DSP Math Problem

3.36 Consider the system

$$H(z) = \frac{1 - 2z^{-1} + 2z^{-2} - z^{-3}}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.2z^{-1})}, \qquad \text{ROC: } 0.5|z| > 1$$

(a) Sketch the pole–zero pattern. Is the system stable?

(b) Determine the impulse response of the system.

Answer:

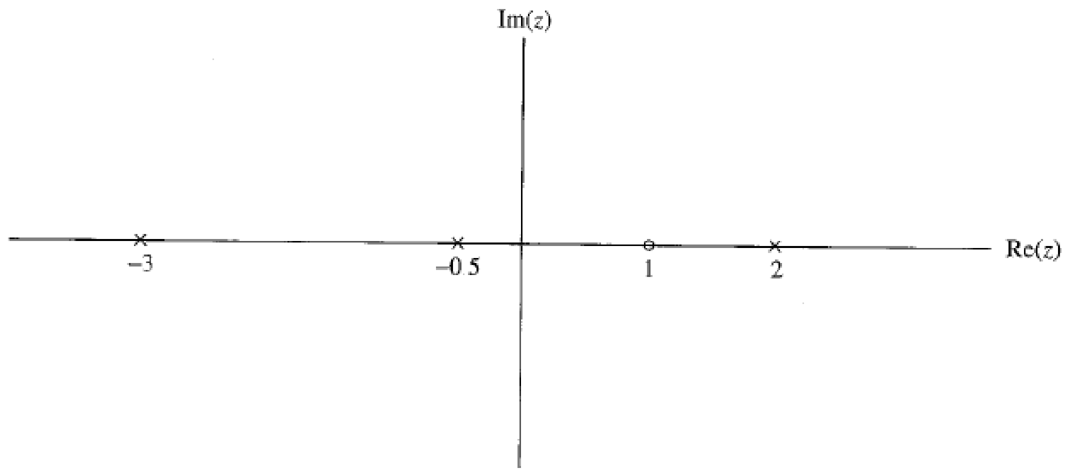
$$\begin{aligned}
H(z) &= \frac{1 - 2z^{-1} + 2z^{-2} - z^{-3}}{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{5}z^{-1})}, \quad \frac{1}{2} < |z| < 1 \\
&= \frac{1 - z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{5}z^{-1})}, \quad \frac{1}{2} < |z| < 1 \\
\text{(a) } Z_{1,2} &= \frac{1 \pm j\sqrt{3}}{2}, \; p_1 = \frac{1}{2}, \; p_2 = \frac{1}{5} \\
\text{(b) } H(z) &= 1 + \left[\frac{\frac{5}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{-2.8}{1 - \frac{1}{5}z^{-1}}\right] z^{-1} \\
h(n) &= \delta(n) + \left[5(\frac{1}{2})^n - 14(\frac{1}{5})^n\right] u(n)
\end{aligned}$$

3.51 Consider an LTI discrete-time system whose pole–zero pattern is shown in Fig. P3.51.

(a) Determine the ROC of the system function $H(z)$ if the system is known to be stable.

(b) It is possible for the given pole–zero plot to correspond to a causal and stable system? If so, what is the appropriate ROC?

(c) How many possible systems can be associated with this pole–zero pattern?



Answer:

(a)

$$H(z) = \frac{z-1}{(z+\frac{1}{2})(z+3)(z-2)}, \qquad \text{ROC: } \frac{1}{2} < |z| < 2$$

(b) The system can be causal if the ROC is $|z| > 3$, but it cannot be stable.

(c)

$$H(z) = \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B}{1+3z^{-1}} + \frac{C}{1-2z^{-1}}$$

(1) The system can be causal; (2) The system can be anti-causal; (3) There are two other noncausal responses.The corresponding ROC for each of these possibilities are :

$\text{ROC}_1 : |z| > 3;$ $\quad \text{ROC}_2 : |z| < 3;$ $\quad \text{ROC}_3 : \frac{1}{2} < |z| < 2;$ $\quad \text{ROC}_4 : 2 < |z| < 3;$

3.55 The step response of an LTI system is

$$s(n) = (\frac{1}{3})^{n-2} u(n+2)$$

(a) Find the system function $H(z)$ and sketch the pole–zero plot.

(b) Determine the impulse response $h(n)$.

(c) Check if the system is causal and stable.

Answer:

$$s(n) = (\frac{1}{3})^{n-2} u(n+2)$$

(a)

$$\begin{aligned} h(n) &= s(n) - s(n-1) \\ &= (\frac{1}{3})^{n-2} u(n+2) - (\frac{1}{3})^{n-3} u(n+1) \\ &= 3^4 \delta(n+2) - 54\delta(n+1) - 18(\frac{1}{3})^n u(n) \\ H(z) &= 81z^2 - 54z + \frac{-18}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{81z(z^{-1})}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

$H(z)$ has zeros at $z = 0, 1$ and a pole at $z = \frac{1}{3}$.

(b) $h(n) = 81\delta(n+2) - 54\delta(n+1) - 18(\frac{1}{3})^n u(n)$

(c) The system is not causal, but it is stable since the pole is inside the unit circle.

4.4 Consider the following periodic signal:

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$$

(a) Sketch the signal $x(n)$ and its magnitude and phase spectra.

Answer:

$$
\begin{aligned}
x(n) &= \left\{\ldots,1,0,1,2,\underset{\uparrow}{3},2,1,0,1,\ldots\right\} \\
N &= 6 \\
c_k &= \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j2\pi kn/6} \\
&= \left[3+2e^{\frac{-j2\pi k}{6}}+e^{\frac{-j2\pi k}{3}}+e^{\frac{-j4\pi k}{3}}+2e^{\frac{-j10\pi k}{6}}\right] \\
&= \frac{1}{6}\left[3+4cos\frac{\pi k}{3}+2cos\frac{2\pi k}{3}\right] \\
\text{Hence, } c_0 &= \frac{9}{6}, c_1=\frac{4}{6}, c_2=0, c_3=\frac{1}{6}, c_4=0, c_5=\frac{4}{6}
\end{aligned}
$$

4.5 Consider the signal

$$
x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}
$$

(a) Determine and sketch its power density spectrum.

(b) Evaluate the power of the signal.

Answer:

$$x(n) = 2 + 2\cos\pi n/4 + \cos\pi n/2 + \frac{1}{2}\cos 3\pi n/4, \Rightarrow N = 8$$

(a)

$$\begin{aligned} c_k &= \frac{1}{8}\sum_{n=0}^{7} x(n)e^{-j\pi kn/4} \\ x(n) &= \left\{\frac{11}{2}, 2+\frac{3}{4}\sqrt{2}, 1, 2-\frac{3}{4}\sqrt{2}, \frac{1}{2}, 2-\frac{3}{4}\sqrt{2}, 1, 2+\frac{3}{4}\sqrt{2}\right\} \\ \text{Hence, } c_0 &= 2, c_1 = c_7 = 1, c_2 = c_6 = \frac{1}{2}, c_3 = c_5 = \frac{1}{4}, c_4 = 0 \end{aligned}$$

(b)

$$\begin{aligned} P &= \sum_{i=0}^{7} |c(i)|^2 \\ &= 4 + 1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} \\ &= \frac{53}{8} \end{aligned}$$

4.19 Let $x(n)$ be a signal with Fourier transform as shown in Fig. P4.19. Determine and sketch the Fourier transforms of the following signals.

(a) $x_1(n) = x(n)\cos(\pi n/4)$

(b) $x_2(n) = x(n)\sin(\pi n/2)$

(c) $x_3(n) = x(n)\cos(\pi n/2)$

(d) $x_4(n) = x(n)\cos\pi n$

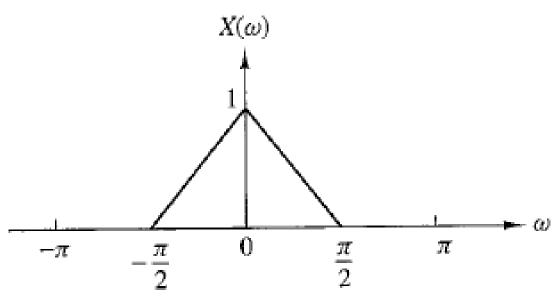


Figure P4.19

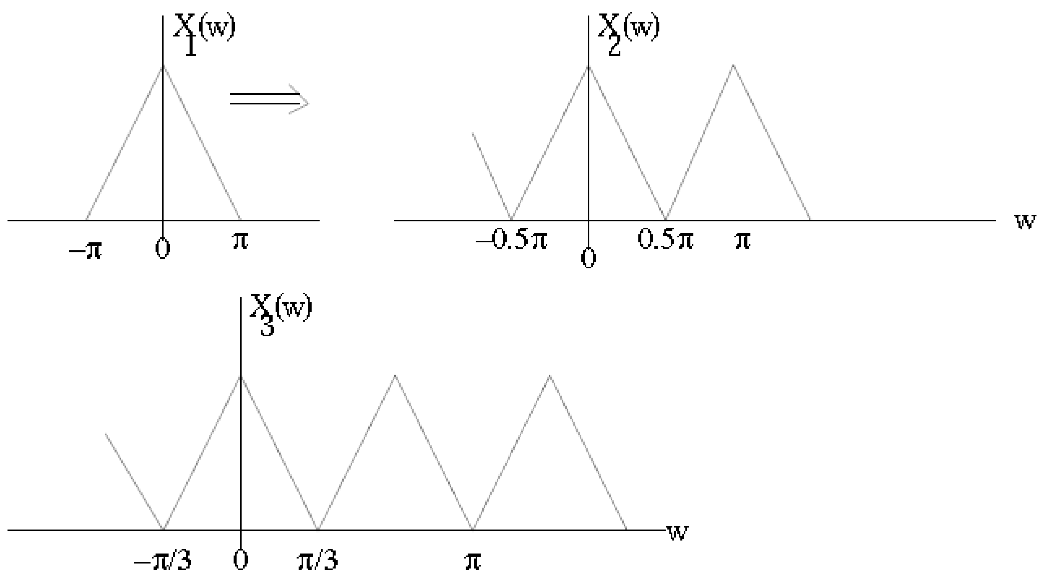
Note that these signal sequences are obtained by *amplitude modulation* of a carrier $\cos\omega_c n$ or $\sin\omega_c n$ by the sequence $x(n)$.

Answer:

(a)

$$x_1(n) = \frac{1}{2}(e^{j\pi n/4} + e^{-j\pi n/4})x(n)$$

$$X_1(w) = \frac{1}{2}\left[X(w - \frac{\pi}{4}) + X(w + \frac{\pi}{4})\right]$$



(b)

$$\begin{aligned} x_2(n) &= \frac{1}{2j}(e^{j\pi n/2} + e^{-j\pi n/2})x(n) \\ X_2(w) &= \frac{1}{2j}\left[X(w-\frac{\pi}{2}) + X(w+\frac{\pi}{2})\right] \end{aligned}$$

(c)

$$\begin{aligned} x_3(n) &= \frac{1}{2}(e^{j\pi n/2} + e^{-j\pi n/2})x(n) \\ X_3(w) &= \frac{1}{2}\left[X(w-\frac{\pi}{2}) + X(w+\frac{\pi}{2})\right] \end{aligned}$$

(d)

$$\begin{aligned} x_4(n) &= \frac{1}{2}(e^{j\pi n} + e^{-j\pi n})x(n) \\ X_4(w) &= \frac{1}{2}\left[X(w-\pi) + X(w+\pi)\right] \\ &= X(w-\pi) \end{aligned}$$

A

4.20 Consider an aperiodic signal $x(n)$ with Fourier transform $X(\omega)$. Show that the Fourier series coefficients C_k^y of the periodic signal

$$y(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

are given by

$$C_k^y = \frac{1}{N} X\left(\frac{2\pi}{N}k\right), \qquad k = 0, 1, \ldots, N-1$$

Answer:

$$c_k^y = \frac{1}{N}\sum_{n=0}^{N-1} y(n)e^{-j2\pi kn/N}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \\
&= \frac{1}{N}\sum_{l=-\infty}^{\infty}\sum_{m=-lN}^{N-1-lN} x(m)e^{-j2\pi k(m+lN)/N}
\end{aligned}
$$

$$
\text{But } \sum_{l=-\infty}^{\infty}\sum_{m=-lN}^{N-1-lN} x(m)e^{-jw(m+lN)} = X(w)
$$

$$
\text{Therefore, } c_k^y = \frac{1}{N}X(\frac{2\pi k}{N})
$$

7.2 Compute the eight-point circular convolution for the following sequences.

(a) $x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$x_2(n) = \sin\frac{3\pi}{8}n, \qquad 0 \le n \le 7$

(b) $x_1(n) = (\frac{1}{4})^n, \qquad 0 \le n \le 7$

$x_2(n) = \cos\frac{3\pi}{8}n, \qquad 0 \le n \le 7$

(c) Compute the DFT of the two circular convolution sequences using the DFTs of $x_1(n)$ and $x_2(n)$.

Answer:

(a)

$$
\begin{aligned}
\tilde{x}_2(l) &= x_2(l), \qquad 0 \le l \le N-1 \\
&= x_2(l+N), \qquad -(N-1) \le l \le -1 \\
\tilde{x}_2(l) &= sin(\frac{3\pi}{8}l), \qquad 0 \le l \le 7 \\
&= sin(\frac{3\pi}{8}(l+8)), \qquad -7 \le l \le -1 \\
&= sin(\frac{3\pi}{8}|l|), \qquad |l| \le 7 \\
\text{Therefore, } x_1(n) \, (8) \, x_2(n) &= \sum_{m=0}^{3} \tilde{x}_2(n-m) \\
&= sin(\frac{3\pi}{8}|n|) + sin(\frac{3\pi}{8}|n-1|) + \ldots + sin(\frac{3\pi}{8}|n-3|) \\
&= \{1.25, 2.55, 2.55, 1.25, 0.25, -1.06, -1.06, 0.25\}
\end{aligned}
$$

(b)

$$\begin{aligned}
\tilde{x}_2(n) &= cos(\frac{3\pi}{8}n), \quad 0 \le l \le 7 \\
&= -cos(\frac{3\pi}{8}n), \quad -7 \le l \le -1 \\
&= [2u(n)-1]\,cos(\frac{3\pi}{8}n), \quad |n| \le 7 \\
\text{Therefore, } x_1(n) \textcircled{8} x_2(n) &= \sum_{m=0}^{3}\left(\frac{1}{4}\right)^m \tilde{x}_2(n-m) \\
&= \{0.96, 0.62, -0.55, -1.06, -0.26, -0.86, 0.92, -0.15\}
\end{aligned}$$

(c)

$$\text{for (a) } X_1(k) = \sum_{n=0}^{7} x_1(n)e^{-j\frac{\pi}{4}kn}$$

$$\begin{aligned}
&= \{4, 1-j2.4142, 0, 1-j0.4142, 0, 1+j0.4142, 0, 1+j2.4142\} \\
\text{similarly,} & \\
X_2(k) &= \{1.4966, 2.8478, -2.4142, -0.8478, -0.6682, -0.8478, \\
&\quad -2.4142, 2.8478\} \\
\text{DFT of } x_1(n) \textcircled{8} x_2(n) &= X_1(k)X_2(k) \\
&= \{5.9864, 2.8478-j6.8751, 0, -0.8478+j0.3512, 0, \\
&\quad -0.8478-j0.3512, 0, 2.8478+j6.8751\} \\
\text{For sequences of part (b)} & \\
X_1(k) &= \{1.3333, 1.1612-j0.2493, 0.9412-j0.2353, 0.8310-j0.1248, \\
&\quad 0.8, 0.8310+j0.1248, 0.9412+j0.2353, 1.1612+j0.2493\} \\
X_2(k) &= \{1.0, 1.0+j2.1796, 1.0-j2.6131, 1.0-j0.6488, 1.0, \\
&\quad 1.0+j0.6488, 1.0+j2.6131, 1.0-j2.1796\} \\
\text{Consequently,} & \\
\text{DFT of } x_1(n) \textcircled{8} x_2(n) &= X_1(k)X_2(k) \\
&= \{1.3333, 1.7046+j2.2815, 0.3263-j2.6947, 0.75-j0.664, 0.8, \\
&\quad 0.75+j0.664, 0.3263+j2.6947, 1.7046-j2.2815\}
\end{aligned}$$

7.25 **(a)** Determine the Fourier transform $X(\omega)$ of the signal

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1, 0\}$$

(b) Compute the six-point DFT $V(k)$ of the signal

$$v(n) = \{3, 2, 1, 0, 1, 2\}$$

Answer:

(a)

$$
\begin{aligned}
X(w) &= \sum_{n=-\infty}^{\infty} x(n)e^{-jwn} \\
&= e^{j2w} + 2e^{jw} + 3 + 2e^{-jw} + e^{-j2w} \\
&= 3 + 2cos(2w) + 4cos(4w)
\end{aligned}
$$

(b)

$$
\begin{aligned}
V(k) &= \sum_{n=0}^{5} v(n)e^{-j\frac{2\pi}{6}nk} \\
&= 3 + 2e^{-j\frac{2\pi}{6}k} + e^{-j\frac{2\pi}{6}2k} + 0 + e^{-j\frac{2\pi}{6}4k} + e^{-j\frac{2\pi}{6}5k} \\
&= 3 + 4cos(\frac{\pi}{3}k) + 2cos(\frac{2\pi}{3}k)
\end{aligned}
$$