Differentiation of Vectors

Introduction
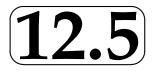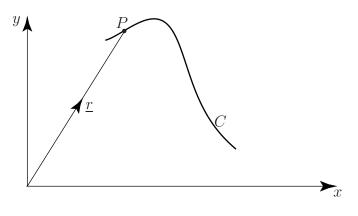
The area of mathematics known as vector calculus is used to model mathematically a vast range of engineering phenomena including electrostatics, electromagnetic fields, air flow around aircraft and heat flow in nuclear reactors. In this Section we introduce briefly the differential calculus of vectors.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of vectors, in Cartesian form

- be able to calculate the scalar and vector products of two vectors

- be able to differentiate and integrate scalar functions

Learning Outcomes

On completion you should be able to . . .

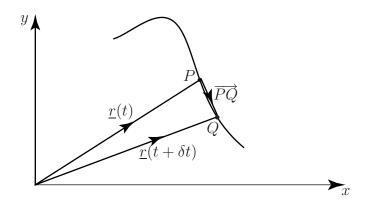- differentiate vectors

1. Differentiation of vectors

Consider Figure 31.

Figure 31

If $\underline{r}$ represents the **position vector** of an object which is moving along a curve C, then the position vector will be dependent upon the time, t. We write $\underline{r} = \underline{r}(t)$ to show the dependence upon time. Suppose that the object is at the point P, with position vector $\underline{r}$ at time t and at the point Q, with position vector $\underline{r}(t + \delta t)$, at the later time $t + \delta t$, as shown in Figure 32.

Figure 32

Then $\overrightarrow{PQ}$ represents the **displacement vector** of the object during the interval of time δt. The length of the displacement vector represents the distance travelled, and its direction gives the direction of motion. The average velocity during the time from t to $t + \delta t$ is defined as the displacement vector divided by the time interval δt, that is,

$$\text{average velocity} = \frac{\overrightarrow{PQ}}{\delta t} = \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t}$$

If we now take the limit as the interval of time δt tends to zero then the expression on the right hand side is the **derivative** of $\underline{r}$ with respect to t. Not surprisingly we refer to this derivative as **the instantaneous velocity**, $\underline{v}$. By its very construction we see that the velocity vector is always tangential to the curve as the object moves along it. We have:

$$\underline{v} = \lim_{\delta t \to 0} \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t} = \frac{d\underline{r}}{dt}$$

Now, since the x and y coordinates of the object depend upon time, we can write the position vector $\underline{r}$ in Cartesian coordinates as:

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$$

Therefore,

$$\underline{r}(t + \delta t) = x(t + \delta t)\underline{i} + y(t + \delta t)\underline{j}$$

so that,

$$
\begin{aligned}
\underline{v}(t) &= \lim_{\delta t \to 0} \frac{x(t + \delta t)\underline{i} + y(t + \delta t)\underline{j} - x(t)\underline{i} - y(t)\underline{j}}{\delta t} \\
&= \lim_{\delta t \to 0} \left\{ \frac{x(t + \delta t) - x(t)}{\delta t}\underline{i} + \frac{y(t + \delta t) - y(t)}{\delta t}\underline{j} \right\} \\
&= \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j}
\end{aligned}
$$

This is often abbreviated to $\underline{v} = \underline{\dot{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$, using notation for derivatives with respect to time. So we see that the velocity vector is the derivative of the position vector with respect to time. This result generalizes in an obvious way to three dimensions as summarized in the following Key Point.

Key Point 8

Given $\qquad \underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$
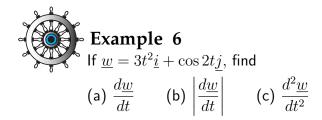
then the velocity vector is

$$\underline{v} = \underline{\dot{r}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$$

The magnitude of the velocity vector gives the speed of the object.

We can define the acceleration vector in a similar way, as the rate of change (i.e. the derivative) of the velocity with respect to the time:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \underline{\ddot{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$
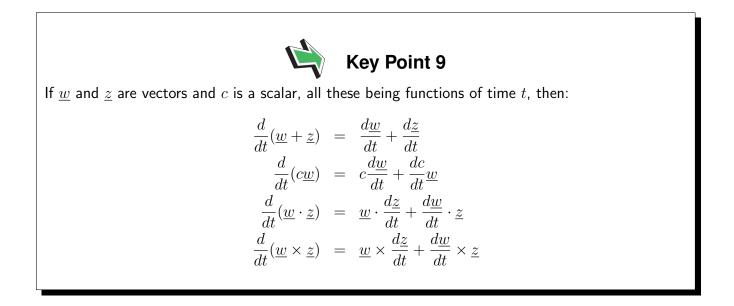
Example 6

If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, find

(a) $\dfrac{d\underline{w}}{dt}$ (b) $\left|\dfrac{d\underline{w}}{dt}\right|$ (c) $\dfrac{d^2\underline{w}}{dt^2}$

Solution

(a) If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, then differentiation with respect to t yields: $\dfrac{d\underline{w}}{dt} = 6t\underline{i} - 2\sin 2t\underline{j}$

(b) $\left|\dfrac{d\underline{w}}{dt}\right| = \sqrt{(6t)^2 + (-2\sin 2t)^2} = \sqrt{36t^2 + 4\sin^2 2t}$

(c) $\dfrac{d^2\underline{w}}{dt^2} = 6\underline{i} - 4\cos 2t\underline{j}$

It is possible to differentiate more complicated expressions involving vectors provided certain rules are adhered to as summarized in the following Key Point.

Key Point 9

If $\underline{w}$ and $\underline{z}$ are vectors and c is a scalar, all these being functions of time t, then:

$$\frac{d}{dt}(\underline{w} + \underline{z}) = \frac{d\underline{w}}{dt} + \frac{d\underline{z}}{dt}$$

$$\frac{d}{dt}(c\underline{w}) = c\frac{d\underline{w}}{dt} + \frac{dc}{dt}\underline{w}$$

$$\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$$

$$\frac{d}{dt}(\underline{w} \times \underline{z}) = \underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z}$$

Example 7

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$$

Solution

$\underline{w} \cdot \underline{z} = (3t\underline{i} - t^2\underline{j}) \cdot (2t^2\underline{i} + 3\underline{j}) = 6t^3 - 3t^2.$

Therefore $\quad \dfrac{d}{dt}(\underline{w} \cdot \underline{z}) = 18t^2 - 6t$ $\hfill (1)$

Also $\quad \dfrac{d\underline{w}}{dt} = 3\underline{i} - 2t\underline{j} \quad$ and $\quad \dfrac{d\underline{z}}{dt} = 4t\underline{i}$

so $\quad \underline{w} \cdot \dfrac{d\underline{z}}{dt} + \underline{z} \cdot \dfrac{d\underline{w}}{dt}$
$\begin{aligned} &= (3t\underline{i} - t^2\underline{j}) \cdot (4t\underline{i}) + (2t^2\underline{i} + 3\underline{j}) \cdot (3\underline{i} - 2t\underline{j}) \\ &= 12t^2 + 6t^2 - 6t \\ &= 18t^2 - 6t \end{aligned}$ $\hfill (2)$

We have verified $\dfrac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \dfrac{d\underline{z}}{dt} + \dfrac{d\underline{w}}{dt} \cdot \underline{z}$ since (1) is the same as (2).

Example 8

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w} \times \underline{z}) = \underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z}$$

Solution

$\underline{w} \times \underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 2t^2 & 3 & 0 \end{vmatrix} = (9t + 2t^4)\underline{k} \quad$ implying $\quad \dfrac{d}{dt}(\underline{w} \times \underline{z}) = (9 + 8t^3)\underline{k}$ $\hfill (1)$

$\underline{w} \times \dfrac{d\underline{z}}{dt} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 4t & 0 & 0 \end{vmatrix} = 4t^3\underline{k}$ $\hfill (2)$

$\dfrac{d\underline{w}}{dt} \times \underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2t & 0 \\ 2t^2 & 3 & 0 \end{vmatrix} = (9 + 4t^3)\underline{k}$ $\hfill (3)$

We can see that (1) is the same as (2) + (3) as required.

Exercises

1. If $r = 3t\underline{i} + 2t^2\underline{j} + t^3\underline{k}$, find   (a) $\dfrac{d\underline{r}}{dt}$ (b) $\dfrac{d^2\underline{r}}{dt^2}$

2. Given $\underline{B} = te^{-t}\,\underline{i} + \cos t\,\underline{j}$ find   (a) $\dfrac{d\underline{B}}{dt}$ (b) $\dfrac{d^2\underline{B}}{dt^2}$

3. If $\underline{r} = 4t^2\underline{i} + 2t\underline{j} - 7\underline{k}$ evaluate $\underline{r}$ and $\dfrac{d\underline{r}}{dt}$ when $t = 1$.

4. If $\underline{w} = t^3\underline{i} - 7t\underline{k}$ and $\underline{z} = (2 + t)\underline{i} + t^2\underline{j} - 2\underline{k}$

     (a) find $\underline{w} \cdot \underline{z}$,   (b) find $\dfrac{d\underline{w}}{dt}$,   (c) find $\dfrac{d\underline{z}}{dt}$,   (d) show that $\dfrac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \dfrac{d\underline{z}}{dt} + \dfrac{d\underline{w}}{dt} \cdot \underline{z}$

5. Given $\underline{r} = \sin t\,\underline{i} + \cos t\,\underline{j}$

     (a) find $\dot{\underline{r}}$,   (b) find $\ddot{\underline{r}}$,   (c) find $|\underline{r}|$

     (d) Show that the position vector $\underline{r}$ and velocity vector $\dot{\underline{r}}$ are perpendicular.

Answers

1. (a) $3\underline{i} + 4t\underline{j} + 3t^2\underline{k}$ (b) $4\underline{j} + 6t\underline{k}$

2. (a) $(-te^{-t} + e^{-t})\underline{i} - \sin t\underline{j}$ (b) $e^{-t}(t - 2)\underline{i} - \cos t\underline{j}$

3. $4\underline{i} + 2\underline{j} - 7\underline{k}, \ 8\underline{i} + 2\underline{j}$

4. (a) $t(t^3 + 2t^2 + 14)$ (b) $3t^2\underline{i} - 7\underline{k}$ (c) $\underline{i} + 2t\underline{j}$

5. (a) $\cos t\underline{i} - \sin t\underline{j}$ (b) $-\sin t\underline{i} - \cos t\underline{j}$ (c) 1 (d) Follows by showing $\underline{r} \cdot \dot{\underline{r}} = 0$.