WAVES AND OSCILLATIONS

(Second Revised Edition)

N. Subrahmanyam, Brij Lal

The present edition of the book has been thoroughly revised and enlarged. Many new topics have been included in the text.

The subject matter is divided into twelve chapters. Each chapter is Self-contained and is treated in a comprehensive way. Using the S.I system of units. Harmonic oscillators, linearity and superposition principle, oscillations with one degree of freedom, resonance and sharpness of resonance, quality factor, Doppler effect in sound and light. medical applications of ultrasonic, acoustic intensity, acoustic measurements, wave velocity and group velocity, Maxwell's equations, propagation of electromagnetic waves in isotropic media, De Broglie waves, Hersenburg's uncertainty principle and special theory of relativity are some of the important topics which have been given special attention.

Solved numerical problems (in S.I. units), wherever necessary, are given in the text solved numerical examples of important topics have also been included so as to provide better understanding and practice to the students.

The book is intended to be a textbook for the undergraduate students of Indian Universities.

S. SUBRAHMANYAM in Reader in the Department of Physics, Kirori Mal College, University of Delhi. He has a teaching experience of over forty years and has had numerous Research Papers published in outstanding Journals.

BRIJ LAL is Reader in the Department of Physics, Hindu College, University of Delhi. He has been teaching in this college for the last thirty three years and has written several textbooks in collaboration with N. Subramanyam.

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CHAPTER 1

Harmonic Oscillators

1.1. Introduction

In every day life we come across numerous things that move. These motions are of two types; *viz.* *(i)* the motion in which the body moves about a mean position *i.e.* a fixed point and *(ii)* the motion in which the body moves from one place to the other with respect of time. The first type of motion of a body about a mean position is called oscillatory motion. A moving train, flying aeroplane, moving ball etc., correspond to the second type of motion. Examples of oscillatory motion are : an oscillating pendulum, vibrations of a stretched string movement of water in a cup, vibration of electrons, movement of light in a laser beam etc.

Sometimes both the types of motion are exhibited in the same phenomenon depending on our point of view. The sea waves appear to move towards the beach but the water moves up and down about the mean position. When a stretched rope is displaced, the displacement pulse travels from one end to the other but the material of the rope vibrates about the mean position without travelling forward.

1.2. Simple Harmonic Motion

Let P be a particle moving on the circumference of a circle of radius a with a uniform velocity v (Fig.1.1). Let ω be the uniform angular velocity of the particle $(v=a\omega)$. The circle along which P moves is called the **circle of reference.** As the particle P moves round the circle continuously with uniform velocity, the foot of the perpendicular M, vibrates along the diameter YY'. If the motion of P is uniform, then the motion of M is periodic *i.e.*, it takes the same time to vibrate once between the

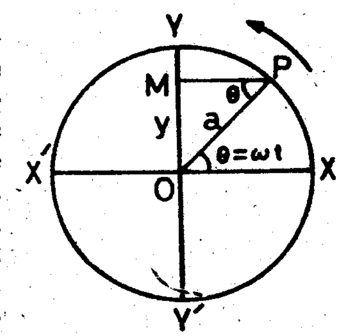


Fig. 1.1.

points Y and Y'. At any instant the distant of M from the centre O of the circle is called the **displacement**. If the particle moved from X to P in time t, then $\angle POX = \angle MPO = \theta = \omega t$.

From the Δ MPO,

$$\sin\theta = \sin\omega t = \frac{OM}{a}$$

or $$OM = y = a\sin\omega t$$

OM is called the displacement of the vibrating particle. The displacement of a vibrating particle at any instant can be defined as its distance from the mean position of rest. The maximum displacement of a vibrating particle is called its **amplitude**.

$$\text{Displacement} = y = a\sin\omega t \qquad \ldots(1)$$

The rate of change of displacement is called the **velocity** of the vibrating particle.

$$\therefore \quad \text{Velocity} = \frac{dy}{dt} = +a\omega\cos\omega t \qquad \ldots(2)$$

The rate of change of velocity of a vibrating particle is called its **acceleration**.

$$\therefore \quad \text{Acceleration} = \text{Rate of change of velocity}$$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d^2y}{dt^2} = -a\omega^2\sin\omega t$$

$$= -\omega^2 . a\sin\omega t = -\omega^2 y \qquad \ldots(3)$$

Angle ωt	Position of the vibrating particle M	Displacement $y = a\sin\omega t$	Velocity $\frac{dy}{dt} = a\omega\cos\omega t$	Acceleration $\frac{d^2y}{dt^2} = -a\omega^2\sin\omega t$
O	O	Zero	$+a\omega$	Zero
$\frac{\pi}{2}$	Y	$+a$	Zero	$-a\omega^2$
π	O	Zero	$-a\omega$	Zero
$\frac{3\pi}{2}$	Y'	$-a$	Zero	$+a\omega^2$
2π	O	Zero	$+a\omega$	Zero

The changes in the displacement, velocity and acceleration of a vibrating particle in one complete vibration are given in the table.

Oscillatory behaviour. At the extreme positions, when y is maximum, dy/dt is zero. The acceleration d^2y/dt^2 is maximum and directed towards the mean position. This return force induces a negative velocity. When the displacement y is zero, the velocity dy/dt is maximum and is –ve. When the displacement is negative maximum, the velocity dy/dt is zero and the acceleration is maximum in the positive direction. This return force again induces a velocity in the positive direction which becomes positive maximum when the displacement is zero. The particle overshoots the mean position due to its velocity. The process repeats itself periodically. Thus the system oscillates. In this process, displacement y, velocity dy/dt and acceleration d^2y/dt^2 continuously change with respect to time.

Thus, the velocity of the vibrating particle is maximum (in the direction OY or OY') at the mean position of rest and zero at the maximum positions of vibration. The acceleration of the vibrating particle is zero at the mean position of rest and maximum at the maximum positions of vibration. The acceleration is always directed towards the mean position of rest and is directly proportional to the displacement of the vibrating particle. This type of motion where the acceleration is directed towards a fixed point (the mean position of rest) and is proportional to the displacement of the vibrating particle is called **simple harmonic motion.**

Further,

$$\text{Acceleration} \quad = \frac{d^2y}{dt^2} = -\omega^2 y$$

$$= -\omega^2 \times \text{displacement}$$

$$\text{Numerically} \quad \omega^2 = \frac{\text{Acceleration}}{\text{Displacement}}$$

or $$\omega = 2\pi n = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or $$\frac{2\pi}{T} = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or $$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

$$= 2\pi\sqrt{K}$$

Thus, in general, the time period of a particle vibrating simple harmonically is given by $T = 2\pi\sqrt{K}$ where K is the displacement per unit acceleration.

If the particle P revolves round the circle, n times per second, then the angular velocity ω is given by

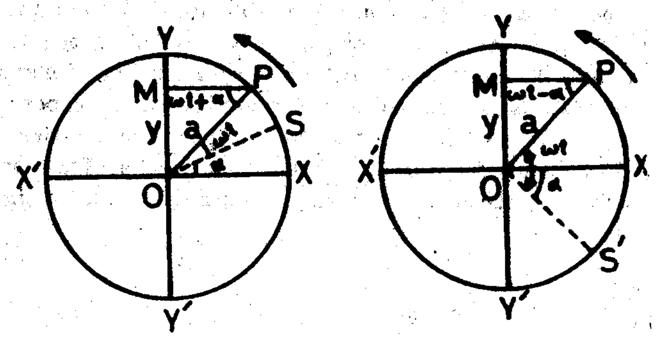
$$\omega = 2\pi n = \frac{2\pi}{T}$$

$$\left(\because \quad n = \frac{1}{T} \text{ where } T \text{ is the time period} \right)$$

or
$$y = a \sin 2\pi\, nt = a \sin 2\pi\, \frac{t}{T}$$

On the other hand, if the time is counted [(Fig. 1.2 (*i*)] from the instant P is at S ($\angle SOX = \alpha$) then the displacement

$$y = a \sin (\omega t + \alpha)$$
$$= a \sin \left(\frac{2\pi t}{T} + \alpha \right)$$



(*i*) (*ii*)

Fig. 1.2.

If the time is counted from the instant P is at S' [Fig. 1.2 (*ii*)], then

$$y = a \sin (\omega t - \alpha)$$
$$= a \sin \left(\frac{2\pi t}{T} - \alpha \right)$$

Phase of the vibrating particle. (*i*) The phase of a vibrating particle is defined as the ratio of the displacement of the vibrating article at any instant to the amplitude of the virbrating particle *(y/a)* or (*ii*) it is also defined as the fraction of the time interval that has lapsed since the particle crossed the mean position of rest in the positive direction or *(iii)* it is also equal to the angle swept by the radius vector since the virbrating particle last crossed its mean position of rest *e.g.*, in the above equations ωt, $(\omega t + \alpha)$ or $(\omega t - a)$ are called phase angles. The initial phase angle when $t = 0$, is called the epoch. Thus α is called the **epoch** in the above expressions.

1.3. Differential Equation of SHM

For a particle vibrating simple harmonically, the general equation of displacement is,

$$y = a \sin (\omega t + \alpha) \qquad \ldots (1)$$

Here y is displacement and a is the amplitude and α is epoch of the vibrating particle.

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = a\, \omega \cos (\omega t + \alpha) \qquad \ldots (2)$$

Here dy/dt represents the velocity of the vibrating particle.

Differentiating equation (2) with respect to time

$$\frac{d^2 y}{dt^2} = - a\, \omega^2 \sin (\omega t + \alpha)$$

But $\quad a \sin (\omega t + \alpha) = y$

$$\therefore \quad \frac{d^2 y}{dt^2} = - \omega^2 y$$

or
$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \qquad \ldots (3)$$

Here $d^2 y/dt^2$ represents the acceleration of the particle. Equation (3) represents the differential equation of simple harmonic motion.

It also shows that in any phenomenon where an equation similar to equation (3) is obtained, the body executes simple harmonic motion. The general solution of equation (3) is

$$y = a \sin (\omega t + \alpha).$$

Also the time period of a vibrating particle can be calculated from equation (3).

Numerically
$$\omega = \sqrt{\frac{d^2 y/dt^2}{y}}$$

or
$$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or
$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

1.4. Graphical Representation of SHM

Let P be a particle moving on the circumference of a circle of radius a. The foot of the perpendicular vibrates on the diameter YY'.

$$y = a \sin \omega t = a \sin 2\pi \frac{t}{T}$$

The displacement graph is a sine curve represented by *ABCDE* (Fig. 1.3).

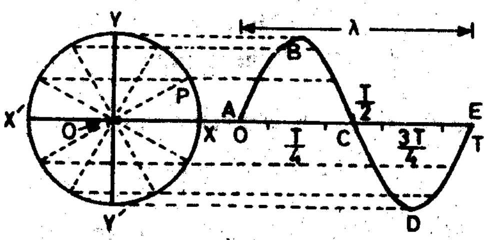


Fig. 1.3. Displacement Time Curve.

The motion of the particle *M* is simple harmonic.
The velocity of a particle moving with simple harmonic motion is

$$v = \frac{dy}{dt} = + a\,\omega \cos \omega t$$

The velocity–time graph is shown in Fig. 1.4. It is a cosine curve.

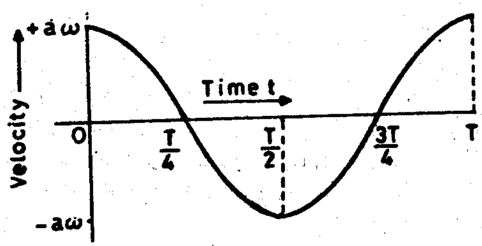


Fig. 1.4. Velocity — Time Curve.

The acceleration of a particle moving with simple harmonic motion is

$$\frac{d^2 y}{dt^2} = - a\,\omega^2 \sin \omega t$$

The acceleration—time graph is shown in Fig. 1.5. It is a negative sine curve.

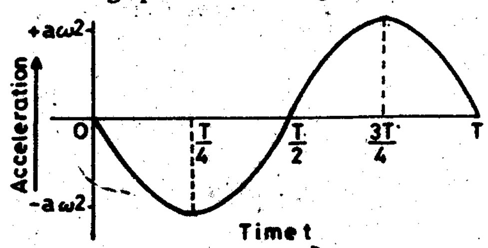


Fig. 1.5. Acceleration — Time Curve.

1.5. Average Kinetic Energy of a Vibrating Particle

The displacement of a vibrating particle is given by

$$y = a \sin (\omega t + \alpha)$$

$$v = \frac{dy}{dt} = a\omega \cos (\omega t + \alpha).$$

If *m* is the mass of the vibrating particle, the kinetic energy at any instant

$$= \frac{1}{2} m v^2 = \frac{1}{2} m \,.\, a^2 \omega^2 \cos^2 (\omega t + \alpha)\,.$$

The average kinetic energy of the particle in one complete vibration

$$= \frac{1}{T} \int_0^T \frac{1}{2} m a^2 \omega^2 \cos^2 (\omega t + \alpha)\, dt$$

$$= \frac{1}{T} \cdot \frac{m a^2 \omega^2}{4} \int_0^T 2 \cos^2 (\omega t + \alpha)\, dt$$

$$= \frac{m a^2 \omega^2}{4T} \int_0^T [1 + \cos 2 (\omega t + \alpha)]\, dt$$

$$= \frac{m a^2 \omega^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2 (\omega t + \alpha)\, dt \right]$$

But $$\int_0^T \cos 2 (\omega t + \alpha)\, dt = 0$$

$$\therefore \quad \text{Average K.E.} = \frac{m a^2 \omega^2}{4T} \cdot T + 0$$

$$= \frac{m a^2 \omega^2}{4} = \frac{m a^2 (4\pi^2 n^2)}{4}$$

$$= \pi^2 m a^2 n^2$$

where *m* is the mass of the vibrating particle, *a* is the amplitude of vibration and *n* is the frequency of vibration. Also, the average kinetic energy of a vibrating particle is directly proportional to the square of the amplitude.

1.6. Total Energy of a Vibrating Particle

$$y = a \sin (\omega t + \alpha)$$

$$\sin (\omega t + \alpha) = \frac{y}{a}$$

$$\cos (\omega t + \alpha) = \sqrt{1 - \frac{y^2}{a^2}} = \sqrt{\frac{a^2 - y^2}{a^2}}$$

$$= \frac{\sqrt{a^2 - y^2}}{a}$$

Velocity $\quad v = a\,\omega \cos \omega t = \frac{a\omega\sqrt{a^2 - y^2}}{a}$

$$= \omega\sqrt{(a^2 - y^2)}$$

$\therefore$ The kinetic energy of the particle at the instant the displacement is y,

$$= \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \,.\, \omega^2 (a^2 - y^2)$$

Potential energy of the vibrating particle is the amount of work done in overcoming the force through a distance y.

Acceleration $\quad = -\omega^2 y$

Force $\quad = -m\omega^2 y$

(The –ve sign shows that the direction of the acceleration and force are opposite to the direction of motion of the virbrating particle.)

$$\therefore \quad \text{P.E.} = \int_0^y m \,.\, \omega^2 y \,.\, dy$$

$$= m\omega^2 \cdot \frac{y^2}{2} = \frac{1}{2} m\omega^2 y^2 .$$

Total energy of the particle at the instant the displacement is y

$$= \text{K.E} + \text{P.E.}$$

$$= \frac{1}{2} m\omega^2 (a^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

$$= \frac{1}{2} m\omega^2 \,.\, a^2$$

$$= \frac{1}{2} m (2\pi n)^2 a^2$$

$$= 2\pi^2 m a^2 n^2.$$

As the average kinetic energy of the vibrating particle $= \pi^2 ma^2n^2$, the average potential energy $= \pi^2 ma^2n^2$. The total energy at any instant is a constant.

Example 1.1. *For a particle vibrating simple harmonically, the displacement is 12 cm at the instant the velocity is 5 cm/s and the displacement is 5 cm at the instant the velocity is 12 cm/s. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

The velocity of a particle executing SHM,

$$v = \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega\sqrt{a^2 - y_1{}^2}$$

Here $\quad v_1 = 5$ cm/s, $y_1 = 12$ cm.

$$5 = \omega\sqrt{a^2 - 144} \qquad \ldots(1)$$

In the second case

$$v_2 = \omega\sqrt{a^2 - y_2^2}$$

Here $\quad v_2 = 12$ cm/s, $y_2 = 5$ cm

$$12 = \omega\sqrt{a^2 - 25} \qquad \ldots(2)$$

Dividing (2) by (1) and squaring

$$\frac{144}{25} = \frac{a^2 - 25}{a^2 - 144}$$

$$a = 13 \text{ cm}$$

The amplitude is **13 cm.**

Substituting the value of $a = 13$ cm in equation (1)

$$5 = \omega\sqrt{(13)^2 - 144}$$

or $\quad \omega = 1$ radian/s

The frequency $n = \frac{\omega}{2\pi} = \frac{1}{2\pi}$ hertz

Time period $\quad T = \frac{1}{n} = 2\pi$ seconds.

Example 1.2. *Show that for a particle executing simple harmonic motion, its velocity at any instant is*

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

The displacement,

$$y = a \sin \omega t \qquad \ldots(1)$$

The velocity at any instant is,

$$\frac{dy}{dt} = a\,\omega \cos \omega t \qquad \ldots(2)$$

From equation (1)

$$\sin \omega t = \frac{y}{a}$$

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t}$$

$$\cos \omega t = \sqrt{1 - \frac{y_1^2}{a^2}}$$

$$\therefore \quad \frac{dy}{dt} = a\,\omega \sqrt{1 - \frac{y^2}{a^2}}$$

or $$\frac{dy}{dt} = \omega \sqrt{a^2 - y^2}.$$

Example 1.3. *For a particle vibrating simple harmonically the displacement is 8 cm at the instant the velocity is 6 cm/s and the displacement is 6 cm at the instant the velocity is 8 cm/s. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

The velocity of a particle executing SHM,

$$v = \frac{dy}{dt} = \omega \sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega \sqrt{a^2 - y_1^2}.$$

Here $v_1 = 6$ cm/s, $y_1 = 8$ cm

$$\therefore \quad 6 = \omega \sqrt{a^2 - 64} \qquad \ldots (1)$$

In the second case,

$$v_2 = \omega \sqrt{a^2 - y_2^2}$$

Here $v_2 = 8$ cm/s $y_2 = 6$ cm ...(2)

$$\therefore \quad 8 = \omega \sqrt{a^2 - 36}$$

Dividing (2) by (1) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

$$a = \mathbf{10\ cm.}$$

The amplitude of vibration = **10 cm**

Substituting the value of

$a = 10$ cm in equation (1)

$$6 = \omega \sqrt{100 - 64}$$

$$\omega = 1 \text{ radian/s}$$

Frequency $n = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}$ **hertz**

Time period $T = \dfrac{1}{n} = 2\pi$ **seconds.**

Example 1.4. *The motion of a particle in simple harmonic motion is given by $x = a \sin \omega t$. If it has a speed u when the displacement is x_1 and speed v when the displacement is x_2, show that the amplitude of the motion is*

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}}$$

(Utkal, 1989)

Here $x = a \sin \omega t$

$$u = \frac{dx_1}{dt} = \omega \sqrt{a^2 - x_1^2} \qquad \ldots (i)$$

and $$v = \frac{dx_2}{dt} = \omega \sqrt{a^2 - x_2^2} \qquad \ldots (ii)$$

Squaring and dividing

$$\frac{u^2}{v^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2}$$

$$u^2 a^2 - u^2 x_2^2 = v^2 a^2 - v^2 x_1^2$$

$$a^2 [v^2 - u^2] = v^2 x_1^2 - u^2 x_2^2$$

$$a = \left[\frac{v^2 x_1^2 - u^2 x_2^2}{v^2 - u^2}\right]^{\frac{1}{2}} \qquad \ldots (iii)$$

Example 1.5. *Show that for a particle executing SHM, the instantaneous velocity is $\omega \sqrt{a^2 - y^2}$ and instantaneous acceleration is $-\omega^2 y$.*

For a particle executing SHM,

$$y = a \sin (\omega t + \alpha) \qquad \ldots (1)$$

The instantaneous velocity,

$$v = \frac{dy}{dt} = + a\,\omega \cos (\omega t + \alpha) \qquad \ldots (2)$$

From equation (1)

$$\sin (\omega t + \alpha) = \frac{y}{a}$$

$$\cos(\omega t+\alpha)=\sqrt{1-\sin^2(\omega t+\alpha)}$$

$$=\sqrt{1-\frac{y^2}{a^2}}$$

$$\therefore \quad v=a\omega\sqrt{1-\frac{y^2}{a^2}}$$

$$v=\omega\sqrt{a^2-y^2} \quad \ldots(3)$$

The instantaneous acceleration,

$$\frac{d^2y}{dt^2}=\frac{dv}{dt}=-a\omega^2\sin(\omega t+\alpha)$$

$$=-\omega^2[a\sin(\omega t+\alpha)]$$

$$=-\omega^2 y \quad \ldots(4)$$

Example 1.6. *A particle performs simple harmonic motion given by the equation*

$$y=20\sin[\omega t+\alpha]$$

If the time period is 30 seconds and the particle has a displacement of 10 cm *at $t=0$, find (i) epoch ; (ii) the phase angle at $t=5$ seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.*

Here $y=20\sin(\omega t+\alpha)$

$T=30$ s

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{30}=\frac{\pi}{15}\text{ radians/s}$$

(i) At $t=0,\quad y=10$ cm

$$\therefore \quad 10=20\sin\left(\frac{\pi}{15}\times 0+\alpha\right)$$

or $\sin\alpha=0{\cdot}5$

or $\sin\alpha=\frac{\pi}{6}$ radian

(ii) At $t=5$ s,

The phase angle $=(\omega t+\alpha)$

$$=\left(\frac{\pi}{15}\times 5+\frac{\pi}{6}\right)$$

$$=\frac{\pi}{2}$$

(iii) At $t=0$

The phase angle $\theta_1=\frac{\pi}{6}$

At $t=15$

The phase angle $\theta_2=(\omega t+\alpha)$

$$\theta_2=\left(\frac{\pi}{15}\times 15+\frac{\pi}{6}\right)$$

$$\theta_2=\frac{7\pi}{6}$$

The phase difference $\theta_2-\theta_1=\frac{7\pi}{6}-\frac{\pi}{6}=\pi$ *radians.*

Example 1.7. *A particle executes simple harmonic motion given by the equation*

$$y=12\sin\left(\frac{2\pi t}{10}+\frac{\pi}{4}\right)$$

Calculate (i) amplitude, (ii) frequency, (iii) epoch, (iv) displacement at $t=1{\cdot}25$ s, *(v) velocity at $t=2{\cdot}5$*s *and (vi) acceleration at $t=5$* s.

Here
$$y=12\sin\left(\frac{2\pi t}{10}+\frac{\pi}{4}\right) \quad \ldots(1)$$

The displacement equation is

$$y=a\sin(\omega t+\alpha) \quad \ldots(2)$$

Comparing equations (1) and (2)

(i) Amplitude $a=12$ units

(ii) $\omega=\frac{2\pi}{10}$

Frequency $n=\frac{\omega}{2\pi}=\frac{1}{10}=0{\cdot}1$ **hertz**

(iii) Epoch $\alpha=\frac{\pi}{4}$

(iv) When $t=1{\cdot}25$ s

$$y=12\sin\left(\frac{2\pi\times 1{\cdot}25}{10}+\frac{\pi}{4}\right)$$

$$y=12\sin\pi/2$$

or $y=12$ **units**

(v) At $t = 2{\cdot}5$ s

$$\text{Velocity} = \frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$$

$$\frac{dy}{dt} = 12 \times \frac{2\pi}{10}\cos\left[\frac{2\pi}{10} \times 2{\cdot}5 + \frac{\pi}{4}\right]$$

$$\frac{dy}{dt} = -\mathbf{5{\cdot}552\ units.}$$

The –ve sign shows that the velocity is directed towards the mean position.

(vi) At $t = 5$ s

$$\text{Acceleration} = \frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t + \alpha)$$

$$\frac{d^2y}{dt^2} = -12 \times \left(\frac{2\pi}{10}\right)^2 \sin\left(\frac{2\pi}{10} \times 5 + \frac{\pi}{4}\right)$$

$$= -0{\cdot}48\,\pi^2 \sin\left(\pi + \frac{\pi}{4}\right)$$

$$= \mathbf{3{\cdot}35\ units.}$$

Example 1.8. *A simple harmonic motion is represented by the equation*

$$y = 10\sin\left(10\,t - \frac{\pi}{6}\right)$$

where y is measured in metres, *t in* seconds *and the phase angle in* radians. *Calculate :*

(i) *the frequency,*
(ii) *the time period,*
(iii *the maximum displacement,*
(iv) *the maximum velocity,*
(v) *the maximum acceleration, and*
(vi) *displacement, velocity and acceleration at time, t=0 and t=1* second.

Here $y = 10\sin\left(10\,t - \frac{\pi}{6}\right)$ ... (1)

The displacement equation is

$y = a\sin(\omega t + \alpha)$ ... (2)

(i) From (1) and (2)

$\omega = 10$

But $\omega = 2\pi n$

$\therefore$ $2\pi n = 10$

or $n = \frac{10}{2\pi}$ hertz

$n = \mathbf{1{\cdot}6\ hertz.}$

(ii) Time period,

$T = \frac{1}{n} = \frac{2\pi}{10}$

or $T = 0{\cdot}63$ s

(iii) Maximum displacement,

$a = \mathbf{10\ m}$

(iv) Velocity, $\frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$

Therefore, maximum velocity

$\frac{dy}{dt} = a\omega$

But $a = 10$ m

and $\omega = 10$

$\therefore$ $\frac{dy}{dt} = 10 \times 10 = \mathbf{100\ m/s}$

(v) Acceleration, $\frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t + \alpha)$

Maximum acceleration

$$\frac{d^2y}{dt^2} = -a\omega^2$$

or $$\frac{d^2y}{dt^2} = -10 \times (10)^2 = \mathbf{-1{,}000\ m/s^2}$$

– ve sign shows that the acceleration is directed towards the mean position.

(vi) From equation (1)

(a) At $t = 0$

$y = 10\sin\left(-\frac{\pi}{6}\right)$

$y = \mathbf{-5\ m}$

$\frac{dy}{dt} = a\omega\cos(\omega t + \alpha)$

(*ii*) In one minute, three cycles are completed.

Therefore, work done in one minute

$$= 3 \times 0{\cdot}5\,\pi \text{ joules}$$

$$= \mathbf{1{\cdot}5\,\pi \text{ joules.}}$$

Example 1.11. *Show that the mean kinetic and potential energies of non-dissipative simple harmonic vibrating systems are equal.*

For free vibration in the absence of damping, the displacement at any instant is given by

$$y = a \sin \omega t$$

$$\frac{dy}{dt} = a\,\omega \cos \omega t$$

Kinetic energy $= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$

$$\text{K.E.} = \frac{1}{2} m\,(a^2 \omega^2 \cos^2 \omega t)$$

or

$$\text{K.E.} = \frac{1}{2} k\,a^2 \cos^2 \omega t \qquad \ldots (1)$$

Here $k = m\,\omega^2$

or

$$\omega^2 = \frac{k}{m}$$

Here k is the force per unit displacement

Potential energy $= \frac{1}{2}\, k\,y^2$

$$\text{P.E.} = \frac{1}{2}\, k\,a^2 \sin^2 \omega t \qquad \ldots (2)$$

Total kinetic energy for one complete cycle

$$= \int_0^T \frac{1}{2} k\,a^2 \cos^2 \omega t\; dt$$

$$= \frac{1}{4} ka^2\,T \qquad \ldots (3)$$

Total potential energy for one complete cycle

$$= \int_0^T \frac{1}{2} k\,a^2 \sin^2 \omega t\; dt$$

$$= \frac{1}{4}\, k\,a^2\,T \qquad \ldots (4)$$

Hence the mean potential and kinetic energies are equal.

Example 1.12. *Write down the equation for a wave travelling along the negative Z direction and having an amplitude 0·01* m, *frequency 550* Hz *and speed 330* m/s. *How would the equation change if a wave with the same parameters was travelling along the positive Z direction. Justify your answer.* **[IAS]**

Here $y = a \sin \frac{2\pi}{\lambda}(vt - z)$

The wave is travelling in the positive z-direction

Here

$$a = 0{\cdot}01 \text{ m}$$

$$\nu = 550 \text{ Hz}$$

$$v = 330 \text{ m/s}$$

$$\lambda = \frac{v}{\nu} = \frac{330}{550}$$

$$\lambda = \mathbf{0{\cdot}6\ m}$$

$$\therefore \quad y = 0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)[330\,t - z] \qquad \ldots (i)$$

The wave is travelling along $+z$ direction

For wave travelling along $-z$ direction

$$y = 0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)[330\,t + z] \qquad \ldots (ii)$$

For $t = 0$, from equation (*i*)

$$y = -0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)z \qquad \ldots (iii)$$

The wave is travelling in $+z$ direction

Similarly for t=0, from equation (*ii*)

$$y = +0{\cdot}01 \sin\left(\frac{2\pi}{0{\cdot}6}\right)z \qquad \ldots (iv)$$

The wave is travelling along $-z$ direction.

1.8. Oscillations with One Degree of Freedom

A pendulum of a clock, a loaded spring and LC circuit have one degree of freedom. In the case of simple pendulum, the swing depends upon the angular dispacement made by the string with the vertical direction. In the case of a loaded spring, the displacement of the mass, and in the case of LC circuit the charge on the condenser plate describes the nature of the oscillations (Fig. 1.6). These oscillations take place about the mean position. All these systems have one degree of freedom.

$$\therefore \quad \frac{d^2 y_1}{dt^2} = -\omega^2 y_1 + A y_1^2 + B y_1^3 + C y_1^4 + \ldots \quad \ldots (5)$$

$$\frac{d^2 y_2}{dt^2} = -\omega^2 y_2 + A y_2^2 + B y_2^3 + C y_2^4 + \ldots \quad \ldots (6)$$

When the two instants are superimposed on each other, the resultant displacement is y. Here if superposition is true it is to be proved that $y = y_1 + y_2$.

$\therefore$ The equation for resultant displacement,

$$\frac{d^2 y}{dt^2} = \frac{d^2}{dt^2}(y_1 + y_2)$$

$$= -\omega^2 (y_1 + y_2) + A(y_1 + y_2)^2 + B(y_1 + y_2)^3 + C(y_1 + y_2)^4 + \ldots \quad \ldots (7)$$

Adding equations (5) and (6)

$$\frac{d^2 y_1}{dt^2} + \frac{d^2 y_2}{dt^2} = -\omega^2 (y_1 + y_2) + A(y_1^2 + y_2^2) + B(y_1^3 + y_2^3) + C(y_1^4 + y_4^4) + \ldots \quad \ldots (8)$$

The equations (7) and (8) are identical, only if

$$\frac{d^2}{dt^2}(y_1 + y_2) = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_2}{dt^2} \quad \ldots (9)$$

$$-\omega^2 (y_1 + y_2) = -\omega^2 y_1 - \omega^2 y_2 \quad \ldots (10)$$

$$A(y_1^2 + y_2^2) = A(y_1 + y_2)^2 \quad \ldots (11)$$

$$B(y_1^3 + y_2^3) = B(y_1 + y_2)^3 \quad \ldots (12)$$

$$C(y_1^4 + y_2^4) = C(y_1 + y_2)^4 \quad \ldots (13)$$

Equations (9) and (10) are true. But equations (11), (12) and (13) are true only, if

$$A = 0, \quad B = 0, \quad C = 0$$

When A, B, C etc. are zero, the equations become linear. Hence superposition principle is true only in the case of homogeneous linear equations. Also the sum of any two solutions is also a solution of the homogeneous linear equation.

All harmonic oscillators given in equations (9) and (10) obey superposition principle.

1.10. Simple Pendulum

A simple pendulum consists of a light string supporting a small sphere and fixed firmly at its upper end. An ideal simple pendulum should consist of a heavy particle suspended by means of a weightless, inextenisble, flexible string from a rigid support.

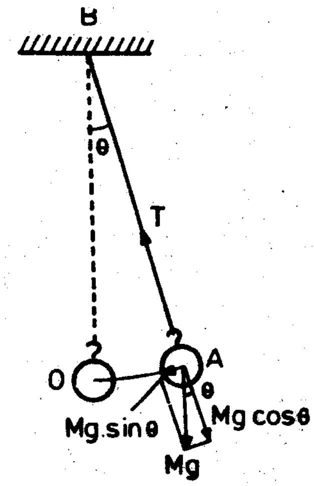


Fig. 1.7.

Let a pendulum be displaced from its mean positioin O and allowed to oscillate (Fig. 1.7). Suppose at any instant of time t, it is at A. The force acting upon the bob vertically downward $= Mg$. Resolve Mg into two rectangular components.

(1) Force along the string $= Mg \cos\theta$

(2) Force perpendicular to the string $= Mg \sin\theta$

Let the tension in the string be T. The component $Mg\cos\theta$ balances the tension T

$$Mg\cos\theta = T$$

Thus the only force acting on the oscillating particle is $-Mg\sin\theta$.

$$\therefore \quad F = -Mg\sin\theta$$

(– ve sign shows that the acceleration is directed towards the mean position)

According to Taylor's series of expansion

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots\ldots$$

For small angular displacements θ, $\quad \sin\theta = \theta$

Tangential force $\quad F = -Mg\theta$

The linear displacement $\quad y = l\theta$

Acceleration $\quad \dfrac{d^2 y}{dt^2} = l\dfrac{d^2\theta}{dt^2}$

$$\therefore \quad \text{Force} = Ml\frac{d^2\theta}{dt^2}$$

From Newton's second law

$$Ml\frac{d^2\theta}{dt^2} = -Mg\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad \ldots (1)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \qquad \ldots (2)$$

From (1) and (2)

$$\omega^2 = \frac{g}{l}$$

$$\omega = \sqrt{\frac{g}{l}}$$

Time period $$T = \frac{2\pi}{\omega}$$

$$T = 2\pi \sqrt{\frac{l}{g}} \qquad \ldots (3)$$

[**Bob of large size.** In the case of a simple pendulum, if the size of the bob is large, a correction has to be applied. In this case

$$t = 2\pi \sqrt{\frac{l + (\frac{2}{5} r^2)/l}{g}}$$

Here $l + \left(\frac{\frac{2}{5} r^2}{l}\right)$ represents the equivalent length of a simple pendulum.

1.11. Compound Pendulum

A compound pendulum is a rigid mass capable of oscillating about a horizontal axis passing through any point of the mass. This point is called the point of suspension. In Fig. 1.8, G is the centre of gravity of the body and S is the point of suspension. At any instant of time, when the mass has been displaced, the force acting vertically downwards = Mg. At this position, the line SG makes an angle θ with the vertical and the restoring moment of this force about the point $S = Mg\ l \sin\theta$. This is the only moment which produces angular acceleration in the pendulum.

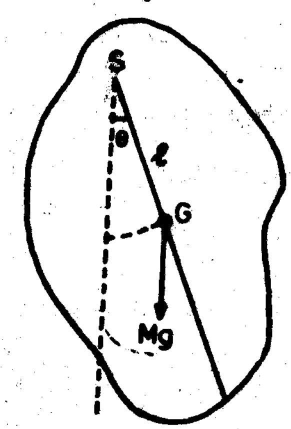


Fig. 1.8.

Let the moment of inertia of the pendulum about an axis passing through S and perpendicular to its length be I. If the angular acceleration at this instant is $\frac{d^2\theta}{dt^2}$, then

$$I\ \frac{d^2\theta}{dt^2} = -Mg\ l\ \sin\ \theta$$

[– ve sign shows that the force is directed towards the mean position].

$$\therefore \qquad I\ \frac{d^2\theta}{dt^2} = + Mg\ l\ \theta = 0$$

For small angular displacements, $\sin\theta = \theta$

$$\therefore \qquad I\ \frac{d^2\theta}{dt^2} + Mg\, l\, \theta = 0$$

The moment of inertia of the pendulum about an axis passing through S and perpendicular to its plane = $M K^2 + Ml^2$.

Here K is the radius of gyration *about an axis passing through the centre of gravity* of the pendulum.

$$\therefore \qquad M(K^2 + l^2)\frac{d^2\theta}{dt^2} + Mg\, l\, \theta = 0$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{lg}{K^2 + l^2}\right)\theta = 0 \qquad \ldots (1)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \qquad \ldots (2)$$

Here y refers to the angular displacement θ.

Comparing (1) and (2)

$$\omega^2 = \left(\frac{lg}{K^2 + l^2}\right)$$

Here ω is the angular frequency

$$\therefore \qquad \text{Time period} \qquad T = \frac{2\pi}{\omega}$$

$$T = 2\pi \sqrt{\frac{K^2 + l^2}{lg}} \qquad \ldots (3)$$

or

$$T = 2\pi \sqrt{\frac{(K^2/l) + l}{g}}$$

Here $(K^2/l + l)$ is called the equivalent length of the simple pendulum.

$$= \frac{A(l_1 - l_2) + B(l_1 + l_2)}{(l_1^2 - l_2^2)}$$

$$= \frac{l_1(A + B) - l_2(A - B)}{(l_1^2 - l_2^2)}$$

$\therefore$ $$A + B = t_1^2$$

$$A - B = t_2^2$$

$\therefore$ $$A = \frac{t_1^2 + t_2^2}{2}$$

and $$B = \frac{t_1^2 - t_2^2}{2}$$

Substituting these values in equation (7)

$$\frac{4\pi^2}{g} = \frac{t_1^2 + t_2^2}{2(l_1 + l_2)} + \frac{t_1^2 - t_2^2}{2(l_1 - l_2)}$$

or $$g = \frac{8\pi^2}{\left(\frac{t_1^2 + t_2^2}{l_1 + l_2}\right) + \left(\frac{t_1^2 - t_2^2}{l_1 - l_2}\right)} \quad \ldots (8)$$

As the values of l_1, l_2, t_1 and t_2 are known by experiment, the value of g can be determined, provided the position of the centre of gravity is accurately known.

However, as it is difficult to locate the position of the centre of gravity in a Kater's pendulum, the time periods t_1 and t_2 are adjusted to be very nearly equal so that in equation (8)

$$\frac{t_1^2 - t_2^2}{l_1 - l_2}$$

is negligibly small.

From equation (8)

$$g = \frac{8\pi^2}{\frac{t_1^2 + t_2^2}{l_1 + l_2}}$$

Taking $t_1 = t_2 = t \quad l_1 + l_2 = L$

$$g = \frac{8\pi^2 L}{2t^2}$$

$$g = \frac{4\pi^2 L}{t^2} \quad \ldots (9)$$

Here L is the distance between the two knife edges. Equation (9) is similar to the equation of a simple pendulum.

[**Note.** When the positions of K_1, K_2, M_1, M_2 and M_3 are finally adjusted, then the time periods about each knife edge must be equal. The positions of M_1, M_2 and M_3 must be the same while determining the time period about each knife edge.]

1.16. Simple Harmonic Oscillations of a Mass between Two Springs

Consider two springs S_1 and S_2 each having a length l in the free position. Mass M is placed midway between the two springs on a frictionless surface [Fig. 1.12 *(i)*]. One end of the spring S_1 is attached to a rigid wall at A and the other end is attached to the mass M. Similarly one end of the spring S_2 is attached to a rigid wall at B and the other end is connected to the mass M. Here $AC = BC = L$. [Fig.1.12 *(ii)*]. At C the mass is equally pulled by both the springs and it is the equilibrium position.

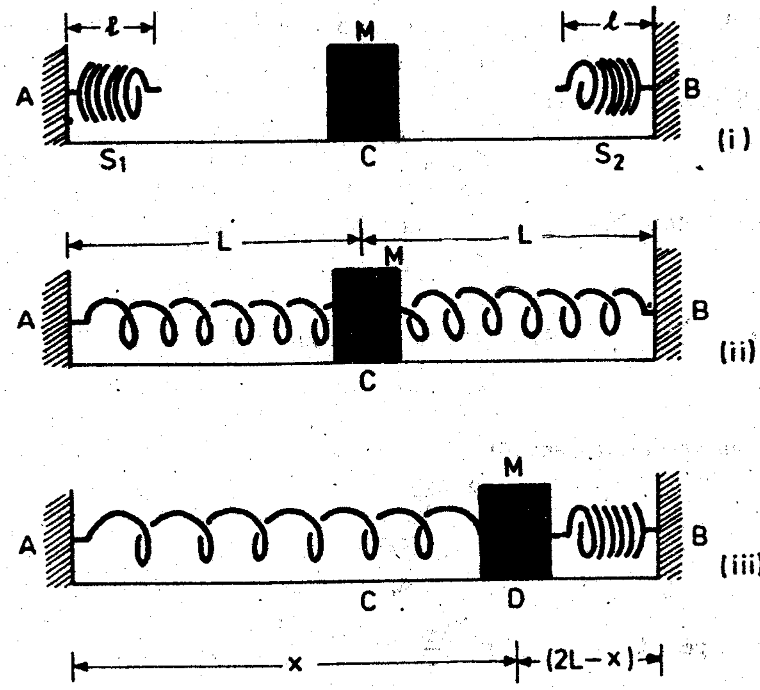


Fig. 1.12.

When the mass M is displaced from its equilibrium position and left, it executes simple harmonic oscillations. Let, at any instant, D be the displaced position of the mass M.

Here $AD = x$, and $BD = (2L - x)$

Let the tension per unit displacement in the spring be K. The displacement of the spring of S_1 is $(x - l)$ and it exerts a force $= K[x - l]$ in the direction DA. The displacement of the spring S_2 is $(2L - x - l)$ and it exerts a force $= K[2L - x - l]$ in the direction DB.

The resultant force on the mass M

$$= K[2L - x - l] - K[x - l] \text{ in the direction } DB$$

$$= -2K[x - L] \text{ in the direction } DB$$

According to Newton's second law of motion

$$F = M\frac{d^2x}{dt^2} = -2K[x - L] \qquad \ldots (1)$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{2K}{M}[x - L]$$

or

$$\frac{d^2x}{dt^2} + \frac{2K}{M}(x - L) = 0 \qquad \ldots (2)$$

Taking the displacement from the mean position

$$x - L = y$$

Differentiating twice,

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

Substituting these values in equation (2)

$$\frac{d^2y}{dt^2} + \frac{2K}{M}y = 0 \qquad \ldots (3)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots (4)$$

From equations (3) and (4)

$$\omega^2 = \frac{2K}{M}$$

$$\omega = \sqrt{\frac{2K}{M}}$$

Time period $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M}{2K}} \qquad \ldots (5)$$

Thus, the mass M executes simple harmonic oscillations and the time period is given by equation (5). Knowing the values of M and K, the time period can be calculated.

1.17. Mass between Two Springs – Transverse Oscillations

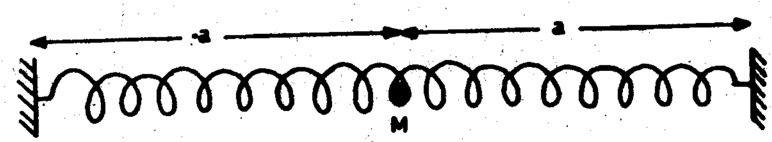


Fig. 1.13 (a)

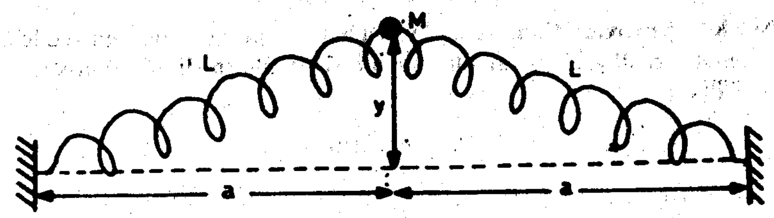


Fig. 1.13 (b)

Consider two springs each having a length a_0 in the free (relaxed) position. Mass M is placed midway between the two springs on a frictionless surface [(Fig. 1.13 (a)]. The length of each spring in the horizontal position is a. It is assumed that a_0 is extremely small as compared to a.

The mass M is displaced along the Y-axis through a displacement y. It is assumed that there is no displacement of the mass along X- or Z-axis.

In the equilibrium position, the tension in each spring is given by

$$T_0 = K(a - a_0) \qquad \ldots (i)$$

In the displaced position, each spring has a length L and tension in each spring is given by

$$T = K[L - a_0] \qquad \ldots (ii)$$

This tension acts along the axis of the spring.

The component along the Y-axis contributes the return force and the transverse oscillations are set up in the system :

The return force for each spring is $T \sin\theta$.

The net force acting on the mass due to both the springs along –ve y-axis is given by

$$F = -2T\sin\theta$$

Here $\sin\theta = \frac{y}{L}$

$$\therefore \quad F = -2K[L-a_0]\left(\frac{y}{L}\right)$$

$$F = -2Ky\left[1-\left(\frac{a_0}{L}\right)\right] \qquad \ldots(iii)$$

$$\therefore \quad M\left(\frac{d^2y}{dt^2}\right) + 2Ky\left[1-\left(\frac{a_0}{L}\right)\right] = 0 \qquad \ldots(iv)$$

$$\omega^2 = \left(\frac{2K}{M}\right)\left[1-\left(\frac{a_0}{L}\right)\right] \qquad \ldots(v)$$

Equation (*iv*) does not represent exact SHM.

Slinky Approximation. A slinky is a helical spring whose relaxed length is extremely small as compared the stretched length. In slinky approximation the quantity

$\left(\frac{a_0}{L}\right)$ is negligible small.

From equation (*iv*)

neglecting $\left(\frac{a_0}{L}\right)$ we get

$$M\left(\frac{d^2y}{dt^2}\right) + 2Ky = 0$$

$$\frac{d^2y}{dt^2} + \left(\frac{2K}{M}\right)y = 0 \qquad \ldots(vi)$$

$$\omega = \left[\frac{2K}{M}\right]^{\frac{1}{2}} \qquad \ldots(vii)$$

$$\nu = \frac{1}{2\pi}\left[\frac{2K}{M}\right]^{\frac{1}{2}} \qquad \ldots(viii)$$

Time period $t = \frac{1}{\nu} = 2\pi\left[\frac{M}{2K}\right]^{\frac{1}{2}}$

The solution of equation (*vi*) is

$$y = A\sin(\omega t + \phi) \qquad \ldots(ix)$$

Also from equation (*i*)

$$K = \frac{T_0}{a\left[1-\left(\frac{a_0}{a}\right)\right]}$$

$\frac{a_0}{a}$ is negligibly small

$$\therefore \quad K = \frac{T_0}{a}$$

Substituting this value of K in equation (*vii*)

$$\omega = \left[\frac{2T_o}{Ma}\right]^{\frac{1}{2}} \qquad \ldots(x)$$

and
$$\nu = \frac{1}{2\pi}\left[\frac{2T_0}{Ma}\right]^{\frac{1}{2}} \qquad \ldots(xi)$$

Time period
$$T = \frac{1}{\nu} = 2\pi\left[\frac{Ma}{2T_0}\right]^{\frac{1}{2}} \qquad \ldots(xii)$$

It may be noted that equation (*ix*) has no restriction on the amplitude A. Even for large amplitude there will be perfect linearity of the return force. This holds good only for slinky approximation.

The frequency is the same for both longitudinal and transverse oscillations.

Example. 1.18 *Show that for a mass connected between two identical springs,*

$$\frac{\omega_{long}}{\omega_{trans}} = \frac{1}{\left[1-\frac{a_0}{a}\right]^{\frac{1}{2}}}$$

For longitudinal oscillations

$$\omega_{long} = \left[\frac{2K}{M}\right]^{\frac{1}{2}} \qquad \ldots(i)$$

For transverse oscillations

$$\omega_{trans} = \left[\frac{2K}{M}\left[1-\left(\frac{a_0}{a}\right)\right]\right]^{\frac{1}{2}} \qquad \ldots(ii)$$

$$\therefore \quad \frac{\omega_{long}}{\omega_{trans}} = \frac{1}{\left[1-\frac{a_0}{a}\right]^{\frac{1}{2}}} \qquad \ldots(iii)$$

1.18. Simple Harmonic Oscillations of a Loaded Spring

Consider a spring S whose upper end is fixed to a rigid support and the lower end is attached to a mass M (Fig.1.14). In the equilibrium position, the mass is at A. When the mass is displaced downwards and left, it oscillates simple harmonically in the vertical direction.

Suppose at any instant the mass is at B. The distance $AB = y$. Let the tension per unit displacement of the spring be K.

Force exerted by the spring$=Ky$

According to Newton's second law

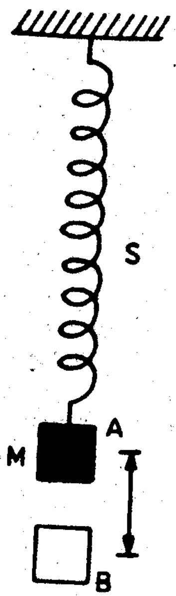


Fig. 1.14.

$$\text{Force} = M\frac{d^2y}{dt^2} = -Ky$$

[– ve sign shows that the force is directed upwards]

$$\therefore \quad M\frac{d^2y}{dt^2} + Ky = 0$$

$$\frac{d^2y}{dt^2} + \left(\frac{K}{M}\right)y = 0 \qquad \ldots(1)$$

This equation is similar to the equation of simple harmonic motion,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \ldots(2)$$

Comparing (1) and (2)

$$\omega^2 = \frac{K}{M}$$

$$\omega = \sqrt{\frac{K}{M}}$$

Time period $\quad T = \dfrac{2\pi}{\omega}$

or $\quad T = 2\pi\sqrt{\dfrac{K}{M}} \qquad \ldots(3)$

Knowing the values of M and K, the value of T can be calculated.

Determination of K. To determine the value of tension per unit displacement of the spring, a small mass m is attached to the free end of the spring. The increase in length of the spring is noted. Let it be x.

Then, $\quad K = \left(\dfrac{mg}{x}\right)$

Substituting the value of K in equation (3),

$$T = 2\pi\sqrt{\frac{Mx}{mg}} \qquad \ldots(4)$$

It is to be noted that $\dfrac{mg}{x}$ is constant for a given spring.

Example 1.19. *A spring is hung vertically and loaded with a mass of 100 grams and allowed to oscillate. Calculate (i) the time period and (ii) the frequency of oscillation. When the spring is loaded with 200 grams it extends by 10 cm.*

Here $\quad M = 100$ grams

$m = 200$ grams

$x = 10$ cm

$g = 980$ cm/s^2

(*i*) $\quad T = 2\pi\sqrt{\dfrac{Mx}{mg}}$

$$T = 2\pi\sqrt{\frac{100\times 10}{200\times 980}}$$

$$T = \frac{2\pi}{14} = \mathbf{0{\cdot}449\ s}$$

(*ii*) Frequency $\quad n = \dfrac{1}{T} = \dfrac{1}{0{\cdot}449}$

$n = $ **2·22 hertz.**

Example 1.20. *The scale of a spring balance reading from 0 — 10* kg *is 0·25* m. *A body suspended from the balance oscillates with a frequency of* $\dfrac{10}{\pi}$ hertz. *Calculate the mass of the body attached to the spring.*

Here $\quad m = 10$ kg

$x = 0{\cdot}25$ m

$M = ?$

$g = 9{\cdot}8$ m/s^2

$n = \dfrac{10}{\pi}$ hertz

or $\quad T = \dfrac{1}{n} = \dfrac{\pi}{10}$ s

$$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

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Consider a spring S whose upper end is fixed to a rigid support and the lower end is attached to a mass M (Fig.1.14). In the equilibrium position, the mass is at A. When the mass is displaced downwards and left, it oscillates simple harmonically in the vertical direction.

Suppose at any instant the mass is at B. The distance $AB = y$. Let the tension per unit displacement of the spring be K.

Force exerted by the spring $= Ky$

According to Newton's second law

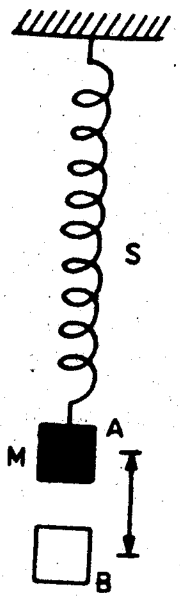

Fig. 1.14.

$$\text{Force} = M\frac{d^2y}{dt^2} = -Ky$$

[– ve sign shows that the force is directed upwards]

$$\therefore \quad M\frac{d^2y}{dt^2} + Ky = 0$$

$$\frac{d^2y}{dt^2} + \left(\frac{K}{M}\right)y = 0 \quad \ldots(1)$$

This equation is similar to the equation of simple harmonic motion,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \ldots(2)$$

Comparing (1) and (2)

$$\omega^2 = \frac{K}{M}$$

$$\omega = \sqrt{\frac{K}{M}}$$

Time period $\quad T = \dfrac{2\pi}{\omega}$

or

$$T = 2\pi\sqrt{\frac{K}{M}} \quad \ldots(3)$$

Knowing the values of M and K, the value of T can be calculated.

Determination of K. To determine the value of tension per unit displacement of the spring, a small mass m is attached to the free end of the spring. The increase in length of the spring is noted. Let it be x.

Then, $\quad K = \left(\dfrac{mg}{x}\right)$

Substituting the value of K in equation (3),

$$T = 2\pi\sqrt{\frac{Mx}{mg}} \quad \ldots(4)$$

It is to be noted that $\dfrac{mg}{x}$ is constant for a given spring.

Example 1.19. *A spring is hung vertically and loaded with a mass of 100 grams and allowed to oscillate. Calculate (i) the time period and (ii) the frequency of oscillation. When the spring is loaded with 200 grams it extends by 10* cm.

Here $\quad M = 100$ grams

$m = 200$ grams

$x = 10$ cm

$g = 980$ cm/s^2

(*i*) $\quad T = 2\pi\sqrt{\dfrac{Mx}{mg}}$

$$T = 2\pi\sqrt{\frac{100\times 10}{200\times 980}}$$

$$T = \frac{2\pi}{14} = \mathbf{0{\cdot}449\ s}$$

(*ii*) Frequency $\quad n = \dfrac{1}{T} = \dfrac{1}{0{\cdot}449}$

$n = \mathbf{2{\cdot}22\ hertz.}$

Example 1.20. *The scale of a spring balance reading from 0 — 10* kg *is 0·25* m. *A body suspended from the balance oscillates with a frequency of* $\dfrac{10}{\pi}$ hertz. *Calculate the mass of the body attached to the spring.*

Here $\quad m = 10$ kg

$x = 0{\cdot}25$ m

$M = ?$

$g = 9{\cdot}8$ m/s^2

$n = \dfrac{10}{\pi}$ hertz

or $\quad T = \dfrac{1}{n} = \dfrac{\pi}{10}$ s

$$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

$$\frac{d^2 x_2}{dt^2} = -\left[\left(\frac{g}{l}\right) + \frac{ky^2}{ml^2}\right] x_2 + \left(\frac{ky^2}{ml^2}\right) x_1 \quad \ldots (vi)$$

Taking

$$\left(\frac{g}{l} + \frac{ky^2}{ml^2}\right) = P$$

and

$$\left(\frac{ky^2}{ml^2}\right) = Q$$

$$\frac{d^2 x_1}{dt^2} = -P x_1 + Q x_2 \quad \ldots (vii)$$

$$\frac{d^2 x_2}{dt^2} = -P x_2 + Q x_1 \quad \ldots (viii)$$

In a normal mode of angular frequency ω and phase ϕ

$$x_1 = A \cos(\omega t + \phi)$$
$$x_2 = B \cos(\omega t + \phi)$$

Differentiating twice,

$$\frac{d^2 x_1}{dt^2} = -\omega^2 x_1$$

and

$$\frac{d^2 x_2}{dt^2} = -\omega^2 x_2$$

Substituting the values in equations (*vii*) and (*viii*) we get,

$$-\omega^2 x_1 = -P x_1 + Q x_2$$

$$\frac{x_1}{x_2} = \left(\frac{Q}{P - \omega^2}\right) \quad \ldots (ix)$$

and

$$-\omega^2 x_2 = -P x_2 + Q x_1$$

$$\frac{x_1}{x_2} = \left(\frac{P - \omega^2}{Q}\right) \quad \ldots (x)$$

Equating right hand sides of equation (*xi*) and (*x*)

$$\left(\frac{Q}{P - \omega^2}\right) = \left(\frac{P - \omega^2}{Q}\right)$$

$$Q^2 = (P - \omega^2)^2$$
$$(P - \omega^2) = \pm Q$$
$$\omega^2 = P \pm Q$$
$$\omega = [P \pm Q]^{1/2}$$

The angular frequencies of the two modes are

CHAPTER 2

Lissajous' Figures

2.1. Lissajous' Figures

When a particle is influenced simultaneously by two simple harmonic motions at right angles to each other, the resultant motion of the particle traces a curve. These curves are called Lissajous' figures. The shape of the curve depends on the time period, phase difference and the amplitude of the two constituent vibrations. Lissajous' figures are helpful in determining the ratio of the time periods of two vibrations and to compare the frequencies of two tuning forks.

2.2. Composition of Two Simple Harmonic Motions in a Straight Line

Analytical method. Let the two simple harmonic vibrations be represented by the equations

$$y_1 = a_1 \sin(\omega t + \alpha_1) \quad \ldots (1)$$

and

$$y_2 = a_2 \sin(\omega t + \alpha_2) \quad \ldots (2)$$

where y_1 and y_2 are the displacements of a particle due to the two vibrations, a_1 and a_2 are the amplitudes of the two vibrations and α_1 and α_2 are the epoch angles. Here, the two vibrations are assumed to be of the same frequency and hence ω is the same for both. The resultant displacement y of the particle is given by

$$y = y_1 + y_2$$
$$= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2)$$
$$= a_1 (\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2 (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)$$
$$y = (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \quad \ldots (3)$$

Since the amplitudes a_1 and a_2 and the angles α_1 and α_2 are constant, the coefficients of sin ωt and cos ωt in equation (3) can be substituted by $A \cos \phi$ and $A \sin \phi$

$$m_1\left(\frac{d^2x_1}{dt^2}\right) = -m_1 g \sin\alpha_1 + m_2 g \sin(\alpha_2 - \alpha_1) \qquad \ldots(i)$$

$$m_2\left(\frac{d^2x_2}{dt^2}\right) = -m_2 g \sin\alpha_2 - m_1\left(\frac{d^2x_1}{dt^2}\right)\cos(\alpha_2 - \alpha_1) \qquad \ldots(ii)$$

For small displacement

$$\sin\alpha_1 = \alpha_1 = \frac{x_1}{l_1}$$

$$\sin\alpha_2 = \alpha_2 = \frac{x_2}{l_2}$$

$$\sin(\alpha_2 - \alpha_1) = \alpha_2 - \alpha_1 = \frac{x_2}{l_2} - \frac{x_1}{l_1}$$

$\cos(\alpha_2 - \alpha_1) = 1$ (For small value of $\alpha_2 - \alpha_1$)

But $l_1 = l_2 = l$ and $m_1 = m_2 = m$

Substituting these values in equations *(i)* and *(ii)* and simplifying

$$\left(\frac{d^2x_1}{dt^2}\right) = -\left(\frac{g}{l}\right)x_1 + \frac{g}{l}(x_2 - x_1) \qquad \ldots(iii)$$

$$\left(\frac{d^2x_2}{dt^2}\right) = -\left(\frac{g}{l}\right)x_2 - \left(\frac{d^2x_1}{dt^2}\right) \qquad \ldots(iv)$$

These equations are not symmetric

Adding *(iii)* and *(iv)* and arranging we get

$$2\left(\frac{d^2x_1}{dt^2}\right) = -2\left(\frac{g}{l}\right)x_1 - \left(\frac{d^2x_2}{dt^2}\right)$$

$$\left(\frac{d^2x_1}{dt^2}\right) = -\left(\frac{g}{l}\right)x_1 - \frac{1}{2}\left(\frac{d^2x_2}{dt^2}\right) \qquad \ldots(v)$$

From equation *(iv)*

$$\left(\frac{d^2x_2}{dt^2}\right) = -\left(\frac{g}{l}\right)x_2 - \left(\frac{d^2x_1}{dt^2}\right) \qquad \ldots(vi)$$

The coupling between these two pendulums is evident and clear. The motion of one pendulum affects the motion of the second pendulum. Such a type of coupling is known as **Inertial coupling.**

Take angular frequency ω and phase ϕ in the normal mode,

$$x_1 = A_1 \cos(\omega t + \phi)$$
$$x_2 = A_2 \cos(\omega t + \phi)$$

Differentiating twice,

$$\frac{d^2x_1}{dt^2} = -\omega^2 x_1$$
$$\frac{d^2x_2}{dt^2} = -\omega^2 x_2$$

Substituting these values in equations *(v)* and *(vi)*

$$-\omega^2 x_1 = -\left(\frac{g}{l}\right)x_1 + \frac{1}{2}\omega^2 x_2$$

$$\left(\frac{g}{l} - \omega^2\right)x_1 = \frac{1}{2}\omega^2 x_2$$

$$\frac{x_1}{x_2} = \frac{\omega^2}{2\left[\frac{g}{l} - \omega^2\right]} \qquad \ldots(vii)$$

and

$$-\omega^2 x_2 = -\left(\frac{g}{l}\right)x_2 + \omega^2 x_1$$

$$\frac{x_1}{x_2} = \frac{\left(\frac{g}{l} - \omega^2\right)}{\omega^2} \qquad \ldots(viii)$$

Equating right hand sides of equations *(vii)* and *(viii)*

$$\frac{\omega^2}{2\left(\frac{g}{l} - \omega^2\right)} = \frac{\frac{g}{l} - \omega^2}{\omega^2}$$

or

$$\omega^4 = 2\left(\frac{g}{l} - \omega^2\right)^2$$

$$\omega^4 = 2\left(\frac{g}{l}\right)^2 + 2\omega^4 - \left(\frac{4g}{l}\right)\omega^2$$

$$\therefore \quad \omega^4 - \left(\frac{4g}{l}\right)\omega^2 + 2\left(\frac{g}{l}\right)^2 = 0 \qquad \ldots(ix)$$

It is a quadratic equation. Its solution is

$$\omega^2 = 2\left(\frac{g}{l}\right) \pm (\sqrt{2})\,\frac{g}{l}$$

$$= (2 \pm \sqrt{2}) \left(\frac{g}{l}\right)$$

$$\omega = \left[(2 \pm \sqrt{2}) \left(\frac{g}{l}\right)\right]^{\frac{1}{2}}$$

Angular frequencies

(*i*) $$\omega_1 = \left[(2 + \sqrt{2}) \left(\frac{g}{l}\right)\right]^{\frac{1}{2}} \quad \ldots (x)$$

(*ii*) $$\omega_2 = \left[(2 - \sqrt{2}) \left(\frac{g}{l}\right)\right]^{\frac{1}{2}} \quad \ldots (xi)$$

Frequencies

(*i*) $$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi} \left[(2 + \sqrt{2}) \left(\frac{g}{l}\right)\right]^{\frac{1}{2}} \quad \ldots (xii)$$

(*ii*) $$\nu_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi} \left[(2 - \sqrt{2}) \left(\frac{g}{l}\right)\right]^{\frac{1}{2}} \quad \ldots (xiii)$$

Equations (x) and (xi) represent the angular frequencies of the two normal modes and equations (xii) and ($xiii$) represent the frequencies of the two normal modes.

EXERCISES

1. Explain simple harmonic motion and discuss its characteristics.
2. Show that for a body vibrating simple harmonically the time period is given by
$$t = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$
3. Calculate the average kinetic energy and the total energy of a body executing simple harmonic motion.
4. Give examples of the systems that oscillate with one degree of freedom. Explain the term damped oscillations.
5. Show that the superposition principle is valid only in the case of homogeneous, linear vibrations.
6. Obtain the expression for the time period of a simple pendulum. Also derive the expression for the time period for a bob of large size.

7. Give the theory of a compound pendulum and derive an expression for its time period.
8. Show that in the case of a compound pendulum, the points of suspension and oscillation are interchangeable.
9. Give the theory of Kater's reversible pendulum.
10. Discuss the case of simple harmonic oscillations of a mass held between two linear springs.
11. Show that the time period of oscillation of a loaded spring is
$$t = 2\pi \sqrt{\frac{Mx}{mg}}.$$
12. Discuss the LC circuit and calculate the expression for the frequency of oscillations.
13. Discuss with examples free oscillations of a system with two degrees of freedom.
14. Discuss the two normal modes of oscillations of coupled LC circuits.
15. How will you determine the value of acceleration due to gravity using a compound pendulum.
16. Derive an expression for the time period of a compound pendulum and show that there are four collinear points on a compound pendulum about which the period of oscillation is the same. Give Bessel's computed time of Kater's pendulum.
17. A particle vibrates simple harmonically with an amplitude of 13 cm. The time period of oscillation is 2π seconds. Calculate the velocity of the vibrating particle at the instant the displacement is 5 cm. Also calculate the frequency of oscillation.
[**Ans.** (*i*) 12 cm/s ; (*ii*) $\frac{1}{2\pi}$ hertz]
18. A simple periodic wave disturbance with an amplitude of 8 units, travels a line of particles in the positive x direction. At a given instant, the displacement of a particle 10 cm from the origin is 6 units, and that of a particle 25 cm from the origin is 4 units, both particles being in positive displacement. What is the wavelength of the disturbance.
[**Ans.** 290 cm.]
19. At time $t = 0$, a train of waves has the form
$$y = 4 \sin 2\pi \left(\frac{x}{100}\right)$$
The velocity of the wave is 30 cm/s. Find the equation giving the waveform at a time $t = 2$ s.

Let n simple harmonic vibrations of the same amplitude a and epoch angles 0, 2α, 4α ... $2(n-1)\alpha$ influence a vibrating particle (Fig. 2.3). If the displacements of the vibrating particle are considered along the y-axis, the individual displacements are given by

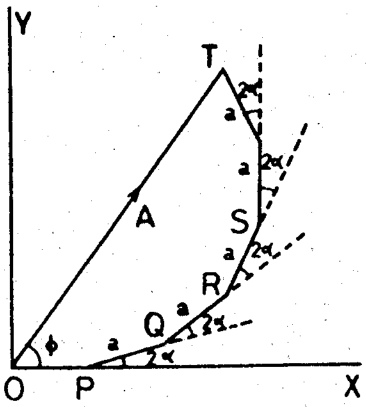

Fig. 2.3.

$$y_1 = a \sin(\omega t - 0)$$
$$y_2 = a \sin(\omega t - 2\alpha)$$

Let A be the amplitude of the resultant. Vibration and ϕ the epoch angle. Then

$$y = A \sin(\omega t - \phi).$$

The projections of the individual vectors OP, PQ, QR etc. on the y-axis are given by

$$0, a \sin 2\alpha, a \sin 4\alpha \text{ etc.}$$

Similarly the projections on the X-axis are given by

$$a, a\cos 2\alpha, a \cos 4\alpha \text{ etc.}$$

If OT represents the resultant vector, then $A \sin\phi$ will give the projection along the Y-axis and $A\cos\phi$ gives the projection along the X-axis.

$$\therefore \quad A \sin\phi = 0 + a\sin 2\alpha + a \sin 4\alpha + \ldots\ldots + a \sin 2(n-1)\alpha$$
$$= a[\sin 2\alpha + \sin 4\alpha + \ldots\ldots + \sin 2(n-1)\alpha] \quad \ldots (1)$$

Similarly,

$$A\cos\phi = a + a\cos 2\alpha + a\cos 4\alpha + \ldots\ldots + a\cos 2(n-1)\alpha$$
$$a[1 + \cos 2\alpha + \cos 4\alpha + \ldots\ldots + \cos 2(n-1)\alpha] \quad \ldots (2)$$

Multiplying equation (2) by $2\sin\alpha$

$$2A\cos\phi\sin\alpha = 2a\sin\alpha[1 + \cos 2\alpha + \cos 4\alpha + \ldots\ldots + \cos 2(n-1)\alpha]$$
$$= a[2\sin\alpha + 2\cos 2\alpha . \sin\alpha + 2\cos 4\alpha \sin\alpha + \ldots\ldots + 2\cos 2(n-1)\alpha\sin\alpha]$$
$$= a[2\sin\alpha + (\sin 3\alpha - \sin\alpha) + (\sin 5\alpha - \sin 3\alpha) + \ldots\ldots + \{\sin(2n-1)\alpha - \sin(2n-3)\alpha\}]$$
$$= a[2\sin\alpha + 2\sin(2n-1)\alpha]$$
$$= a[2 . \sin n\alpha . \cos(n-1)\alpha]$$
$$\therefore \quad 2A\cos\phi\sin\alpha = 2a\sin n\alpha . \cos(n-1)\alpha$$
$$\therefore \quad A\cos\phi = \frac{a\sin n\alpha . \cos(n-1)\alpha}{\sin\alpha} \quad \ldots (3)$$

Multiplying equation (1) by $2\sin\alpha$ and proceeding in a similar way, it can be shown that

$$A\sin\phi = \frac{a\sin n\alpha . \sin(n-1)\alpha}{\sin\alpha} \quad \ldots (4)$$

quaring equations (3) and (4) and adding

$$A^2(\sin^2\phi + \cos^2\phi) = A^2$$
$$= \frac{a^2\sin^2 n\alpha}{\sin^2\alpha}[\sin^2(n-1)\alpha + \cos^2(n-1)\alpha]$$
$$= \frac{a^2\sin^2 n\alpha}{\sin^2\alpha} \quad \ldots (5)$$
$$\therefore \quad A = \frac{a\sin n\alpha}{\sin\alpha} \quad \ldots (6)$$

Dividing equation (4) by (3)

$$\frac{A\sin\phi}{A\cos\phi} = \tan\phi = \frac{a\sin n\alpha . \sin(n-1)\alpha . \sin\alpha}{\sin\alpha . a\sin n\alpha . \cos(n-1)\alpha}$$
$$= \tan(n-1)\alpha$$
$$\phi = (n-1)\alpha \quad \ldots (7)$$

ere ϕ represents the epoch angle for the resultant vibration, n is the mber of simple harmonic vibrations influencing a particle and α is half the crease in the epoch angle between successive vibrations.

.4. Composition of Two Simple Harmonic Vibrations of Equal Time Periods Acting at Right Angles

Let

$$x = a\sin(\omega t + \alpha) \quad \ldots (1)$$

nd

$$y = b\sin\omega t \quad \ldots (2)$$

present the displacements of a particle along the X and Y-axes due to the fluence of two simple harmonic vibrations acting simultaneously on a particle in perpendicular directions. Here, the two vibrations are of the same time eriod but are of different amplitudes and different phase angles.

From equation (2),

$$\sin\omega t = \frac{y}{b}$$
$$\therefore \quad \cos\omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

From equation (1),

$$\frac{x}{a} = [\sin\omega t\cos\alpha + \cos\omega t\sin\alpha] \quad \ldots (3)$$

Substituting the values of $\sin\omega t$ and $\cos\omega t$ in equation (3)

$$\frac{x}{a} = \left[\frac{y}{b}\cos\alpha + \sqrt{1 - \frac{y^2}{b^2}} . \sin\alpha\right]$$

r

$$\frac{x}{a} - \frac{y}{b}\cos\alpha = \sqrt{1 - \frac{y^2}{b^2}}\sin\alpha$$

Squaring

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\cos^2\alpha-\frac{2xy}{ab}\cos\alpha=\left(1-\frac{y^2}{b^2}\right)\sin^2\alpha$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\,[\sin^2\alpha+\cos^2\alpha]-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha$$

$\therefore$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha \qquad \ldots(4)$$

This represents the general equation of an ellipse. Thus, due to the super imposition of two simple harmonic vibrations, the displacement of the particle will be along a curve (Fig. 2.4) given by equation (4).

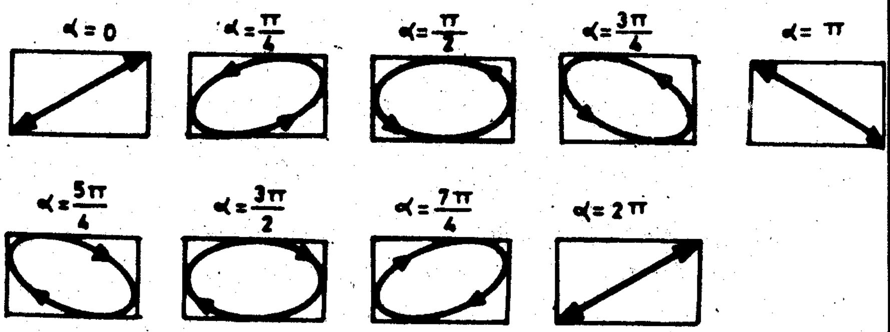

Fig. 2.4.

The resultant vibration of the particle will depend upon the value of α. Figure 2.4 represents the resultant vibration for values of α changing from 0 to 2 π.

Special cases

(*i*) If $\alpha = 0$ or 2π; $\cos\alpha = 1$; $\sin\alpha = 0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$$

or
$$\frac{x}{a}-\frac{y}{b}=0$$

or
$$y=\frac{b}{a}x.$$

This represents the equation of the straight line *BD* (Fig. 2.5) *i.e.*, the particle vibrates simple harmonically along the line *DB*.

(*ii*) *If* $\alpha=\pi$; $\sin\alpha = 0$; $\cos\alpha = -1$

$\therefore$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}=0$$

$$\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$y=-\frac{b}{a}x. \qquad \ldots(6)$$

This represents the equation of the straight line *AC* (Fig. 2.5).

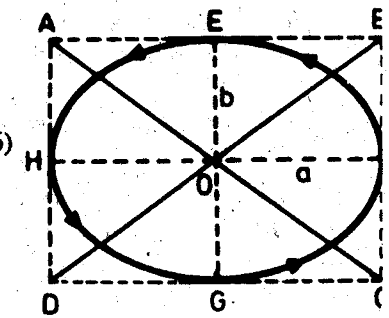

Fig. 2.5.

(*iii*) if $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

sin $\alpha = 1$; $\cos\alpha = 0$

$\therefore$
$$\frac{x^2}{d^2}+\frac{y^2}{b^2}=1$$

This represents the equation of the ellipse *EHGF* (Fig. 2.5) with *a* and *b* as the semi-major and semi-minor axes.

(*iv*) If $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

and $a = b$;

then
$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$$

or
$$x^2+y^2=a^2$$

This represents the equation of a circle of radius *a* (Fig. 2.6).

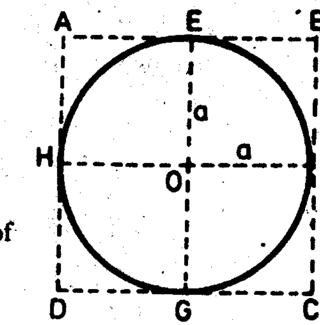

Fig. 2.6.

(*v*) if $\alpha = \frac{\pi}{4}$ or $\frac{7\pi}{4}$, the resultant vibration is an oblique ellipse *KLMN* as shown in Fig. 2.7(*i*).

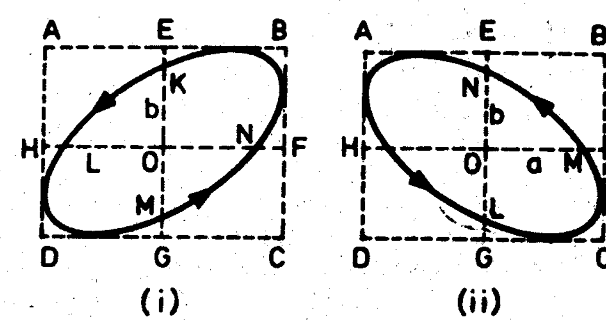

Fig. 2.7.

On the other hand if $\alpha = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$, the resultant vibration is again an oblique ellipse *KLMN* as shown in Fig. 2.7 (*ii*). The cycle of changes is repeated after every time period.

2.5. Composition of Two SHMs at Right Angles of Equal Periods

Graphical method. (1) Let a particle be influenced simultaneously by two simple harmonic vibrations at right angles to each other. The two vibrations are represented by the equations

$$x = a \sin \omega t$$
$$x = b \sin \omega t$$

Here the phase difference between the two vibrations is zero, time periods are equal and the amplitudes are unequal.

Draw two circles of reference with centres C_1 and C_2 and radii a and b respectively. Divide each circle into eight equal parts, marked 0, 1, 2, . . . 7, 8. The angular frequency in each case is ω. If the particle *O* is subjected to the simple harmonic motion along the *X*-axis only, the particle will vibrate along *XX'*. Similarly, if the particle *O* is subjected to the simple harmonic motion along the *Y*-axis only, the particle will vibrate along *YY'* (Fig. 2.8).

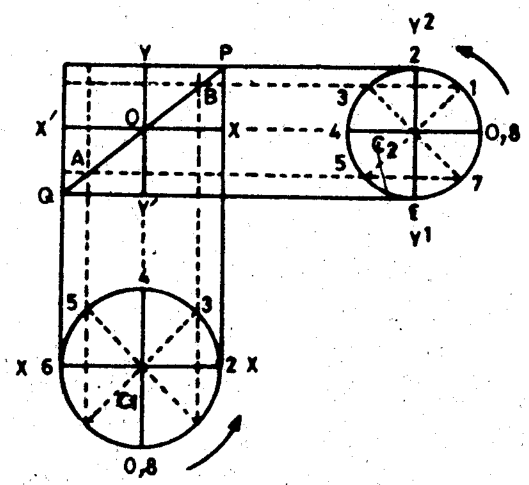

Fig. 2.8.

When the particle *O* is subjected to the two simple harmonic motions simultaneously ; the resultant vibration of *O* will be along the straight line *PQ*. At zero, zero position, the particle is at the mean position *O*. After *T*/8 seconds, corresponding to 1, 1 position in each circle, the particle will be at *B*. After *T*/4 seconds, corresponding to 2, 2 positions in each circle, the particle will be at *P*.

In this way the positions of the resultant displacement of the particle will be, after 3*T*/8 seconds, *O* after *T*/2 seconds, *A* after 5*T*/8 seconds, *Q* after 3*T*/4 conds, *A* after 7*T*/8 seconds and again at *O* after *T* seconds. In this way the ultant vibration is along *POQ*. The amplitude of the resultant vibration is P.

Here
$$(OP)^2 = (OX)^2 + (OY)^2$$
$$(OP)^2 = a^2 + b^2$$
$$OP = \sqrt{a^2 + b^2}$$

e angular frequency and time period remain the same as for the two constituent vibrations.

(2) Let a particle be influenced simultaneously by two simple harmonic brations at right angles to each other. The two vibrations are represented by e equations

$$x = a \sin (\omega t + \alpha)$$
nd
$$y = b \sin \omega t$$

uppose the phase difference $\alpha = \pi/2$ and the time periods are equal. The mplitudes are *a* and *b*.

Draw two circles of reference with centres C_1 and C_2 and radii *a* and *b* espectively. Divide each circle into eight equal parts, marked 0, 1, 2, . . .7, 8.

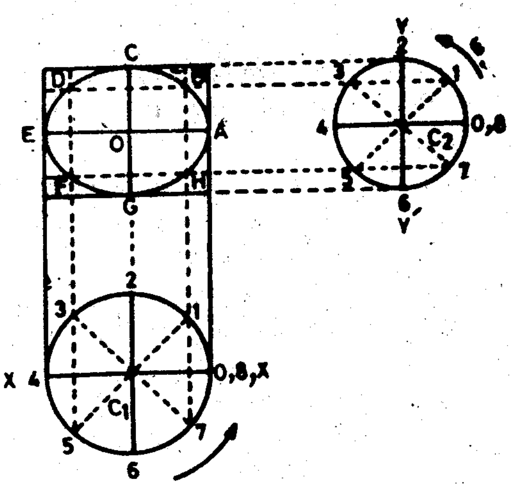

Fig. 2.9.

e angular frequency in each case is ω. If the particle *O* is subjected to the mple harmonic motion along the *X*-axis only, the particle will vibrate along X'. Similarly if the particle *O* is subjected to the simple harmonic motion long the *Y*-axis only, the particle will vibrate along *YY'*. Here, the initial position for vibration along XX' is at the extreme position at $t = 0$. The points on the circle of reference are marked showing that there is a phase difference of $\pi/2$ between the two vibrations (Fig. 2.9).

When the particle O is subjected to the two SHMs simultaneously, the resultant vibration of O will be along an ellipse having a and b as semi-major and semi-minor axes. The resultant position after T/8 seconds corresponding to 1, 1 position in each circle of reference will be the point B. Similarly C, D, E, F etc. will be the resultant positions after successive time intervals of $T/8$ seconds. The motion of the particle O will be along the ellipse $ABCDEFGHA$. The angular frequency and time period remain the same as for the two constituent vibrations.

Here at any instant

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If a and b are equal, then the resultant motion is given by the equation

$$x^2 + y^2 = a^2$$

which represents a circle with centre O and radius a.

[**Note.** If a particle is moving along a circular path of radius a with angular frequency ω, it can be considered to be under the influence of two rectangular SHMs having equal amplitude of value a and angular frequency ω, and a phase difference of $\pi/2$.]

2.6. Composition of Two Simple Harmonic Motions at Right Angles to each other and having Time Periods in the Ratio 1 : 2

Let two simple harmonic motions be given by the equations

$$x = a \sin (2\omega t + \alpha) \quad \ldots (1)$$

and

$$y = b \sin \omega t \quad \ldots (2)$$

Here a is the amplitude for the motion along the X-axis and b is the amplitude for the motion along the Y- axis. The phase difference between the two vibrations is α.

From equation (2)

$$\frac{y}{b} = \sin \omega t$$

and

$$\cos \omega t = \sqrt{1 - \sin^2 \omega t} \quad \ldots (3)$$

or

$$\cos \omega t = \sqrt{1 - \frac{y^2}{b^2}} \quad \ldots (4)$$

From equation (1)

$$\frac{x}{a} = \sin (2\omega t + \alpha)$$

$$= \sin 2\omega t \cos \alpha + \cos 2\omega t \sin \alpha$$

$$= 2 \sin \omega t \cos \omega t \cos \alpha + (1 - 2 \sin^2 \omega t) \sin \alpha$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$,

$$\frac{x}{a} = 2 \cdot \frac{y}{b} \cdot \sqrt{1 - \frac{y^2}{b^2}} \cos \alpha + \left(1 - 2\frac{y^2}{b^2}\right) \sin \alpha$$

$$\left[\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \alpha\right] = \frac{2y}{b} \cos \alpha \sqrt{1 - \frac{y^2}{b^2}}$$

$$\left[\left(\frac{x}{a} - \sin \alpha\right) + \frac{2y^2}{b^2} \sin \alpha\right] = \frac{2y \cos \alpha}{b} \sqrt{1 - \frac{y^2}{b^2}}$$

Squaring both sides

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^2} \sin^2 \alpha + 2\left(\frac{x}{a} - \sin \alpha\right) \frac{2y^2}{b^2} \sin \alpha$$

$$= \frac{4y^2 \cos^2 \alpha}{b^2} \left(1 - \frac{y^2}{b^2}\right)$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^4} (\sin^2 \alpha + \cos^2 \alpha) - \frac{4y^2}{b^2} (\sin^2 \alpha + \cos^2 \alpha) + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin \alpha = 0$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin \alpha = 0$$

$$\left(\frac{x}{a} - \sin \alpha\right)^2 + \frac{4y^2}{b^2} \left[\frac{y^2}{b^2} + \frac{x}{a} \sin \alpha - 1\right] = 0 \quad \ldots (5)$$

Equation (5) represents the general equation of a curve having two loops. The resultant motion of the particle for different values of α is given in

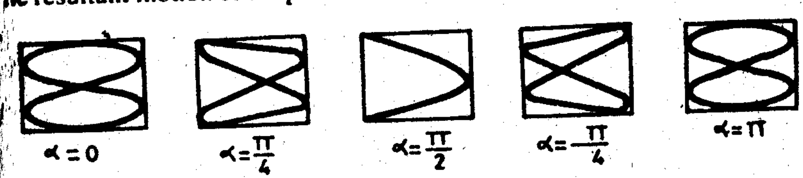

Fig. 2.10

Fig. 2.10 For a phase difference of 0, π and 2 π, the resultant motion gives the figure of eight.

Special cases

(*i*) When $\alpha = 0$, π , 2π etc.

$\sin\alpha = 0$

From equation (5)

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0 .$$

This equation represents the figure of eight and has two loops.

(*ii*) When $\alpha = \frac{\pi}{2}$

$\sin\alpha = +1$

From equation (5)

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

or
$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{x}{a} - 1\right) + \frac{4y^4}{b^4} = 0$$

or
$$\left[\left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2}\right]^2 = 0$$

or
$$\left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2} = 0$$

or
$$\frac{2y^2}{b^2} = -\left(\frac{x}{a} - 1\right)$$

$$y^2 = -\frac{b^2}{2}\left(\frac{x}{a} - 1\right)$$

$$y^2 = -\frac{b^2}{2a}(x - a)$$

This represents the equation of a parabola, with vertex at $(a, 0)$.

2.7. Composition of Two SHMs at Right Angles with Time Periods in the Ratio 1 : 2

Graphical Method

Let a particle be influenced simultaneously by two simple harmonic vibrations at right angles to each other. The two vibrations are represented by the equations

$$x = a \sin 2\,\omega t$$
$$y = b \sin\,\omega t$$

Here the phase difference between the two vibrations is zero, amplitudes are unequal and the time periods are in the ratio of 1 : 2.

Draw two circles of reference with centres C_1 and C_2 and radii a and b respectively. Divide the circle with centre C_1 into 4 equal parts and the circle with centre C_2 into 8 equal parts. The angular frequencies are 2ω and ω. If the particle O is subjected to the SHM along the X-axis only, the particle will vibrate along xx' (Fig. 2.11).

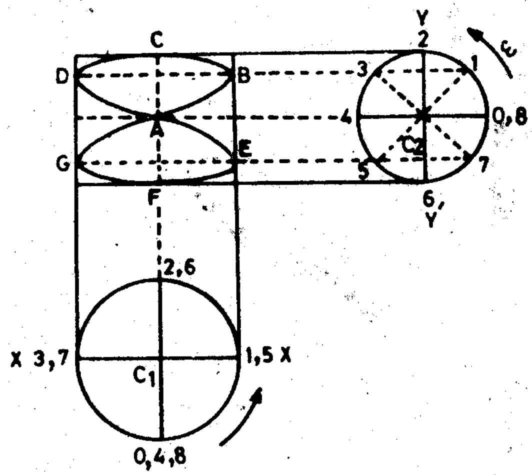

Fig. 2.11.

Similarly, if the particle O is subjected to the SHM along the Y-axis only, the particle will vibrate along yy'.

When the particle O is subjected to the two SHMs simultaneously, the resultant vibration of O will be along a curve *ABCDAEFGA* which represents the figure of eight. At zero-zero position the particle is at the mean position A. After equal intervals of time the positions B, C, D etc. are obtained in the same order from the two circles of reference (Fig. 2.11). The angular frequency of the resultant vibration is ω.

CHAPTER 3

Free, Forced and Resonant Vibrations

3.1. Free Vibrations

When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. In such cases there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude of swing remains constant.

3.2. Undamped Vibrations

For a simple harmonically vibrating particle, the kinetic energy for displacement y, is given by

$$\frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

At the same instant, the potential energy of the particle is $\frac{1}{2} Ky^2$ where K is the restoring force per unit displacement.

The total energy at any instant,

$$= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} Ky^2$$

For an undamped harmonic oscillator, this total energy remains constant.

$$\therefore \quad = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} Ky^2 = \text{constant} \quad \ldots (1)$$

Differentiating equation (1) with respect to time,

$$m \frac{d^2 y}{dt^2} + Ky = 0 \quad \ldots (2)$$

$$\frac{d^2 y}{dt^2} + \left(\frac{K}{m}\right) y = 0 \quad \ldots (3)$$

Equation (3) is similar to the equation

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \quad \ldots (4)$$

Here $\omega^2 = \left(\frac{K}{m}\right)$

The solution for equation (4) is

$$y = a \sin (\omega t - \alpha)$$

$$\therefore \quad y = a \sin \left[\sqrt{\frac{K}{m}}\, t - \alpha\right]$$

The frequency of oscillation is

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

Thus, in the case of undamped free vibrations, the differential equation is

$$\frac{d^2 y}{dt^2} + \left(\frac{K}{m}\right) y = 0 \quad \ldots (5)$$

This is only an ideal case. In the first chapter, for the motion of a pendulum, loaded spring, LC circuit etc., it has been assumed that the vibrations are free and undamped.

3.3. Damped Vibrations

In actual practice, when the pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. Such vibrations are called **free damped** vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium. The dissipative force due to friction etc. (resistance in LCR circuit) is proportional to the velocity of the particle at that instant. Let $\mu \frac{dy}{dt}$ be the dissipative force due to friction etc.

This term is to be introduced in equation (2).

Therefore, the differential equation in the case of Free-damped vibrations is,

$$m \frac{d^2 y}{dt^2} + Ky + \mu \frac{dy}{dt} = 0 \quad \ldots (6)$$

or $$\frac{d^2y}{dt^2}+\left(\frac{\mu}{m}\right)\frac{dy}{dt}+\left(\frac{K}{m}\right)y=0 \qquad \ldots(7)$$

This equation is similar to a general differential equation,

$$\frac{d^2y}{d^2}+2b\,\frac{dy}{dt}+k^2\,y=0 \qquad \ldots(8)$$

The solution of this equation is

$$y=ae^{-bt}\sin(\omega t-\alpha) \qquad \ldots(9)$$

The general solution of equation (7) is also given by

$$y=A\,e^{(-b+\sqrt{b^2+k^2})\,t}+B\,e^{(-b-\sqrt{a^2-k^2})\,t}$$

Here $$b=\frac{\mu}{2m} \text{ and } k^2=\frac{K}{m}$$

and $$\omega=\sqrt{k^2-b^2}$$

or $$\omega=\sqrt{\frac{K}{m}-\frac{\mu^2}{4m^2}}$$

or $$n=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{k^2-b^2}$$

3.4. Damped SHM in an Electrical Circuit

In the case of an electrical circuit, the force equation is replaced by the voltage equation. The circuit consists a condenser C, inductance L and resistance R (Fig. 3.1).

When the condenser C is charged by pressing the morse key, it gets discharged through an inductance L and resistance R when the key is released (Fig. 3.1).

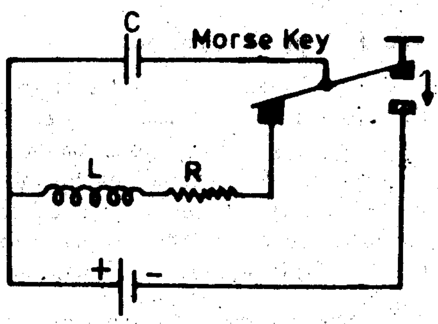

Fig. 3.1:

Suppose, during discharge, at any instant, the charge on the condenser $= Q$, current flowing $= I$ and rate of fall of current $=\frac{dI}{dt}$.

In this case, $$\frac{Q}{C}+RI+L\frac{dI}{dt}=0$$

But, $$I=\frac{dQ}{dt} \text{ and } \frac{dI}{dt}=\frac{d^2Q}{dt^2}$$

$\therefore$ $$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=0$$

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC}=0 \qquad \ldots(1)$$

Taking $$\frac{R}{L}=2b \text{ and } \frac{1}{LC}=k^2$$

we get $$\frac{d^2Q}{dt^2}+2b\frac{dQ}{dt}+k^2Q=0$$

The general solution of this equation is

$$Q=A\,e^{(-b+\sqrt{b^2-k^2})\,t}+B\,e^{(-b-\sqrt{b^2-k^2})\,t} \qquad \ldots(2)$$

When $t=0$, $Q=Q_0$ and from equation (2)

$$A+B=Q_0 \qquad \ldots(3)$$

Differentiating equation (2),

$$\frac{dQ}{dt}=A(-b+\sqrt{b^2-k^2})\,e^{(-b+\sqrt{b^2-k^2})\,t}+B(-b-\sqrt{b^2-k^2})\,e^{(-b-\sqrt{b^2-k^2})\,t}$$

When, $$t=0,\ \frac{dQ}{dt}=0$$

$\therefore$ $$A(-b+\sqrt{b^2-k^2})+B(-b-\sqrt{b^2-k^2})=0$$

$$-b(A+B)+\sqrt{b^2-k^2}\,(A-B)=0$$

$$-bQ_0+\sqrt{b^2-k^2}\,(A-B)=0$$

$\therefore$ $$A-B=\frac{bQ_0}{\sqrt{b^2-k^2}} \qquad \ldots(4)$$

Adding (3) and (4),

$$2A=Q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)$$

$$A=\frac{Q_0}{2}\left(1+\frac{b}{\sqrt{b^2-k^2}}\right) \qquad \ldots(5)$$

Subtracting (4) from (3),

$$2B=Q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right)$$

$$B=\frac{Q_0}{2}\left(1-\frac{b}{\sqrt{b^2-k^2}}\right) \qquad \ldots(6)$$

$$Q = \frac{Q_0 e^{-\frac{R}{2L}t}}{\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)LC}} \sin\left(\sqrt{\left(\frac{1}{LC} - \frac{r^2}{4L^2}\right)} \cdot t + \theta\right) \quad \ldots (8)$$

In this case, the discharge is oscillatory as represented by the curve c (Fig. 3.2). This dischage is of simple harmonic type and the natural frequency of the circuit,

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \ldots (9)$$

When $$R = 0,\ f = \frac{1}{2\pi\sqrt{LC}} \quad \ldots (10)$$

Results. (1) When $R^2 > \frac{4L}{C}$, the discharge is non-oscillatory and dead beat.

(2) When $R^2 = \frac{4L}{C}$, the discharge is 'aperiodic' and critically damped.

(3) When $R^2 < \frac{4L}{C}$, the discharge is oscillatory and the natural frequency of the circuit.

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Example 3.1. *A condenser of capacity 1 uF, an inductance of 0·2* henry *and a resistance of 800* ohms *are joined in series. Is the circuit oscillatory?*

Here $L = 1\ \mu F = 10^{-6} F,\ R = 800$ ohms, $L = 0{\cdot}2$ henry.

$$\therefore \quad R^2 = (800)^2 = 640000 = 6{\cdot}4 \times 10^5 \quad \ldots (1)$$

$$\frac{4L}{C} = \frac{4 \times 0{\cdot}2}{10^{-6}} = 8 \times 10^5 \quad \ldots (2)$$

As $R^2 <= \frac{4L}{C}$, the *circuit is oscillatory.*

Example 3.2. *In an oscillatory circuit, $L = 0{\cdot}2$* henry, *$C = 0{\cdot}0012\ \mu F$. What is the maximum value of resistance for the circuit to be oscillatory?*

Here, $L = 0{\cdot}2$ henry, $C = 0{\cdot}0012\ \mu F = 12 \times 10^{-10}$ F

$$R^2 = \frac{4L}{C}$$

$$R = \sqrt{\frac{4L}{C}} = \sqrt{\frac{4 \times 0{\cdot}2}{12 \times 10^{-10}}} = 2{\cdot}582 \times 10^4 \text{ ohms.}$$

A maximum resistance of $R = 2{\cdot}582 \times 10^4$ ohms can be connected in series so that the discharge remains oscillatory. For the discharge to be oscillatory R should be less than $2{\cdot}582 \times 10^4$ ohms.

Example 3.3. *Find whether the discharge of a condenser through the following inductive circuit is oscillatory:*

$C = 0{\cdot}1\ \mu F$, $L = 10$ millihenry, *$R = 200$* ohms.

If the circuit is oscillatory, calculate its frequency.

Here $C = 0{\cdot}1 \times 10^{-6} F,\ L = 10 \times 10^{-3} H,\ R = 200$ ohms

$$R^2 = (200)^2 = 40000 = 4 \times 10^4$$

$$\frac{4L}{C} = \frac{4 \times 10 \times 10^{-3}}{0{\cdot}1 \times 10^{-6}} = 4 \times 10^5$$

As $R^2 < \frac{4L}{C}$, the circuit is oscillatory

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{10 \times 10^{-3} \times 0{\cdot}1 \times 10^{-6}} - \frac{40000}{4 \times (10 \times 10^{-3})^2}}$$

$$= \frac{1}{2\pi}\sqrt{10^9 - 10^8} = \frac{1}{2\pi}\sqrt{9 \times 10^8} = \frac{3}{2\pi} \times 10^4 = 4772{\cdot}7 \text{ Hz.}$$

3.5. Forced Vibrations

The time period of a body executing simple harmonic motion depends on the dimensions of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, it continues to oscillate under the influence of such external forces. Such vibrations of the body are called *forced vibrations.*

Initially, the amplitude of the swing increases, then decreases with time, becomes minimum and again increases. This will be repeated if the external periodic force is constantly applied on the system. In such cases the body will finally be forced to vibrate with the same frequency as that of the applied force. The frequency of the forced vibration is different from the natural frequency of vibration of the body. The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency of the applied force. The amplitude will be large if difference in frequencies is small and *vice versa*.

For forced vibrations, equation (6) Art. 3.3 is modified in the form,

$$m\frac{d^2 y}{dt^2}+Ky+\mu\frac{dy}{dt}=F\sin pt \qquad \ldots (1)$$

Here p is the angular frequency of the applied periodic force.

The particular solution of equation (1) representing the forced vibrations is

$$y=a\sin(pt-\alpha) \qquad \ldots (2)$$

$$\therefore \quad \frac{dy}{dt}=ap\cos(pt-\alpha) \qquad \ldots (3)$$

$$\frac{d^2y}{dt^2}=-ap^2\sin(pt-\alpha)=-p^2 y \qquad \ldots (4)$$

Substituting these values in equation (1)

$$-mp^2 a\sin(pt-\alpha)+Ka\sin(pt-\alpha)+\mu ap\cos(pt-\alpha)=F\sin pt$$

$$-mp^2 a[\sin pt\cos\alpha-\cos pt\sin\alpha]+Ka[\sin pt\cos\alpha-\cos pt\sin\alpha]$$

$$+\mu ap[\cos pt\cos\alpha+\sin pt\sin\alpha]-F\sin pt=0 \qquad \ldots (5)$$

When $\sin pt=1$; $\cos pt=0$

$$\therefore \quad -mp^2 a\cos\alpha+Ka\cos\alpha+\mu ap\sin\alpha-F=0 \qquad \ldots (6)$$

When $\cos pt=1$; $\sin pt=0$

$$\therefore \quad +mp^2 a\cos\alpha-Ka\sin\alpha+\mu ap\cos\alpha=0 \qquad \ldots (7)$$

Dividing equation (7) by $\cos\alpha$ and simlifying

$$\tan\alpha=\frac{\mu p}{K-mp^2}=\frac{A}{B} \qquad \ldots (8)$$

From equation (8)

$$\sin\alpha=\frac{A}{\sqrt{A^2+B^2}} \qquad \ldots (9)$$

$$\cos\alpha=\frac{B}{\sqrt{A^2+B^2}} \qquad \ldots (10)$$

Dividing equation (6) by $\cos\alpha$

$$-mp^2 a+Ka+\mu ap\tan\alpha-\frac{F}{\cos\alpha}=0$$

or $$a[(K-mp^2)+\mu p\tan\alpha]=\frac{F}{\cos\alpha}$$

But $(K-mp^2)=B$, and $\mu p=A$

Substituting the values of $\tan\alpha$ and $\cos\alpha$

$$a\left[B+\frac{A^2}{b}\right]=\frac{F\sqrt{A^2+B^2}}{B}$$

$$a=\frac{F}{\sqrt{A^2+B^2}}$$

Substituting the values of A and B

$$a=\frac{F}{\sqrt{\mu^2p^2+(K-mp^2)^2}} \qquad \ldots (11)$$

$$\therefore \quad y=a\sin(pt-\alpha)$$

$$y=\frac{F}{\sqrt{\mu^2p^2+(K-mp^2)^2}}\sin(pt-\alpha) \qquad \ldots (12)$$

Applying the boundary conditions, another solution is obtained when $F=0$. This corresponds to free vibrations. In the case of free vibrations the solution is

$$y=a\,e^{-bt}\sin(\omega t-\alpha) \qquad \ldots (13)$$

The general solution will include both the particular solutions for free and forced vibrations.

$$\therefore \quad y=a\,e^{-bt}\sin(\omega t-\alpha)+\frac{F}{\sqrt{\mu^2p^2+(K-mp^2)^2}}\sin(pt-\alpha) \qquad \ldots (14)$$

Here $b=\dfrac{\mu}{2m}$.

Example 3.4. *The equation for displacement of a point on a damped oscillator is given by*

$$x=5\,e^{-0.25t}\sin\left(\frac{\pi}{2}\right)t \text{ metre}$$

Find the velocity of the oscillating point at

$t=\dfrac{T}{4}$ and T, where T is the time period of the oscillator.

(IAS, 1987)

Here, $$x=5\,e^{-0.25t}\sin\left(\frac{\pi}{2}\right)t \qquad \ldots (i)$$

This equation is similar to the equation

$$x=a\,e^{-bt}\sin\omega t \qquad \ldots (ii)$$

Comparing (*i*) and (*ii*)

$$\omega=\frac{\pi}{2}$$

Time period $$T=\frac{2\pi}{\omega}=\frac{2\pi}{\pi/2}=4\text{ s}$$

Differentiating equation *(i)* with respect to time t,

$$\text{velocity, } \frac{dx}{dt} = 5(-0{\cdot}25)\, e^{-0{\cdot}25t} \sin\left(\frac{\pi}{2}\right)t$$

$$+ 5 \times \left(\frac{\pi}{2}\right) e^{-0{\cdot}25t} \cos\left(\frac{\pi}{2}\right)t \qquad \ldots (iii)$$

(*i*) When $t = \frac{T}{4}$,

$$t = \frac{4}{4} = 1 \text{ s}$$

$$\frac{dx}{dt} = -1{\cdot}25\, e^{-0{\cdot}25} \sin\left(\frac{\pi}{2}\right) + \left(\frac{5\pi}{2}\right) e^{-0{\cdot}25} \cos\left(\frac{\pi}{2}\right)$$

$$\frac{dx}{dt} = -1{\cdot}25\, e^{-0{\cdot}25} = (-1{\cdot}25) \times (0{\cdot}368)^{0{\cdot}25}$$

$$= -1{\cdot}25 \times 0{\cdot}779$$

$$= \mathbf{-0{\cdot}974 \text{ m/s.}}$$

–ve sign shows that velocity is in opposite direction.

(*ii*) When $t = T = 4$ s

$$\frac{dx}{dt} = -1{\cdot}25\, e^{-0{\cdot}25\times 4} \sin\left(\frac{\pi}{2}\right)4 + \left(\frac{5\pi}{2}\right) e^{-0{\cdot}25\times 4} \cos\left(\frac{\pi}{2}\right)4$$

$$= \left(\frac{5\pi}{2}\right) e^{-1}$$

$$= \left(\frac{5\pi}{2}\right) \times 0{\cdot}368$$

$$= \mathbf{2{\cdot}89 \text{ m/s.}}$$

3.6. Resonance and Sharpness of Resonance

In the case of forced vibrations, the general solution for the displacement at any instant is given by

$$y = a\, e^{-bt} \sin(\omega t - \alpha) + \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha)$$

If the effect of viscocity of the medium is small, the amplitude,

$$\frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}},$$

under the action of the driving force is maximum when the denominator is minimum. It is possible if $K - mp^2 = 0$ or $K = mp^2$

or $$p = \sqrt{\frac{K}{m}}$$

Further, the amplitude will be infinite if μ is also zero. The oscillations will have maximum amplitude and this state of vibration of a system is called *resonance*. It means that, when the forced frequency is equal to the natural frequency of vibration of the body, resonance takes place. If friction is present, the amplitude at resonance

$$= \frac{F}{\mu p} = \frac{F}{\mu\sqrt{K/m}}$$

or amplitude at resonance $$= \frac{F}{\mu}\sqrt{\frac{m}{K}}.$$

In the case of sound, the study of sharpness of resonance is of great importance. Sharpness of resonance refers to the fall in amplitude with change in frequency on each side of the maximum amplitude.

The particular solution for displacement in the case of forced vibrations is,

$$y = \frac{F}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \sin(pt - \alpha) \qquad \ldots (1)$$

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \cos(pt - \alpha) \qquad \ldots (2)$$

The velocity *(dy/dt)* is maximum when cos *(pt – α)* is maximum *i.e.* the instant at which the particle crosses the mean position.

$$\left(\frac{dy}{dt}\right)_{max} = \frac{Fp}{\sqrt{\mu^2 p^2 + (K - mp^2)^2}} \qquad \ldots (3)$$

Kinetic energy of the vibrating particle at the instant of crossing the mean position is given by

$$\text{K.E.} = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2_{max}$$

$$\text{K.E.} = \frac{\frac{1}{2} m F^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \qquad \ldots (4)$$

The mean square of the driving force per unit mass

$$= \frac{\left(\frac{O + F^2}{2}\right)}{m} = \frac{F^2}{2m} \qquad \ldots (5)$$

Dividing equation (4) by $\frac{F^2}{2m}$ we get *kinetic energy per unit force* which is called the response R.

$$R = \frac{\frac{1}{2} mF^2 p^2}{\mu^2 p^2 + (K - mp^2)^2} \div \frac{F^2}{2m}$$

$$R = \frac{m^2 p^2}{\mu^2 p^2 + (K - mp^2)^2}$$

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2} + \left(\frac{K}{m} - p^2\right)^2} \quad \ldots (6)$$

The natural frequency of the system in the absence of damping is $\sqrt{\frac{K}{m}}$

Therefore, the term $\left(\frac{K}{m} - p^2\right)$ in equation (6) represents the extent to which the natural frequency of the system deviates from the forced frequency.

When $\frac{K}{m} = p^2$

the natural frequency coincides with the forced frequency, and the value of R will be maximum. From equation (6)

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2}} = \frac{m^2}{\mu^2} = \left(\frac{m}{\mu}\right)^2 \quad \ldots (7)$$

The response $R \propto \frac{1}{\mu}$

It means that the response R is inversely proportional to the frictional force. In the absence of friction, the response is maximum.

The term $\left(\frac{K}{m} - p^2\right)$ in equation (6), refers to mistuning. The larger is its value, the greater is the system away from resonance.

The graph between p/ω along the X-axis and the response R along the Y-axis is shown in Fig. 3.3.

(*i*) When p/ω is equal to 1 the response is maximum. For curve A, μ is large and for curve C, μ is less. The response decreases for values of p/ω greater than 1 or less than 1.

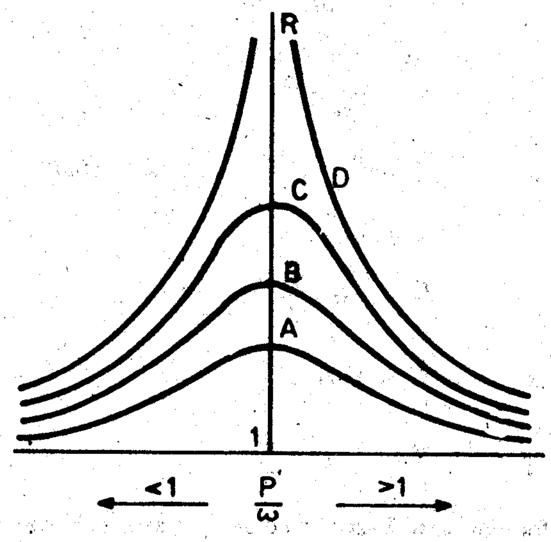

Fig. 3.3.

(*ii*) When the frictional forces are absent, *i.e.*, $\mu = 0$, R is infinite and the sharpness of resonance is maximum.

(*iii*) The sharpness of resonance decreases with increase in the value of μ.

(*iv*) The sharpness of resonance dies rapidly even for a very small change in the value of p/ω from 1, in the case, where μ is minimum.

In the case of the resonance tube, the damping force is large and the graph will be similar to the curve A in Fig. 3.3. The resonance persists over a wide range and it is difficult to exactly locate the position of maximum sharpness of resonance. Hence the results obtained with the resonance tube apparatus are not very accurate.

In the case of the sonometer wire, the damping forces are small and the graph will be similar to curve C in Fig. 3.3. In this case the sharpness of resonance is maximum in a very narrow region. Even a slight variation in length or tension reduces the sharpness considerably. The vibrations die out rapidly. Thus, the results obtained with a sonometer are accurate.

3.7. Phase of Resonance

Considering the phase lead of the forced vibrations with reference to the driving force, in equation (8) of Art. 3.5,

$$\tan \alpha = \frac{\mu p}{K - mp^2}$$

17. Discuss the phenomenon of sharpness of resonance and show how it depends on the damping factor. *(Kanpur, 1975)*
18. What are free, damped and forced vibrations? Give the theory of forced vibrations and discuss the condition of resonance. *[Delhi (Sub.), 1976]*
19. What are damped vibrations? Obtain an expression for the displacement in the case of a damped oscillatory motion. Discuss the effect of damping on the natural frequency. *[Delhi (Sub.) Supp., 1976]*
20. What is a forced vibration? Discuss mathematically, the vibration of a system executing damped simple harmonic motion when subjected to an external periodic force. What is sharpness of resonance? *(Delhi, 1976)*
21. Define quality factor and bandwidth of the sharpness of resonance. Obtain quality factor for a driven harmonic oscillator at resonance. *(Bhagalpur, 1990)*
22. Explain, in brief, (*i*) free oscillations, (*ii*) forced oscillation and (*iii*) phenomena of resonance. *(Bhagalpur, 1990)*
23. What do you understand by damped vibrations? Obtain an expression for displacement as a function of time for a damped oscillator. What is the effect of damping on the natural frequency of the oscillator? *(Delhi, 1991)*
24. Discuss the phenomenon of sharpness of resonance and show how it depends on the damping factor? *(Delhi, 1992)*
25. (*a*) Derive the differential equation of damped oscillatory motion and give its general solution.
    (*b*) What type of motion do you get when the damping is small? *(Delhi, 1991)*

CHAPTER 4

Wave Motion

4.1. Wave Motion

Wave motion is a form of disturbance which travels through the medium due to the repeated periodic motion of the particles of the medium about their mean positions, the disturbance being handed over from one particle to the next. When a stone is dropped into a pond containing water, waves are produced at the point where the stone strikes the water in the pond. The waves travel outward, the particles of water vibrate only up and down about their mean positions. Water particles do not travel along with the wave. Similarly when a tuning fork is set into vibration, it produces waves in air. The wave travels from one particle to the next but the particles of air vibrate about their mean positions.

It is essential to understand the concept of wave motion in the study of various branches in Physics. Wave motion, in general, refers to the transfer of energy from one point to another point of the medium. Transference of various forms of energy like sound, heat, light, X-rays, γ-rays, radio-waves etc. takes place in the form of wave motion. For the transference of energy through a medium, the medium must possess the properties of *elasticity, inertia and negligible frictional resistance.*

4.2. What Propagates in Wave Motion?

Before studying the characteristics of the different forms of wave motion, it is essential to clearly understand–*what is propagated in a wave motion* ? The answer to this question is that the physical condition due to a disturbance generated at some point in the medium is propagated to other points in the medium. In all the waves, the particles of the medium vibrate about their mean positions. Hence, in the case of wave motion, it is not matter that is propagated but it is only state of motion of the matter that is propagated. It is a form of *dynamic condition* that is propagated from one point to the other point in the medium.

According to the laws of Physics, any dynamic condition is related to momentum and energy. To conclude, it may be said that in wave motion *momentum and energy* are transferred or propagated. It is not a case of propagation of matter as a whole.

4.3. Characteristics of Wave Motion

1. Wave motion is a disturbance produced in the medium by the repeated periodic motion of the particles of the medium.

2. Only the wave travels forward whereas the particles of the medium vibrate about their mean positions.

3. There is a regular phase change between the various particles of the medium. The particle ahead starts vibrating a little later than a particle just preceding it.

4. The velocity of the wave is different from the velocity with which the particles of the medium are vibrating about their mean positions. The wave travels with a uniform velocity whereas the velocity of the particles is different at different positions. It is maximum at the mean position and zero at the extreme position of the particles.

There are two types of wave motions :
(*i*) Transverse and (*ii*) Longitudinal.

Sound waves are longitudinal waves and light waves are transverse waves.

4.4. Transverse Wave Motion

In this type of wave motion, the particles of the medium vibrate at right angles to the direction of propagation of the wave.

To understand the propagation of transverse waves in a medium consider nine particles of the medium and the circle of reference (Fig. 4.1). The particles are vibrating about their mean positions up and down and the wave is travelling from left to right. The disturbance takes $T/8$ seconds to travel from one particle to the next.

(1) At $t = 0$, all the particles are at their mean position.

(2) After $T/8$ seconds, particle 1 travels a certain distance upward and the disturbance reaches particle 2.

(3) After $2T/8$ seconds, particle 1 has reached its extreme position and the disturbance has reached particle 3.

(4) After $3T/8$ seconds, particle 1 has completed 3/8 of its vibration and the disturbance has reached particle 4. The positions of particles 2 and 3 are also shown in Fig. 4.1.

(5) In this way after $T/2$ seconds, particle 1 has come back to its mean position and the particles 2, 3 and 4 are at the positions shown in the diagram.

The disturbance has reached particle 5.

In this way the process continues and the positions of the particles after *5T/8, 6T/8, 7T/8* and *T* seconds are shown in the diagram.

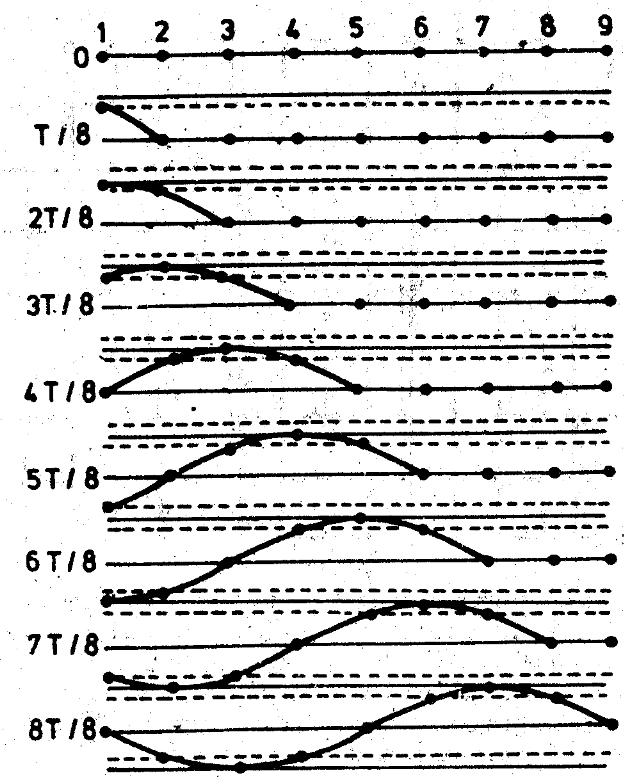

Fig. 4.1.

After T seconds, the particles 1, 5 and 9 are at their mean positions. The wave has reached particle 9. Particles 1 and 9 are in the same phase. The wave has travelled a distance between particles 1 and 9 in the time in which the particle 1 has completed one vibration.

The top point on the wave at the maximum distance from the mean position is called *crest*, while the point at the maximum distance below the mean position is called trough. Thus in a transverse wave, crests and troughs are alternately formed. The contour of the displaced particles of the medium represents the wave. In the case of transverse (or longitudinal) progressive waves, this contour continuously changes position in space and the wave seems to advance in the direction of propagation.

4.5. Longitudinal Wave Motion

In this type of wave motion, particles of the medium vibrate along the direction of propagation of the wave.

Consider nine particles of the medium and the circle of reference (Fig. 4.2).

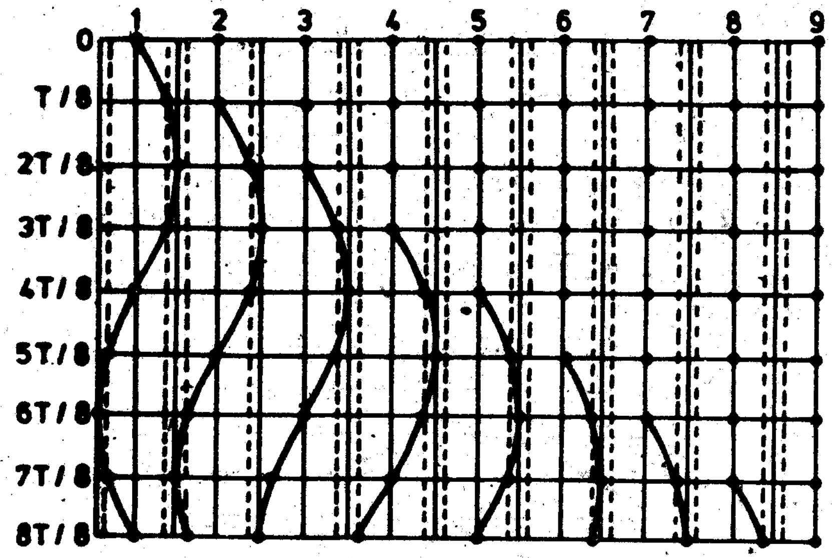

Fig. 4.2.

The wave travels from left to right and the particles vibrate about their mean positions. After $T/8$ seconds, the particle 1 goes to the right and completes 1/8 of its vibration. The disturbance reaches the particle 2. After $T/4$ seconds the particle 1 has reached its extreme right position and completes 1/4 of its vibration and the particle 2 completes 1/8 of its vibration. The disturbance reaches the particle 3. The process continues.

After one complete time period, the positions of the various particles is as shown in the diagram. The wave has reached particle 9. Here 1 and 9 are again in the same phase. Here particles 1, 5 and 9 are at their mean positions. The particles 1 and 3 are close to the particle 2. This is the position of condensation. Similarly particles 9 and 8 are close to the particle 7. This is also the position of condensation or compression. On the other hand, particles 4 and 6 are far away from the particle 5. This is the position of rarefaction. Hence in a longitudinal wave motion, condensations and rarefactions are alternately formed.

4.6. Definitions

Wavelength. It is the distance travelled by the wave in the time in which the particle of the medium completes one vibration. It is also defined as the distance between two nearest particles in the same phase.

The distance AB (Fig. 4.3) is equal to the wavelength λ.

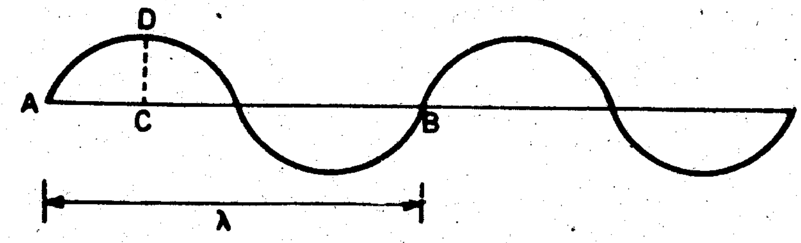

Fig. 4.3.

Frequency. It is the number of vibrations made by a particle in one second.

Amplitude. It is the maximum displacement of the particle from its mean position of rest. In the diagram CD is the amplitude.

Time period. It is the time taken by a particle to complete one vibration.

Suppose frequency $= n$

Time taken to complete n vibrations = 1 second.

Time taken to complete 1 vibration $= \frac{1}{n}$ second.

From the definition of time period, time taken to complete one vibration is the time period (T)

$$\therefore \qquad T = \frac{1}{n} \qquad \text{or} \qquad nT = 1$$

$$\therefore \qquad \text{Frequency} \times \text{Time period} = 1$$

Vibration. It is the to and fro motion of a particle from one extreme position to the other and back again. It is also equal to the motion of a particle from the mean position to one extreme position, then to the other extreme position and finally back to the mean position.

Phase. It is defined as the ratio of the displacement of the vibrating particle at any instant to the amplitude of the vibrating particle or it is defined as the fraction of the time interval that has elapsed since the particle crossed the mean position of rest in the positive direction or it is also equal to the angle swept by the radius vector since the vibrating particle last crossed its mean position of rest.

4.7. Relation between Frequency and Wavelength

Velocity of the wave is the distance travelled by the wave in one second.

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}}$$

Wavelength (λ) is the distance travelled by the wave in one time period (T).

$$\text{Velocity} = \frac{\text{Wavelength}}{\text{Time period}} = \frac{\lambda}{T}$$

But, frequency × time period = 1

$$n \times T = 1$$

$$T = \frac{1}{n}$$

$$v = \frac{\lambda}{T} = \frac{\lambda}{\frac{1}{n}}$$

$$v = n\lambda$$

Example 4.1. *If the frequency of a tuning fork is 400 and the velocity of sound in air is 320* metres/s, *find how far sound travels while the fork completes 30 vibrations.*

Here, $n = 400$, $v = 320$ metres/second, $\lambda = ?$

$$v = n\lambda$$

or $$\lambda = \frac{v}{n} = \frac{320}{400} = \mathbf{0{\cdot}8\ metre}$$

∴ Distance travelled by the wave when the fork completes 1 vibration
= 0·8 metre

Distance travelled by the wave when the fork completes 30 vibrations
= 0·8 × 30 = **24 metres.**

4.8. Properties of Longitudinal Progressive Waves

1. The particles of the medium vibrate simple harmonically along the direction of propagation of the wave.

2. All the particles have the same amplitude, frequency and time period.

3. There is a gradual phase difference between the successive particles.

4. All the particles vibrating in phase will be at a distance equal to $n\lambda$. Here $n = 1, 2, 3$ etc. It means the minimum distance between two particles vibrating in phase is equal to the wavelength.

5. The velocity of the particle is maximum at their mean position and it is zero at their extreme positions.

6. When the particle moves in the same direction as the propagation of the wave, it is in a region of compression.

7. When the particle moves in a direction opposite to the direction of propagation of the wave, it is in a region of rarefaction.

8. When the particle is at the mean position, it is a region of maximum compression or rarefaction.

9. When the particle is at the extreme position, the medium around the particles has its normal density, with compression on one side and rarefaction on the other.

10. Due to the repeated periodic motion of the particles, compressions and rarefactions are produced continuously. These compressions and rarefactions travel forward along the wave and transfer energy in the direction of propagation of the wave.

4.9. Demonstration of Transverse Waves

(*a*) Wave Apparatus

The formation of transverse waves can be demonstrated in the laboratory with the help of the wave motion apparatus. The apparatus consists of a vertical rectangular frame fixed on a horizontal base (Fig. 4.4). The axle passes escentrically through a number of circular discs with grooved edges. The circular discs are equidistant. Vertical rods carrying spherical balls at their upper ends rest on the circumference of their respective discs. When the axle is rotated, the rods are displaced vertically through different distances. It is adjusted that there is a gradual phase difference between the successive rods. Each ball will execute simple harmonic motion and will complete one vibration in one rotation of the discs. When the axle is continuously rotated with a uniform speed, the balls show a wave pattern and the transverse wave appears to progress in the forward direction with the formation of alternate crests and troughs. It is observed that the balls move in the vertical direction and the wave advances in the horizontal direction. Therefore, transverse waves are produced.

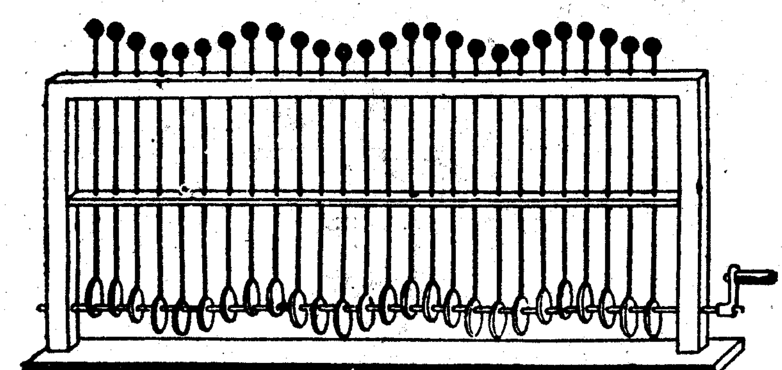

Fig. 4.4.

(*b*) Ripple Tank

The formation of transverse waves on the surface of water can be demonstrated in the laboratory with the help of ripple tank apparatus.

The ripple tank apparatus consists of a shallow rectangular dish with a glass bottom. The edges of the dish are sloping outwards so as to avoid interference between the direct and the reflected waves. An electrically maintained tuning fork having a fine style fixed to one of its prongs is adjusted so that the tip of the style just dips in water. The field of view is illuminated from below with the help of a strong source of light S and a condensing lens system C (Fig. 4.5).

The tuning fork is set into vibration. The style oscillates vertically up and down and transverse waves are produced on the surface of water. These waves originate at the tip of the style and travel radially outwards. The image of the water surface is obtained on the screen by reflection from the mirror M. The wave pattern is observed on the screen.

If a stroboscope is used so that the dish is illuminated intermittently with the same frequency as that of the tuning fork, a stationary pattern is obtained on the screen.

Using the ripple tank apparatus, interference phenomenon can be demonstrated fixing two styles to the same prong of the tuning fork.

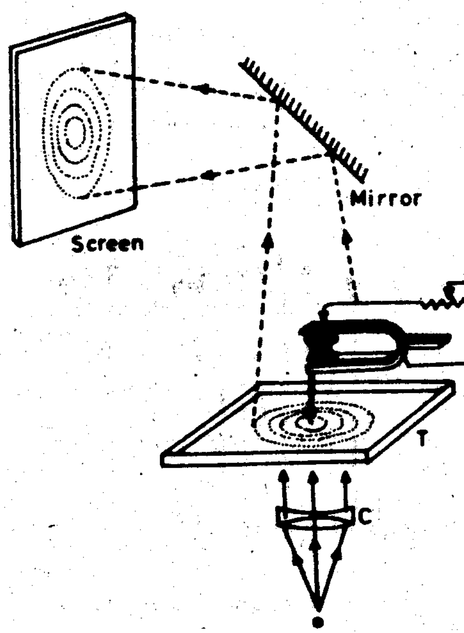

Fig. 4.5.

In this case, amplitude and frequency of vibrations produced by the two styles will be the same. The interference pattern can be observed on the screen.

If instead of a style, a thin edged blade is used, plane progressive transverse waves are produced on the surface of water.

4.10. Demonstration of Longitudinal Waves

The formation of longitudinal waves can be demonstrated with the help of a spring. One end of the spring is fixed to a handle H and the other end is free, A small push is given to the handle. The first three turns are compressed, the rest of the spring is in the relaxed position (Fig. 4.6). When the handle is brought back to its original position, the compression travels forward and there is rarefaction between the handle and the compression. If the handle is kept fixed at the initial position it is seen that with time, the compression and rarefaction travel forward as shown in Fig. 4.6. This demonstrates the formation of longitudinal waves in which any particle on the spring vibrates simple harmonically along the direction of propagation of the wave.

If the handle is vibrating continuously, continuous compressions and rarefactions are produced alternately all along the spring.

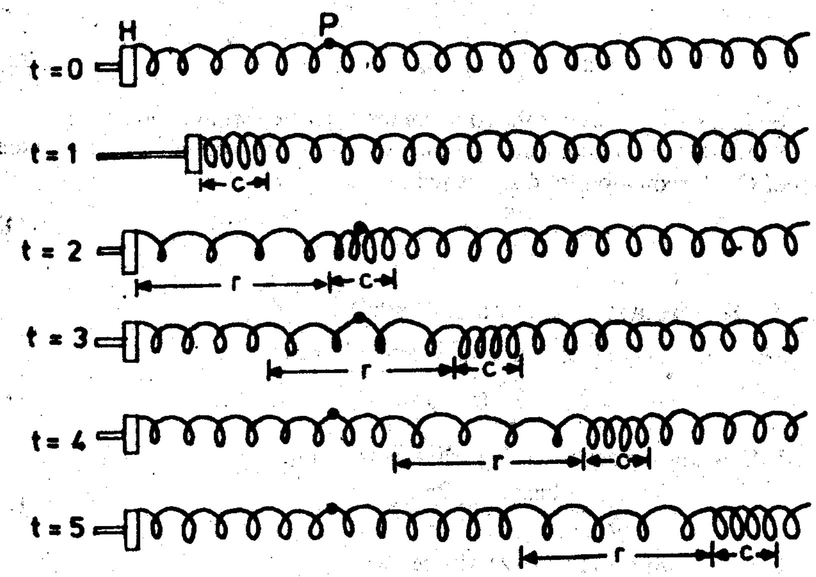

Fig. 4.6.

4.11. Equation of a Simple Harmonic Wave

Consider a particle O in a medium. Let the displacement at any instant of time be given by

$$y = a \sin \omega t \qquad \ldots (1)$$

Consider another particle A at a distance x from the particle O to its right. Here it is assumed that the wave is travelling with a velocity v from left to right *i.e.* from particle O towards A. The displacement at A is given by

$$y = a \sin (\omega t - \alpha) \qquad \ldots (2)$$

where α is the phase difference between the particles O and A. For a phase difference of 2π, the path difference is λ. Suppose for a phase difference of α, the path difference is x.

$$\therefore \quad \frac{\alpha}{x} = \frac{2\pi}{\lambda}$$

$$\therefore \quad \alpha = \frac{2\pi x}{\lambda}$$

Also $$\omega = \frac{2\pi}{T} = \frac{2\pi v}{\lambda}$$

Substituting the values of α and ω in equation (2),

$$y = a \sin\left(\frac{2\pi vt}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (3)$$

Equation (3) represents the equation for a simple harmonic wave.

Similarly for a particle at a distance x in the negative direction (*i.e.* to the left of O), the equation for displacement is,

$$y = a \sin \frac{2\pi}{\lambda}(vt + x) \quad \ldots (4)$$

4.12. Differential Equation of Wave Motion

The general equation of a simple harmonic wave is,

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (1)$$

Differentiating equation (1) with respect to time,

$$\frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (2)$$

Differentiating equation (2) with respect to time,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (3)$$

To find the value of compression, differentiate equation (1) with respect to x,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (4)$$

To find the rate of change of compression with respect to distance, differentiate equation (4) with respect to x,

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (5)$$

From equations (2) and (4)

$$\frac{dy}{dt} = -v\frac{dy}{dx} \quad \ldots (6)$$

From equations (3) and (5)

$$\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2} \quad \ldots (7)$$

Equation (7) represents the differential equation of wave motion.

The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K\frac{d^2y}{dx^2} \quad \ldots (8)$$

Here $$K = v^2$$

or $$v = \sqrt{K}$$

Thus, knowing the value of K, the value of the wave velocity can be calculated.

4.13. Particle Velocity and Wave Velocity

The equation for a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (1)$$

Here v is the velocity of the wave and y is the displacement of the particle.

The velocity of the particle $U = dy/dt$.

$\therefore$ Differentiating equation (1) with respect to time t,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \quad \ldots (2)$$

The maximum value of the particle velocity is

$$U_{max} = \frac{2\pi av}{\lambda} \quad \ldots (3)$$

$\therefore$ [Maximum Particle Velocity] $= \frac{2\pi a}{\lambda}$ [Wave Velocity]

To find the particle acceleration, differentiate equation (2) with respect to time

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \quad \ldots (4)$$

$$f = -\frac{4\pi^2 v^2}{\lambda^2}\left[a \sin \frac{2\pi}{\lambda}(vt - x)\right]$$

$$\therefore \quad f = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right)y$$

16. Obtain an expression for the velocity of sound in a gas discussing in detail Newton's formula and Laplace's correction. What is the effect of temperature variation on the velocity of sound in a gas.
(Delhi, 1976)
17. Obtain an expression for the velocity of sound in air. How does the velocity depend on humidity, temperature and pressure? What are the other factors which influence the velocity of sound?
[Delhi (Suppl.), 1976]
18. At what temperature is the velocity of sound in nitrogen gas is equal to its velocity in oxygen at 20°C. The atomic weights of oxygen and nitrogen are in the ratio 16 : 14. *(Delhi, 1971)*
[**Hint.** $T_2 \rho_1 = T_1 \rho_2$] [**Ans.** 16.7°]
19. Derive an expression for the excess pressure at a point in compression waves in a fluid and hence obtain the velocity of propagation of waves. *(Bhagalpur, 1990)*
20. Find an expression for the velocity of longitudinal waves through a homogeneous, elastic medium. *(Guahati, 1992)*

CHAPTER 6

Stationary Waves, Interference and Beats

6.1. Stationary Waves

When two simple harmonic waves of the same amplitude, frequency and time period travel in *opposite directions* in a straight line, the resultant wave obtained is called a stationary or a standing wave. The formation of stationary waves is due to the superposition of the two waves on the particles of the medium.

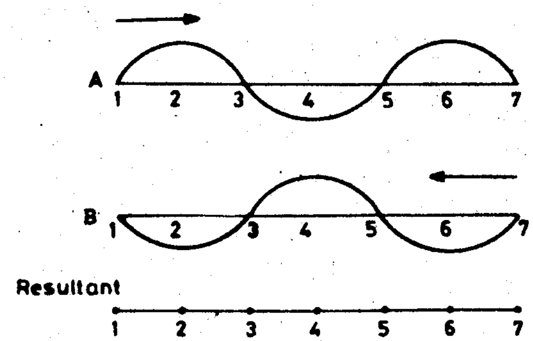

Fig. 6.1.

The formation of stationary waves can be represented graphically as follows :

Consider two wave trains A and B of the same amplitude, frequency and wavelength travelling in opposite directions. At an instant of time $t = 0$, the waves are as shown in Fig. 6.1. The resultant displacement curve is a straight line. All the particles of the medium are at their mean positions.

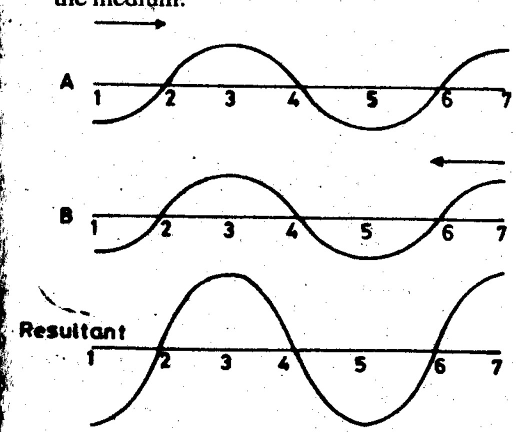

Fig. 6.2.

At time $t = T/4$, the wave A will advance through a distance $\lambda/4$ towards right, and the wave B will advance through a distance $\lambda/4$ towards left. The resultant displacement pattern is shown in Fig. 6.2.

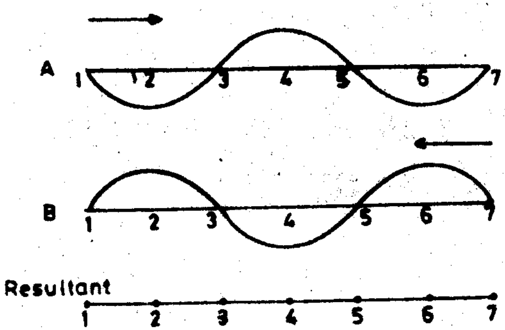

Fig. 6.3.

The particles 1, 3, 5, and 7 are at their extreme positions and particles 2, 4 and 6 are at their mean positions.

At time $t=T/2$, the wave A will advance through a distance $\lambda/2$ towards right and the wave B will advance through a distance $\lambda/2$ towards left (with reference to zero time).

The resultant displacement pattern is shown in Fig. 6.3.

All the particles of the medium are at their mean positions.

At time $t = 3T/4$, the wave A will advance through a distance $3\lambda/4$ towards right and the wave B will advance through a distance $3\lambda/4$ towards left (with reference to zero time).

The resultant displacement pattern is shown in Fig. 6.4.

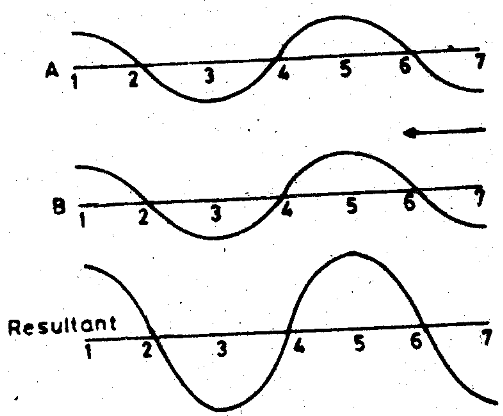

Fig. 6.4.

The particles 1, 3, 5 and 7 are at their extreme positions and 2, 4, 6 are at their mean positions.

At time $t = T$, the wave A will advance through a distance λ towards right and the wave B will advance through a distance λ towards left (with reference to zero time). The resultant displacement pattern is shown in Fig. 6.5.

All the particles are at their mean positions.

From the patterns discussed above it is clear that the particles of the medium such as 2, 4, 6 etc. always remain at their mean positions. The particles such as 1, 3, 5, 7 etc. continue to vibrate simple harmonically about their mean positions with double the amplitude of each wave. It appears as though the wave pattern is stationary in space. The resultant displacement patterns at intervals of time,

$$0, \frac{T}{4}, \frac{T}{2}, \frac{3T}{4}, T$$

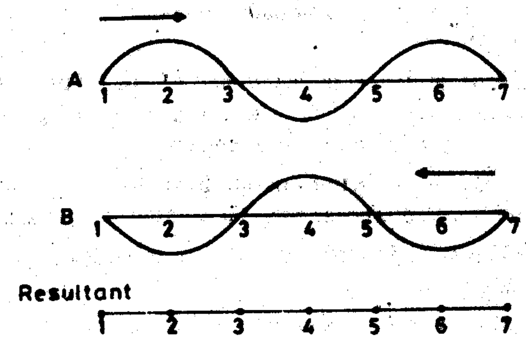

Fig. 6.5.

are shown in Fig. 6.6.

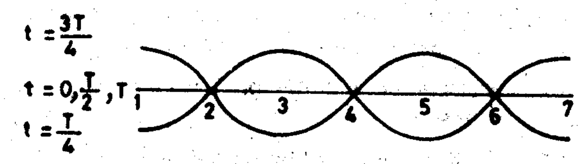

Fig. 6.6.

The positions of the particles 2, 4, 6 etc. which always remain at their mean ositions are called *nodes*. Node is a position of zero displacement and maxi-um strain.

The positions of the particles 1, 3, 5, 7 etc. which vibrate simple harmonical- with maximum amplitude (twice the amplitude of each wave) are called tinodes. At the antinodes, the strain is minimum. The distance between any o consecutive nodes or antinodes is equal to $\lambda/2$. Between a node and an tinode, the amplitude gradually increases from zero to maximum.

2. Properties of Stationary Longitudinal Waves

The stationary waves are formed due to the superposition of two simple rmonic longitudinal progressive waves of the same amplitude and periodic e and travelling in opposite directions. The important properties of these aves are :

(1) In these waves, nodes and antinodes are formed alternately. Nodes are e positions where the particles are at their mean positions having maximum rain. Antinodes are the positions where the particles vibrate with maximum nplitude having minimum strain.