 pounds at some time during her 12 years of life, since her weight is a continuous function of time.

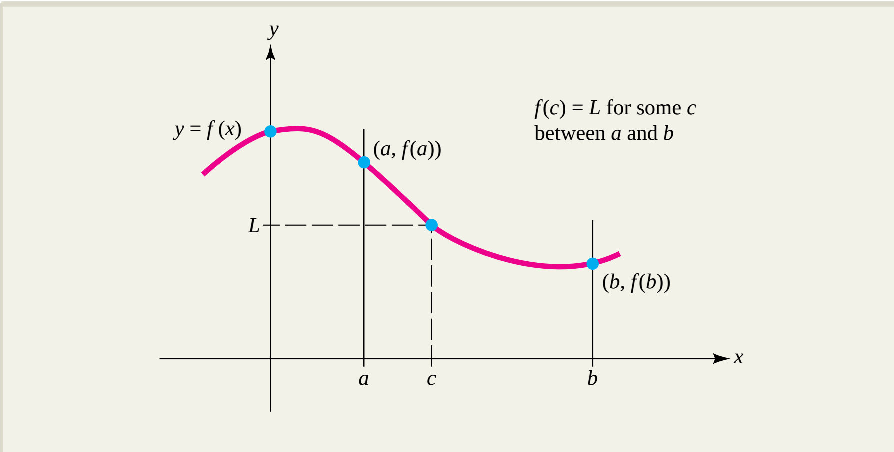


FIGURE 1.61 The intermediate value property.

The intermediate value property has a variety of applications. In Example 1.6.9, we show how it can be used to estimate a solution of a given equation.

EXAMPLE 1.6.9

Show that the equation $x^2 - x - 1 = \dfrac{1}{x+1}$ has a solution for $1 < x < 2$.

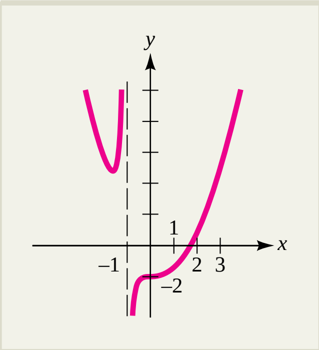


FIGURE 1.62 The graph of $y = x^2 - x - 1 - \dfrac{1}{x+1}$.

Solution

Let $f(x) = x^2 - x - 1 - \dfrac{1}{x+1}$. Then $f(1) = -\dfrac{3}{2}$ and $f(2) = \dfrac{2}{3}$. Since $f(x)$ is continuous for $1 \le x \le 2$ and the graph of f is below the x axis at $x = 1$ and above the x axis at $x = 2$, it follows from the intermediate value property that the graph must cross the x axis somewhere between $x = 1$ and $x = 2$ (see Figure 1.62). In other words, there is a number c such that $1 < c < 2$ and $f(c) = 0$, so

$$c^2 - c - 1 = \frac{1}{c+1}$$

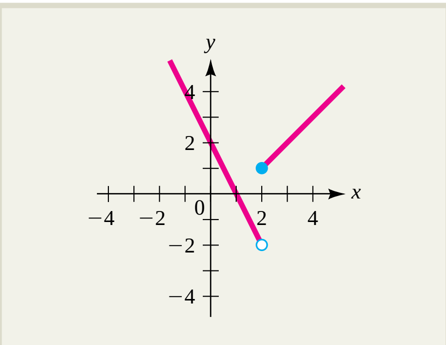
NOTE The root-location procedure described in Example 1.6.9 can be applied repeatedly to estimate the root c to any desired degree of accuracy. For instance, the midpoint of the interval $1 \le x \le 2$ is $d = 1.5$ and $f(1.5) = -0.65$, so the root c must lie in the interval $1.5 < x < 2$ (since $f(2) > 0$), and so on.

"That's nice," you say, "but I can use the solve utility on my calculator to find a much more accurate estimate for c with much less effort." You are right, of course, but how do you think your calculator makes its estimation? Perhaps not by the method just described, but certainly by some similar algorithmic procedure. It is important to understand such procedures as you use the technology that utilizes them. ■

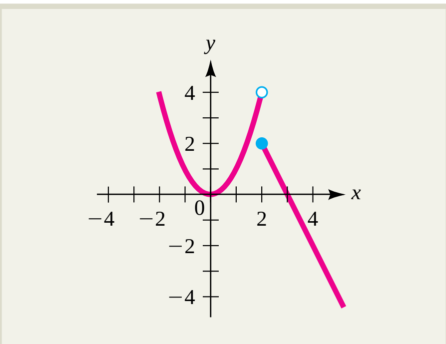
EXERCISES ■ 1.6

In Exercises 1 through 4, find the one-sided limits $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ from the given graph of f and determine whether $\lim\limits_{x\to 2} f(x)$ exists.

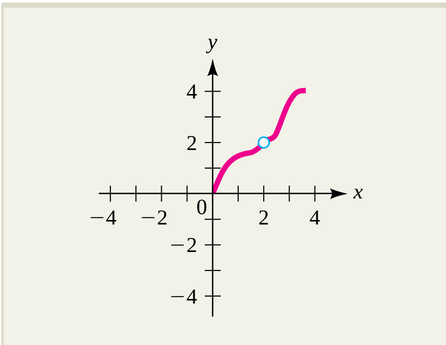
1.



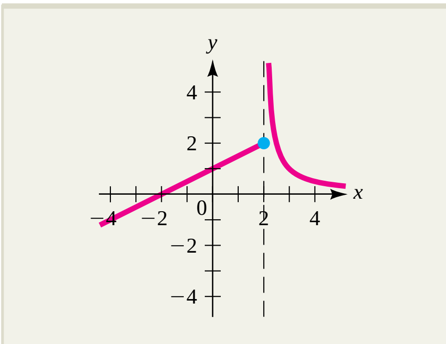
2.



3.



4.



In Exercises 5 through 16, find the indicated one-sided limit. If the limiting value is infinite, indicate whether it is $+\infty$ or $-\infty$.

5. $\lim_{x \to 4^+} (3x^2 - 9)$

6. $\lim_{x \to 1^-} x(2 - x)$

7. $\lim_{x \to 3^+} \sqrt{3x - 9}$

8. $\lim_{x \to 2^-} \sqrt{4 - 2x}$

9. $\lim_{x \to 2^-} \dfrac{x + 3}{x + 2}$

10. $\lim_{x \to 2^-} \dfrac{x^2 + 4}{x - 2}$

11. $\lim_{x \to 0^+} (x - \sqrt{x})$

12. $\lim_{x \to 1^-} \dfrac{x - \sqrt{x}}{x - 1}$

13. $\lim_{x \to 3^+} \dfrac{\sqrt{x + 1} - 2}{x - 3}$

14. $\lim_{x \to 5^+} \dfrac{\sqrt{2x - 1} - 3}{x - 5}$

15. $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$,

where $f(x) = \begin{cases} 2x^2 - x & \text{if } x < 3 \\ 3 - x & \text{if } x \geq 3 \end{cases}$

16. $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$

where $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x < -1 \\ x^2 + 2x & \text{if } x \geq -1 \end{cases}$

In Exercises 17 through 28, decide if the given function is continuous at the specified value of x.

17. $f(x) = 5x^2 - 6x + 1$ at $x = 2$

18. $f(x) = x^3 - 2x^2 + x - 5$ at $x = 0$

19. $f(x) = \dfrac{x + 2}{x + 1}$ at $x = 1$

20. $f(x) = \dfrac{2x - 4}{3x - 2}$ at $x = 2$

21. $f(x) = \dfrac{x + 1}{x - 1}$ at $x = 1$

22. $f(x) = \dfrac{2x + 1}{3x - 6}$ at $x = 2$

23. $f(x) = \dfrac{\sqrt{x} - 2}{x - 4}$ at $x = 4$

24. $f(x) = \dfrac{\sqrt{x} - 2}{x - 4}$ at $x = 2$

25. $f(x) = \begin{cases} x + 1 & \text{if } x \leq 2 \\ 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

26. $f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$ at $x = 0$

27. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 3 \\ 2x + 4 & \text{if } x > 3 \end{cases}$ at $x = 3$

28. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x < -1 \\ x^2 - 3 & \text{if } x \geq -1 \end{cases}$ at $x = -1$

In Exercises 29 through 42, list all the values of x for which the given function is not continuous.

29. $f(x) = 3x^2 - 6x + 9$

30. $f(x) = x^5 - x^3$

31. $f(x) = \dfrac{x + 1}{x - 2}$

32. $f(x) = \dfrac{3x - 1}{2x - 6}$

33. $f(x) = \dfrac{3x + 3}{x + 1}$

34. $f(x) = \dfrac{x^2 - 1}{x + 1}$

35. $f(x) = \dfrac{3x - 2}{(x + 3)(x - 6)}$

36. $f(x) = \dfrac{x}{(x + 5)(x - 1)}$

37. $f(x) = \dfrac{x}{x^2 - x}$

38. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - x - 2}$

39. $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ 6x - 1 & \text{if } x > 1 \end{cases}$

40. $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 9 & \text{if } x > 2 \end{cases}$

41. $f(x) = \begin{cases} 3x - 2 & \text{if } x < 0 \\ x^2 + x & \text{if } x \geq 0 \end{cases}$