WAVES & OSCILLATIONS

S.H.M (Simple Harmonic Motion)

Whenever a force acting on a particle and hence the acceleration of the particle is proportional to its displacement from its equilibrium position or any other fixed point in its path, but it is always directed in a direction opposite to the direction of the displacement and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute a simple harmonic motion.

Differential Equation of a Simple Harmonic Motion:

If F be the force acting on a particle and y its displacement from its equilibrium position,

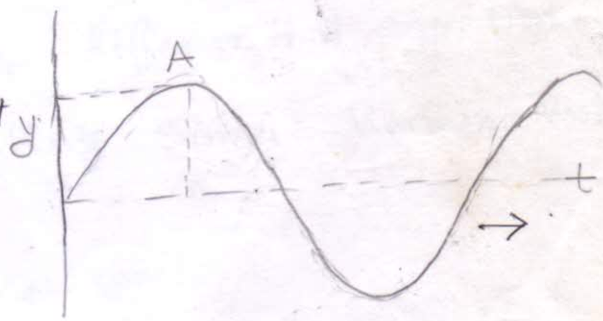


Fig - Simple harmonic Motion

$F \propto y$

then $F = -ky$ ——— (i)

According to Newtons second law of motion, $F = ma'$ ——— (ii)
(where a' is the acceleration.)

From (i) and (ii) we get.

$$ma' = -ky$$

$$\text{or,} \quad a' = -\frac{ky}{m}, \quad \text{——— (iii)}$$

where k is a force constant.

when y has its maximum positive value, the acceleration has its maximum negative value; and at the instant the particle passes the equilibrium position ($y=0$), the acceleration is zero. but velocity of course, not zero at this point.

The acceleration in differential form in this position $a' = \frac{d^2y}{dt^2}$, and hence, we can write,

$$-ky = m\frac{d^2y}{dt^2} \qquad (\because F = ma')$$

$$\text{or,} \quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad \text{———} \quad (iv)$$

Equation (iv) is called the differential equation of motion of a body executing simple harmonic motion.

Rearranging equ^n (iv), we can write

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega^2 y \quad \text{———} \quad (v)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular velocity of the particle.

To obtain general solution of differential equation of simple harmonic motion, let us multiply both sides of equ^n (v) by $2\frac{dy}{dt}$, when we get

$$2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = -\omega^2 y . 2\frac{dy}{dt}$$

or, $\cancel{2\frac{dy}{dt}\cdot\left(\frac{dy}{dt}\right)^2} = \cancel{-\omega^2 . 2 . \frac{dy}{dt} . y}$

or. $2\frac{dy}{dt}\frac{d^2y}{dt^2} = -\omega^2 . 2\frac{dy}{dt} . y = -2\omega^2 y\frac{dy}{dt}$

Integrating with respect to time we have

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + c \qquad \text{(vi)}$$

where c is a constant of integration. To evaluate c we recall, at maximum ~~amplitude~~ displacement

$\frac{dy}{dt} = 0$ (when Y = amplitude 'a')

$\therefore \quad 0 = -\omega^2 a^2 + c$

or. $c = \omega^2 a^2 \qquad \text{(vii)}$

Therefore from equⁿ (vi) we can write

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + \omega^2 a^2$$
$$= \omega^2(a^2 - y^2)$$

or, $\frac{dy}{dt} = \pm \omega\sqrt{a^2 - y^2}$

$= \pm\sqrt{\frac{k}{m}(a^2 - y^2)}$

or, $\frac{dy}{\sqrt{a^2 - y^2}} = \omega . dt \qquad \text{(viii)}$

~~integr~~

Integrating again with respect to time we have

$$\left\{ \frac{d}{dt}\left(\sin^{-1}\frac{y}{a}\right) = \frac{dy}{\sqrt{a^2-y^2}} \right\}$$

$$\sin^{-1}\frac{y}{a} = \omega t + \phi$$

or $y = a\sin(\omega t + \phi)$ —— (1x)

~~This is the final equation of motion of a SHM.~~
This is the general solution of the differential equation of SHM

Energy of a body Executing SHM:

Total Energy:

If K be the kinetic energy and U be the potential energy of a particle executing SHM then total energy,

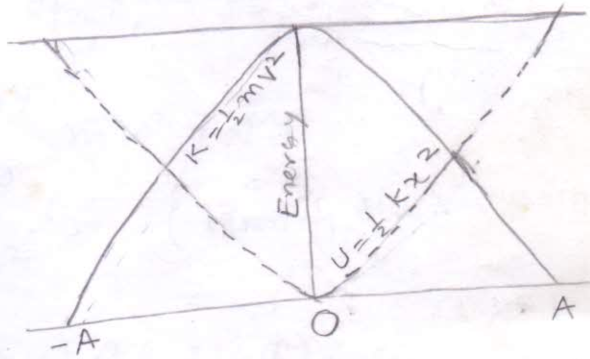


Fig:- Energy of SHM.

$$E = K + U \quad \text{—— (1)}$$

Let the displacement of a particle executing SHM at any instant be y. If the mass of the particle be m and velocity v then its kinetic energy is $\frac{1}{2}mv^2$.

The potential energy at the same instant is the amount of work that must be done in overcoming the force through a displacement y and is given by the relation $\int_0^y F.dy$ where F is the force required to maintain the displacement and dy is a small displacement.

Now the displacement is given by the relation

$$y = a \sin(\omega t + \phi)$$

Therefore velocity $\frac{dy}{dt} = a\omega \cos(\omega t + \phi)$ and the

acceleration $\frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \phi) = -\omega^2 y$

Then, force F = mass x acceleration

$$= m(-\omega^2 y) = -m\omega^2 y$$

Then, the potential energy of the particle is —

$$P.E. = \int_0^y F.\,dy = \int_0^y m\omega^2 y\, dy$$

(ignoring the minus sign)

$$= \frac{1}{2} m \omega^2 y^2$$

$$= \frac{1}{2} m \omega^2 a^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} K a^2 \sin^2(\omega t + \phi) \quad \text{—— (ii)}$$

$$\left[\because \omega^2 = \frac{K}{m}\right]$$

Now kinetic energy of the particle

$$KE = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2} m \left[\omega a \cos(\omega t + \phi)\right]^2$$

$$= \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} K a^2 \cos^2(\omega t + \phi) \quad \text{—— (iii)}$$

Therefore the total energy

$$E = K + U = \frac{1}{2} K a^2 \sin^2(\omega t + \phi) + \frac{1}{2} K a^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} K a^2 \left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2} K a^2$$

again,

$$E = \frac{1}{2} \cancel{k} a^2$$
$$= \frac{1}{2} m\omega^2 a^2$$
$$= \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 a^2 = \frac{2\pi^2 m a^2}{T^2} \quad , \left[\because \omega = \frac{2\pi}{T}\right]$$

But $\frac{1}{T} = n$, the frequency of the oscillation.

$$E = \frac{1}{2} K a^2 = 2\pi^2 m\, a^2 n^2$$

= maximum value of potential energy

= maximum value of kinetic energy.

Average Values of Energies of Harmonic Oscillator:

The potential energy of the particle at a displacement y is given by.

$$PE = \frac{1}{2} m\omega^2 y^2 \qquad [y = a\sin(\omega t + \phi)]$$
$$= \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t + \phi),$$

So the average potential energy of a particle over a complete cycle

$$= \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t + \phi)\, dt$$

$$= \frac{1}{T} \cdot \frac{m\omega^2 a^2}{4} \int_0^T 2\sin^2(\omega t + \phi)\, dt$$

$$= \frac{m\omega^2 a^2}{4T} \int_0^T [1 - \cos 2(\omega t + \phi)]\, dt$$

$$= \frac{m\omega^2 a^2}{4T} \left\{ \int_0^T dt - \int_0^T \cos 2(\omega t + \phi)\, dt \right.$$

The average value of both a sine and a cosine function for a complete cycle is zero.

Therefore,

$$P.E. = \frac{1}{4T} m\omega^2 a^2 [t]_0^T - 0$$

$$= \frac{1}{4} m\omega^2 a^2$$

$$= \frac{1}{4} k a^2 \qquad \left[\omega^2 = \frac{k}{m}\right] \text{ ——— (1)}$$

The kinetic energy of the particle at displacement y is given by

$$KE = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

$$KE = \frac{1}{2} m \left[\frac{d}{dt} a\sin(\omega t + \phi)\right]^2$$

$$= \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi)$$

The average KE of the particle for a complete cycle is

$$= \frac{1}{T}\int_0^T \frac{1}{2} m\omega^2 a^2 \cos^2(\omega t + \phi)\, dt$$

$$= \frac{1}{T} \frac{m\omega^2 a^2}{4} \int_0^T \left[1 + \cos 2(\omega t + \phi) dt\right]$$

$$= \frac{m\omega^2 a^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2(\omega t + \phi)\right.$$

The average value of both sine and cosine function over a complete ~~of~~ cycle is zero.

$$\therefore \text{ Avercge } K.E. = \frac{m\omega^2 a^2}{4T}[t]_0^T = \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} k a^2$$

Average PE = average $KE = \frac{1}{4} ka^2$ = half the total energy.

Combination of SHM:

* Composition of two SHM of same frequency but different phase and amplitude:

Let two equation of SHV are,

$$y_1 = a_1 \sin(\omega t + \phi_1) \text{ and}$$

$$y_2 = a_2 \sin(\omega t + \phi_2)$$

where y_1 and y_2 are the displacement of the particles due to the individual vibration of amplitudes a_1 and a_2 respectively. If Y be the resultant displacement then,

$$Y = y_1 + y_2 = a_1 \sin(\omega t + \phi_1) + a_2 \sin(\omega t + \phi_2)$$

$$= a_1\{\sin\omega t \cos\phi_1 + \cos\omega t \sin\phi_1\}$$

$$+ a_2\{\sin\omega t \cos\phi_2 + \cos\omega t \sin\phi_2\}$$

$$= (a_1 \cos\phi_1 + a_2 \cos\phi_2)\sin\omega t$$

$$+ (a_1 \sin\phi_1 + a_2 \sin\phi_2)\cos\omega t$$

a_1 and a_2 and ϕ_1 and ϕ_2 are constant

Hence putting

$$a_1 \cos\phi_1 + a_2 \cos\phi_2 = A\cos\phi$$

and $a_1 \sin\phi_1 + a_2 \sin\phi_2 = A\sin\phi$

The resultant amplitude is

$$Y = A\cos\phi \sin\omega t + A\sin\phi\cos\omega t$$

$$= A\sin(\omega t + \phi)$$

* Composition of two SHV at right angles to eachother having equal frequencies but different phases and amplitudes.

Let us consider two Simple harmonic motions of the same frequency but of amplitude a and b and having their vibrations mutually perpendicular to one another. If ϕ is the phase difference between two then equations can be written as

$$x = a\sin(\omega t + \phi) \quad \text{——— (i)}$$

$$y = b\sin\omega t \quad \text{——— (ii)}$$

From equ (i),

$$\frac{x}{a} = \sin(\omega t + \phi)$$

$$= \sin\omega t\cos\phi + \cos\omega t\sin\phi$$

$$= \sin\omega t\cos\phi + \sqrt{1-\sin^2\omega t}\,\sin\phi \quad \text{——— (iii)}$$

from equ (ii)

$$\sin\omega t = \frac{y}{b} \quad \text{——— (iv)}$$

so, from equ (iii),

$$\frac{x}{a} = \frac{y}{b}\cos\phi + \sqrt{1-\frac{y^2}{b^2}}\,\sin\phi$$

or, $\left(\frac{x}{a} - \frac{y}{b}\cos\phi\right) = \sqrt{1-\frac{y^2}{b^2}}\,\sin\phi$

squaring both sides, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\phi - 2\frac{xy}{ab}\cos\phi = \sin^2\phi - \frac{y^2}{b^2}\sin^2\phi$$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2}(\sin^2\phi + \cos^2\phi) - 2\frac{xy}{ab}\cos\phi = \sin^2\phi$

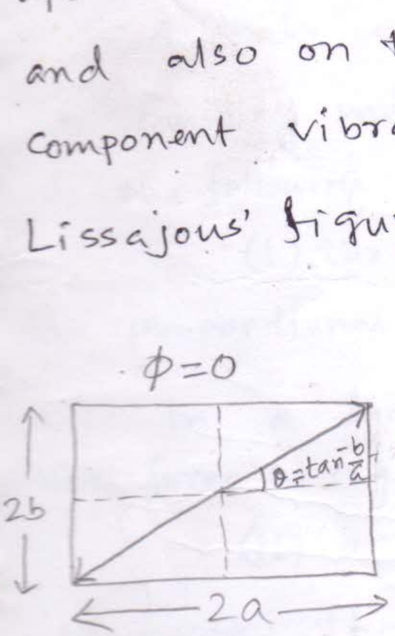
or, $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\phi=\sin^2\phi \quad \text{———} \quad (v)$$

This is the general equation of a conic whose shape will depend upon the value of the phase difference between the two vibrations.

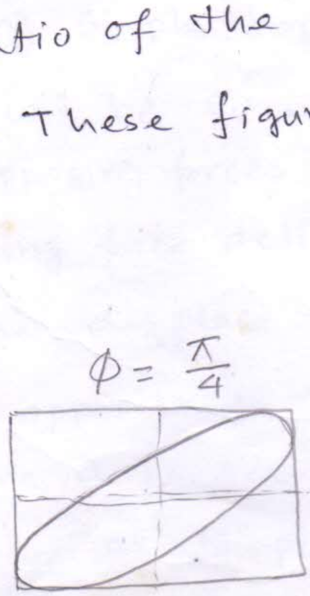
LISSAJOUS' FIGURES :

The composition of two simple harmonic vibrations in mutually perpendicular directions gives rise to an ellipse. The resultant motion is thus, in general, along an elliptical path. The actual shape of the curve will, however, depend upon the phase difference ϕ between the two vibrations and also on the ratio of the frequencies of the component vibrations. These figures are known as Lissajous' figures.

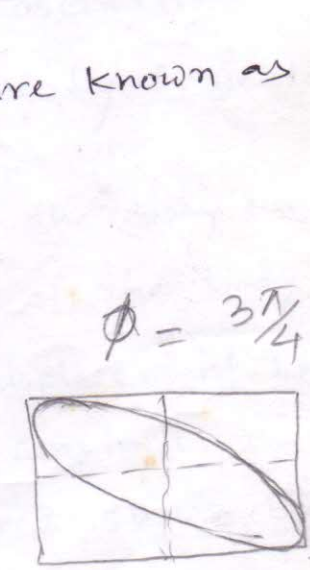
$\phi=0$



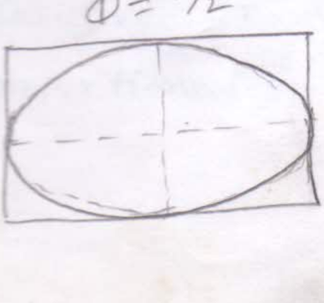
$\phi=\frac{\pi}{4}$



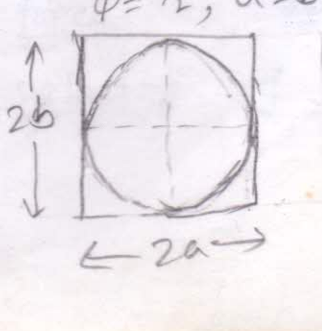
$\phi=3\pi/4$



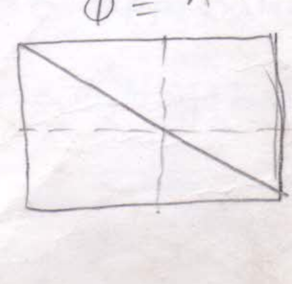
$\phi=\pi/2$



$\phi=\pi/2,\ a=b$



$\phi=\pi$



DAMPED VIBRATION / Oscillations:

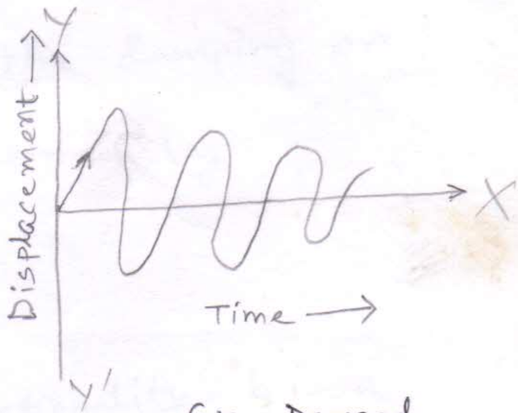


Fig – Damped oscillator

In actual practice a simple harmonic oscillator almost always vibrates in a resisting medium. Consequently when the oscillator vibrates in such a medium, energy is dissipated in each vibration, therefore, goes on decreasing progressively with time. Such forces, which are non-conservative in nature have thus a damping effect on the oscillation.

Damping Coefficient:

A body executing simple harmonic oscillations in a damping medium will be simultaneously subjected to the following two opposing forces:

(i) the restoring force acting on the body which is proportional to the displacement of the body and acts in a direction opposite to the displacement. Let this force $-ay$, where a is the force constant.

(ii) a resistive or damping force shown by Mayevski that at ordinary velocities, the opposing resistive or damping force is, to a first approximation, proportional to the velocity of the oscillating body.

As most cases of interest to ~~to~~ us fall in the category of ordinary velocities, the damping or resistive force may thus be represented by

$$F = -bv = -b\frac{dy}{dt},$$

Where b is a constant of proportionality. b is a constant called damping coefficient of the medium.

Thus the differential equation may be written as

$$m.\frac{d^2y}{dt^2} = -ay - b\frac{dy}{dt}$$

$$\text{or,}\quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = 0$$

$$\text{or,}\quad \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = 0$$

$$\text{a}\quad \frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0 \qquad \text{——— (1)}$$

where $2\lambda = \frac{b}{m}$ and $\omega^2 = \frac{a}{m}$

Eqn. (1) is referred to as the differential equation of a damped harmonic oscillator.

Spring Mass System :

Simple Hermonic Oscillations of a loaded Spring :

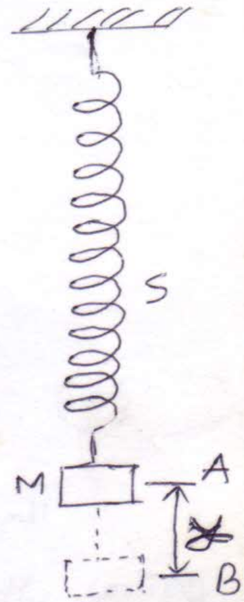


Fig - Spring with load

Consider a spring S whose upper end is fixed to a rigid support and lower end is attached to a mass M. In the equilibrium position the mass is at A.

suppose at any instant the mass is at B. The distance $AB = y$. Let the tension per unit displacement of the Spring be K.

$\therefore$ Force exerted ~~per unit~~ by the spring $= Ky$

According to Newton's second law:

$$\text{Force} = M\frac{d^2y}{dt^2} = -Ky$$

[$-$ ve sign shows that force is directly upward]

$$\therefore \quad M\frac{d^2y}{dt^2} + Ky = 0$$

$$\text{or,} \quad \frac{d^2y}{dt^2} + \frac{K}{M}y = 0 \quad \text{——— (1)}$$

This equation is similar to the equ^n of SHM;

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad \text{——— (11)}$$

$\therefore$ from (1) and (11) we can write

$$\omega^2 = \frac{K}{M}$$

$$\text{or } \omega = \sqrt{\frac{K}{M}} \quad \text{———— (iii)}$$

Again we know, Time period, $T = \frac{2\pi}{\omega}$

$$\therefore \cancel{T = 2\pi\sqrt{\frac{K}{M}}}$$

$$\therefore T = 2\pi\sqrt{\frac{M}{K}} \quad \text{———— (iv)}$$

To determine the value of K, a small mass m is attached to the free end of the spring. Let the increase in length is x.

Then $K = \dfrac{mg}{x}$

Therefore from equ^n (IV),

$$T = 2\pi\sqrt{\frac{Mx}{mg}} \quad \text{———— (v)}$$

Ex: The scale of a spring balance reading from 0 — 10kg is 0.25 m. A body suspended from the balance ~~reading from~~ oscillates with a frequency of $\frac{10}{\pi}$ hertz. Calculate the mass of the body attached to the spring.

Solution: We know,

$$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

Here, $m = 10\,kg$
$x = 0.25\,m$
$g = 9.8\,m/s^2$
frequ. $n = \frac{10}{\pi}$ Hz
$M = ?$

again $T = \frac{1}{n}$

$$\therefore \frac{1}{n} = 2\pi\sqrt{\frac{Mx}{mg}}$$

$$\text{or, } \frac{1}{n^2} = 4\pi^2 \frac{Mx}{mg}$$

$$\text{or, } M = \frac{mg}{4\pi^2 x n^2}$$

$$= \frac{10 \times 9.8}{4\pi^2 \times \left(\frac{10}{\pi}\right)^2 \times 0.25}$$

$$= \frac{10 \times 9.8 \times \pi^2}{4\pi^2 \times 100 \times 25}$$

$$= 0.98 \text{ kg} \quad \text{An.}$$

Calculation of Minimum Time Period of Compound Pendulum:

The time period of a compound pendulum

$$t = 2\pi\sqrt{\frac{k^2 + l_1^2}{l_1 g}} \quad \text{—— (1)}$$

Here l_1 is the distance of the point of suspension from the centre of gravity and k is the radius of gyration about the C.G. Suppose the distance of the point of oscillation from the C.G is l_2.

$$\therefore \frac{K^2}{l_1} = l_2$$

or, $K^2 = l_1 l_2$ ——— (II)

For the time period to be minimum, the value of $\left(\frac{K^2}{l_1} + l_1\right)$ should be minimum.

Differentiating $\left(\frac{K^2}{l_1} + l_1\right)$ with respect to l_1

$$\frac{d}{dl_1}\left(\frac{K^2}{l_1} + l_1\right) = -\frac{K^2}{l_1^2} + 1 \quad \text{——— (III)}$$

This value should be zero, for the ~~time~~ time period to be minimum.

$$\therefore -\frac{K^2}{l_1^2} + 1 = 0$$

$$\text{or, } -\frac{K^2}{l_1^2} = -1$$

$$\text{or } K^2 = l_1^2$$

$$\text{But } K^2 = l_1 l_2$$

$$\therefore l_1^2 = l_1 l_2$$

$$\text{or } l_2 = l_1 \quad \text{— — — — (IV)}$$

It means that the time period will be minimum when the points of suspension and oscillation are equidistant from the centre of gravity.

Ex: A uniform circular disc of radius 10 cm vibrates about a horizontal axis perpendicular to its plane and at a distance of 5 cm from the centre. Calculate, the time period of oscillation and the equivalent length of the simple pendulum.

Sol^n

We know

$$t = 2\pi\sqrt{\frac{K^2 + l_1^2}{l_1 g}}$$

$$= 2\pi\sqrt{\frac{50+25}{5 \times 980}}$$

$$= 0.782 \text{ Sec.}$$

Here,

$R = 10$ cm.

Radius of gyration (For circular disc) $K^2 = \frac{R^2}{2} = 50 \text{ cm}^2$

Distance the point of suspension from CG $= l_1 = 5$ cm.

$g = 980 \text{ cm/sec}^2$

Equivalent Length of the simple Pendulum $= L$

Here. $L = \frac{K^2 + l_1^2}{l_1}$

$\therefore L = \frac{50+25}{5}$

$= 15$ cm.

Forced Vibration:

The time period of a body executing simple harmonic motion depends upon the dimensions of the body and its elastic properties. When the body oscillates in a medium like air, its oscillations, as we know get damped.

If however, an external periodic force is applied to the oscillator, of a frequency not necessarily the same as the natural frequency of the oscillator, a sort of tussle ensues between the damping force and the applied force. The damping force ~~retard~~ tends to retard the motion of the body and the applied force tends to maintain it. Initially the amplitude of the oscillations increases, then decreases with time, becomes minimum and again increases. After some initial erratic movements, the body ultimately succumbs to the applied or driving force and settles down to oscillating with the frequency of the applied or driving force and a constant amplitude and phase so long as the applied force remain operative. Such vibration of the body are called forced vibrations. The amplitude of this vibration of the body depends on the difference between the natural and applied force frequency.

Let the periodic force to which a damped harmonic oscillator is subjected be $F = F_0 \sin pt$. F_0 is amplitude and frequency $p/2\pi$.

Now the damping and restoring force on the oscillator are $-b\frac{dy}{dt}$ and $-ay$ respectively where b and a have the same meanings. Hence the equation of motion may be written as

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ay + F$$

or, $$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = F_0 \sin pt$$

or, $$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = \frac{F_0}{m}\sin pt$$

or, $$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = f_0 \sin pt \quad \text{——— (1)}$$

Where $2\lambda = \frac{b}{m}$, $\omega^2 = \frac{a}{m}$ and $f_0 = \frac{F}{m}$

After the steady state has been attained be

$$y = A\sin(pt - \theta) \quad \text{——— (ii)}$$

where A is the amplitude and θ is the possible phase difference between the applied force and the displacement of the oscillator, Then we have

$$\frac{dy}{dt} = Ap\cos(pt - \theta)$$

and $$\frac{d^2y}{dt^2} = -Ap^2\sin(pt - \theta)$$

$$= -p^2 y \quad \text{——— (iii)}$$

Putting the value of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in eqn (1), we get

$$-Ap^2 \sin(pt-\theta) + 2\lambda A p \cos(pt-\theta) + \omega^2 A \sin(pt-\theta) = f_0 \sin pt$$

$$= f_0 \sin[(pt-\theta)+\theta]$$

$$= f_0 \sin(pt-\theta)\cos\theta + f_0 \cos(pt-\theta)\sin\theta$$

or, $A(\omega^2-p^2)\sin(pt-\theta) + 2\lambda A p \cos(pt-\theta)$

$$= f_0 \cos\theta \sin(pt-\theta) + f_0 \sin\theta \cos(pt-\theta) \quad \text{——— (iv)}$$

If this solution is to hold good for all values of t, the respective coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ on either side of equn (iv) must be equal. Thus we must have

$$A(\omega^2-p^2) = f_0 \cos\theta \quad \text{——— (a)}$$

and $$2\lambda A p = f_0 \sin\theta \quad \text{——— (b)}$$

squaring and adding equ^n (a) and (b) we have

$$A^2(\omega^2-p^2)^2 + 4\lambda^2 A^2 p^2 = f_0^2 \cos^2\theta + f_0^2 \sin^2\theta$$

a $$A^2[(\omega^2-p^2)^2 + 4\lambda^2 p^2] = f_0^2$$

a $$A^2 = \frac{f_0^2}{(\omega^2-p^2)^2 + 4\lambda^2 p^2} \quad \text{——— (v)}$$

Thus the amplitude of the driven or forced oscillator is

$$A = \frac{f_0}{\sqrt{(\omega^2-p^2)^2 + 4\lambda^2 p^2}} \quad \text{——— (vi)}$$

The phase difference between the driven force and applied force is

$$\theta = \tan^{-1}\frac{2\lambda p}{(\omega^2-p^2)} \quad \text{——— (vii)}$$

Sharpness of Resonance:

Sharpness of resonance may be regarded, in a way, as a measure of the rate of fall of amplitude from its maximum value at the resonant frequency, on either side of it, the smaller the damping, the sharper the resonance. Sharpness of resonance is inversely proportional to the square of the damping constant λ.

Expression for a plane progressive wave:

A progressive wave is one which travels onward through the medium in a given direction without attenuation i.e., with its amplitude constant.

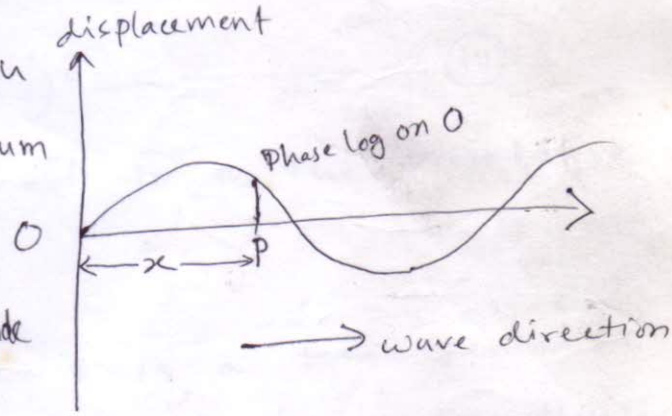


Fig → progressive wave

A typical wave form is shown in fig. Let a wave is originating at O, travel to the right along the x-axis. The equation of motion of this particle at O is obviously

$$y = a \sin \omega t \quad \text{——— (1)}$$

where y is the displacement of the particle at time t, a its amplitude and ω its angular velocity.

For a particle at P which is at a distance x away from O, let this phase difference be ϕ. Hence the equation of motion of the particle at P is

$$Y = a\sin(\omega t - \phi) \qquad \text{(11)}$$

For a difference in path of λ, the difference in phase is 2π. Hence for a distance x, the corresponding phase difference is $\frac{2\pi}{\lambda} \cdot x$. Substituting this value in equ^n (11) we get

$$Y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

or, $$Y = a\sin(\omega t - Kx) \qquad \text{(111)}$$

Where $K = \frac{2\pi}{\lambda}$ is referred to as the propagative constant.

Now $\omega = \frac{2\pi}{T}$, where T is a time period for a complete oscillation. n is the frequency, therefor. $v = n\lambda$, or $\frac{1}{T} = n = \frac{v}{\lambda}$, Then equ^n (111) becomes.

$$Y = a\sin\left(\frac{2\pi v}{\lambda}t - \frac{2\pi}{\lambda}x\right)$$

or $$Y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

The most commonly used equation of ~~wave~~ a progressive wave.

③

Differential equation of wave motion (from general equ^n of a prograssive wave)

=> The general equation of a prograssive simple harmonic wave is

$$Y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{—— (I)}$$

Differentiating equ^n (I) with respect to time

$$\frac{dy}{dt} = a \frac{2\pi v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \text{—— (II)}$$

Again differentiating equ^n (II) w r t time

$$\frac{d^2y}{dt^2} = - a \cdot \frac{2\pi v}{\lambda} \cdot \frac{2\pi v}{\lambda} \sin (vt - x)$$

$$= - a \frac{4\pi^2 v^2}{\lambda^2} \sin (vt - x) \quad \text{—— (III)}$$

To find the value of compression, differentiate equ^n (I) with respect to x,

$$\frac{dy}{dx} = - a \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \text{—— (IV)}$$

differentiate equ^n (IV) with respect to x

$$\frac{d^2y}{dx^2} = - \frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{—— (V)}$$

From equ's (II) and (IV) we get

$$\frac{dy}{dt} = - v \frac{dy}{dx} \quad \text{—— (VI)}$$

from equ^{n}s (III) and (V) we get

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \text{——— (VII)}$$

Equation (VII) represents the differential equation of wave motion.

The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K \frac{d^2y}{dx^2} \quad \text{——— (VIII)}$$

Where, $K = v^2$

and $v = \sqrt{K}$

Thus knowing the value of K, the value of v can be calculated

Particle velocity and wave velocity:

The equation for a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{——— (1)}$$

Differentiating equation (1) with respect to t, we get

$$\frac{dy}{dt} = \text{Particle velocity} = \frac{2\pi a v}{\lambda} \cos (vt - x) \quad \text{——— (2)}$$

The maximum particle velocity $u_{max} = \frac{2\pi a v}{\lambda}$ ——— (3)

or,

Maximum particle velocity = $\frac{2\pi a}{\lambda}$ × wave velocity.

* To find the particle acceleration, differentiating equ^n (2) ~~by~~ with respect to t we get,

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)$$

$$f = -\frac{4\pi^2 v^2}{\lambda^2}\left[a \sin \frac{2\pi}{\lambda}(vt - x)\right]$$

$$= -\left[\frac{4\pi^2 v^2}{\lambda^2}\right] y$$

Maximum acceleration ~~is~~ will be when $Y = a$

$$f_{max} = -\left[\frac{4\pi^2 v^2}{\lambda^2}\right] a$$

The negative sign shows that the acceleration of the particle is directed towards its mean position.

Prob: A simple harmonic wave of amplitude 8 units tranverses a line of particles in the direction of the positive X-axis. At any instant of time, for a particle at a distance of 10 cm from the origin, the displacement is +6 units, and for a particle at a distance of 25 cm from the origin, the displacement is +4 units, Calculate the wavelength.

Sol^n

$$Y = a \sin \frac{2\pi}{\lambda}(vt - x)$$

or, $$\frac{Y}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

(1) In the first case

$$\frac{Y_1}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

Here, $Y_1 = +6$, $a = 8$, $x_1 = 10$ cm.

$$\therefore \frac{6}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) \quad \text{——— (1)}$$

(2) in the second case

$$\frac{Y_2}{a} = \sin 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right)$$

$Y_2 = +4$, $a = 8$, $x_2 = 25$

$$\therefore \frac{4}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) \quad \text{——— (2)}$$

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From equ^n (1)

$$0.75 = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right)$$

But $$0.75 = \sin\left(\frac{48.6\pi}{180}\right)$$

$$\therefore \quad 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) = \frac{48.6\pi}{180}$$

or, $$\frac{t}{T} - \frac{10}{\lambda} = \frac{48.6}{360} \qquad \text{——— (3)}$$

From equation (2) we get

$$0.5 = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right)$$

But $$\sin\frac{\pi}{6} = 0.5$$

$$\therefore \quad 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) = \frac{\pi}{6}$$

or $$\frac{t}{T} - \frac{25}{\lambda} = \frac{1}{12} \qquad \text{——— (4)}$$

subtracting (4) from (3) we get.

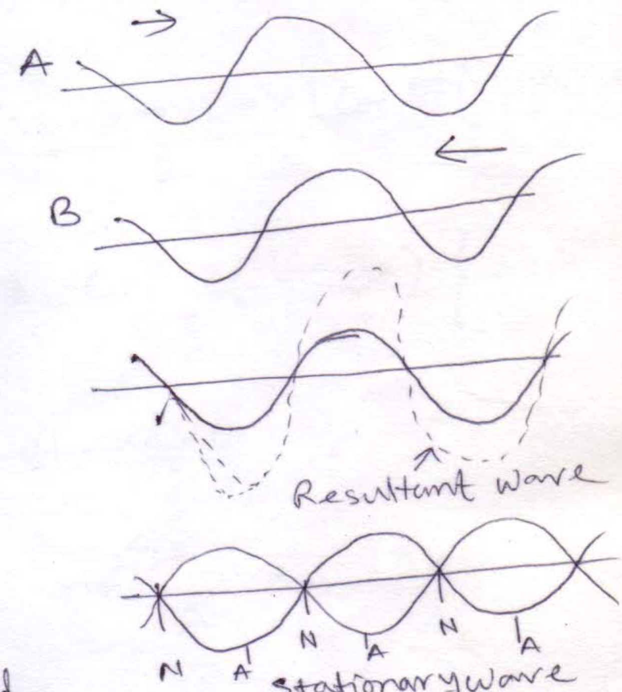
$$\frac{25}{\lambda} - \frac{10}{\lambda} = \frac{48.6}{360} - \frac{1}{12}$$

$$\lambda = 290.8 \text{ cm.}$$

Stationary Waves :

When two simple harmonic waves of the same amplitude, frequency and time period travel in opposite directions in a straight line, the resultant wave obtained is called a stationary or a Standing wave.

Stationary waves are formed in an open end organ pipe or a closed end organ pipe. Stationary waves are also formed with a stretched string fixed at one end and free at the other end or fixed at the other end.



N → Node point
A → Anti node point

fig → ①

Formation of Stationary wave :

Consider a simple harmonic wave given by the equation,

$$Y_1 = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{——— (1)}$$

The displacement of the same particle at the same instant due to the reflected wave is given by

$$Y_2 = a \sin \frac{2\pi}{\lambda}(vt + x) \quad \text{——— (11)}$$

The resultant displacement of the wave

$$\bar{Y} = \bar{Y}_1 + \bar{Y}_2$$

or, $$Y = a \sin \frac{2\pi}{\lambda}(vt - x) + a \sin \frac{2\pi}{\lambda}(vt + x)$$

$$= a\left\{ \sin \frac{2\pi}{\lambda}(vt - x) + \sin \frac{2\pi}{\lambda}(vt + x) \right\}$$

$$= a\left\{ 2 \sin \frac{2\pi}{\lambda}\left(\frac{vt - x + vt + x}{2}\right) \cos \frac{2\pi}{\lambda}\left(\frac{vt - x + vt +}{2}\right.\right.$$

$$= -a\left\{ 2 \sin \frac{2\pi}{\lambda} x \cos \frac{2\pi vt}{\lambda} \right\}$$

$$= \cancel{2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}}$$

$$= \cancel{A \sin} = +2a \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

$$= A \cos \frac{2\pi vt}{\lambda}$$

Where A is the resultant amplitude of stationary wave, and $A = +2a \sin \frac{2\pi x}{\lambda}$,

Acoustics:

The branch of physics that deals with the process of generation, reception and propagation of sound is called acoustics. This ~~field~~ branch infact covers many fields and is closely related to various branches of engineering. Some of the important fields of acoustics are

(i) Design of acoustical instruments

(ii) Electro-acoustics viz: the branch relating to the methods of sound production and recording (microphones, amplifires, loud speakers etc)

(iii) Architectural acoustics dealing with the design and construction of buildings, operas, music halls recording rooms in radio and television broadcasting station, and

(iv) Musical acoustics deals with the design of musical instruments.

Reverberation:

It is observed that for a listener in a room or an auditorium, whenever a sound pulse is produced, he receives directly compressional sound waves from the source as well as sound waves from the walls, celling and other materials present in the room. The sound

~~or waves from the source as well as sound waves from the walls.~~

or waves received by the listener are: ~~(i) direct~~

(i) Direct waves

(ii) Reflected waves due to multiple reflections at the various surfaces. The quality of the note received by the listener will be the combined effect of these two sets of waves. There is also a time gap between the direct wave received by the listener and the waves received by successive reflection. Due to this, the sound persists for some time even after the source has stopped. This persistence of sound is termed as reverberation. The time gap between the initial direct note and the reflected note upto the minimum audibility is called reverberation time.

Sabine's Reverberation Formula:

Sabine developed the reverberation formula to express the rise and fall of sound in an auditorium. The main assumptions are:

(1) The average energy per unit volume is uniform. It is represented as σ.

(ii) The energy is not lost in the auditorium.

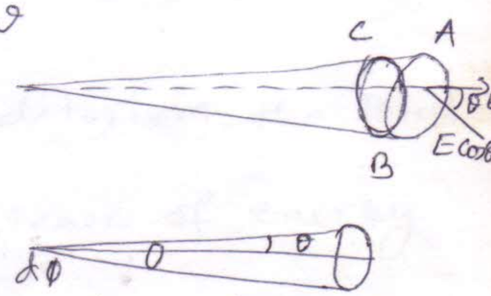
The energy ~~is~~ lost is only due to the absorption of the material of the walls and celing and also due to escape through the windows and ventilators. Both these factors are included in the term absorption of energy.

Suppose a source is producing sound continuously. This sound energy is propagated in all directions. Let σ be the energy contained in a unit volume. The energy that is contained in a solid angle.

$$\cancel{d\phi =} \frac{\sigma . d\phi}{4\pi}$$

Let this energy be incident on a unit surface area of the wall at an angle θ. If the velocity of sound is v, then the total energy falling per second on a unit surface area of the wall.

$$= \frac{\sigma . d\phi}{4\pi} \cos\theta . v$$



$\phi = 2\pi(1 - \cos\theta)$

Fig- 1

The total energy falling per ~~sec~~ second with in a hemisphere

$$= \frac{\sigma v}{4\pi} \int \cos\theta . d\phi$$

But $\phi = 2\pi(1 - \cos\theta)$

or $d\phi = 2\pi \sin\theta \, d\theta$, So, the total energy falling per second within a hemisphere

$$\frac{\sigma v}{4\pi}\int_0^{\pi/2} 2\pi \sin\theta . \cos\theta \, d\theta$$

$$= \frac{\sigma v}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\sigma v}{4}$$

Suppose α is the absorption coefficient of the walls that refers to the fraction of the incident energy not reflected from the walls. The amount of energy absorbed per second per unit area $= \frac{\alpha \sigma v}{4}$. If A is the area of the walls and the other absorbing materials including ceiling, windows and ventilators etc., the amount of energy absorbed per second

$$= \frac{A \alpha \sigma v}{4}$$

Let V be the volume of the auditorium, the total energy $= V\sigma$. The rate of increase of energy

$$= \frac{d}{dt}(V\sigma)$$

$$= V\frac{d\sigma}{dt} \quad \text{- - - - (1)}$$

Suppose, the source supplies energy at the rate of Q units per second.

Then, the rate of increase of energy

$$= Q - \frac{A \alpha \sigma v}{4} \quad \text{———} \quad (2)$$

Equating (1) and (2)

$$v \frac{d\sigma}{dt} = Q - \frac{A \alpha \sigma v}{4} \quad \text{———} \quad (3)$$

Let $\frac{A \alpha v}{4} = K$

and $\frac{K}{v} = \beta$ and $B = \frac{Q}{K} = \frac{4Q}{A \alpha v}$

From equation (3)

$$v. \frac{d\sigma}{dt} = Q - K\sigma$$

$$\frac{d\sigma}{dt} = \frac{Q}{V} - \frac{K}{V} . \sigma \quad \text{———} \quad (4)$$

The general solution of this equation is

$$\sigma = B + b e^{-\beta t} \quad \text{———} \quad (5)$$

when $t = 0$, and $\sigma = 0$

$\therefore$ from equation (5)

$$0 = B + b$$

or, $b = -B$

$$\sigma = B - B e^{-\beta t}$$

$$\sigma = B[1 - e^{-\beta t}]$$

Substituting the values of B and β

$$\sigma = \frac{4Q}{A \alpha v} \left[1 - e^{-\frac{(A \alpha v) . t}{4V}} \right] \quad \text{- - - -} \quad (6)$$

Equation (6) represents the rise of average sound energy per unit time from the time the source commences to produce sound.

The maximum value of average energy per unit volume

$$\sigma_{max} = \frac{4Q}{A\alpha v} \quad \text{——— (7)}$$

similarly, after the source ceases to emit sound, the decay of the ~~ev~~ everage energy per unit volume is given by

$$\sigma = \frac{4Q}{A\alpha v} e^{-\frac{A\alpha v.t}{4V}} \quad \text{——— (8)}$$

$$\sigma = \sigma e^{-\frac{(A\alpha v)t}{4V}} \quad \text{——— (9)}$$

The factor $\frac{A\alpha v}{4V}$ gives the reverberation time in the auditorium. If σ_0 represents the minimum audible intensity after a time t_1, then from equ (9)

$$\sigma_0 = \sigma_{max} e^{-\frac{(A\alpha v)t_1}{4V}} \quad \text{——— (10)}$$

Here t_1 is the time interval between the cutting off the sound and the time at which intensity falls below the minimum audible level.

From equ (10)

$$\sigma_{max} = \sigma_0 e^{+\left(\frac{A\alpha v . t_1}{4V}\right)}$$

Taking logarithms

$$\log_e\left(\frac{\sigma_{max}}{\sigma_0}\right) = \frac{A\alpha v}{4V} t_1 \quad \text{———} \quad (11)$$

Here α and σ_0 change with the frequency of sound.

For calculating the reverberation time, a standard steady intensity is required. Sabine took the value of

$$\frac{\sigma_{max}}{\sigma_0} = 10^6$$

From equation (11)

$$\log_e(10^6) = \frac{A\alpha v}{4V} t_1$$

$$2.303 \times 6 = \frac{A\alpha v}{4V} t_1$$

Taking velocity of sound approximately at room temperature as 350 m/S

$$2.303 \times 6 = \frac{A\alpha \times 350}{4V} t_1$$

$$\text{or,} \quad t_1 = \frac{2.303 \times 24V}{350 A\alpha}$$

$$t_1 = \frac{0.158V}{A\alpha} \quad \text{———} \quad (12)$$

In general $t_1 = \frac{0.158V}{\Sigma a\alpha}$ ——— (13)

Equ (12) represents the Sabine's reverberation time formula.