$$y = f(x) = x^2 + 3x + 2 \qquad \text{[Say]}$$
$$y = x^2 + 3x + 2$$
$$x^2 + 3x + (2 - y) = 0$$

In the above equation the value of x will be real if and only if

$$3^2 - 4.1.(2 - y) \geq 0$$
$$9 - 4(2 - y) \geq 0$$
$$9 - 8 + 4y \geq 0$$
$$1 + 4y \geq 0$$
$$4y \geq -1$$
$$y \geq -\frac{1}{4}$$

Therefore, the range of the given function is $R_f = [-\frac{1}{4}, \infty)$. (Ans)

09 Differentiation

If y and x are two variables related to one another, then the rate of change of y in terms of x is denoted by $\frac{dy}{dx}$ and is calculated by the first principle rule $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

First Principle Rules: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Problem 01: Find the derivative of $y = f(x)$ the followings by using First principle rule:

(a) $f(x) = x^n$	(b) $f(x) = e^x$	c) $f(x) = a^x$	d) $f(x) = \ln x$
(e) $f(x) = \cos x$	(f) $f(x) = \sin a x$	(g) $f(x) = \tan x$	(h) $f(x) = x$

(i) $f(x)=c$	(j) $f(x)=cg(x)$	(k) $f(x)=\sin^{-1}x$	(l) $f(x)=\tan^{-1}x$

(a) Given function is, $f(x)=x^n$.

Now, $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$

$$=\lim_{h\to 0}\frac{\left\{x\left(1+\frac{h}{x}\right)\right\}^n-x^n}{h}=\lim_{h\to 0}\frac{x^n\left(1+\frac{h}{x}\right)^n-x^n}{h}=x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n-1}{h}$$

$$=x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n-1}{h}=x^n\lim_{h\to 0}\frac{1+n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\cdots+\infty-1}{h}$$

$$=x^n\lim_{h\to 0}\frac{n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\cdots+\infty}{h}$$

$$=x^n\lim_{h\to 0}\frac{h\left\{n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right)+\cdots+\infty\right\}}{h}$$

$$=x^n\lim_{h\to 0}\frac{h\left\{n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right)+\cdots+\infty\right\}}{h}$$

$$=x^n\lim_{h\to 0}\left\{n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{h^2}{x^3}\right)+\cdots+\infty\right\}$$

$$=x^n\times n\left(\frac{1}{x}\right)=nx^{n-1}$$ **(As desired)**

(b) Given function is, $f(x)=e^x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x.e^h-e^x}{h}$$

$$=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x\lim_{h\to 0}\frac{1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots+\infty-1}{h}=e^x\lim_{h\to 0}\frac{h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots+\infty}{h}$$

$$=e^x\lim_{h\to 0}h\times\frac{1+\frac{h}{2!}+\frac{h^2}{3!}+\cdots+\infty}{h}=e^x\lim_{h\to 0}\left(1+\frac{h}{2!}+\frac{h^2}{3!}+\cdots+\infty\right)=e^x$$ **(As desired)**

(c) Given function is, $f(x)=a^x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\frac{a^x.a^h-a^x}{h}$$

$$=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x\lim_{h\to 0}\frac{1+h\ln a+\frac{h^2}{2!}(\ln a)^2+\frac{h^3}{3!}(\ln a)^3+\cdots\infty-1}{h}$$

$$=a^x\lim_{h\to 0}\frac{h\ln a+\frac{h^2}{2!}(\ln a)^2+\frac{h^3}{3!}(\ln a)^3+\cdots\infty}{h}$$

$$=a^x\lim_{h\to 0}\left(\ln a+\frac{h}{2!}(\ln a)^2+\frac{h^2}{3!}(\ln a)^3+\cdots\infty\right)$$

$$=a^x\ln a$$ **(As desired)**

(d) Given function is, $f(x)=\ln x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}=\lim_{h\to 0}\frac{\frac{h}{x}-\frac{1}{2}\left(\frac{h}{x}\right)^2+\frac{1}{3}\left(\frac{h}{x}\right)^3-\cdots\infty}{h}=\lim_{h\to 0}\left(\frac{1}{x}-\frac{1}{2}\frac{h}{x^2}+\frac{1}{3}\frac{h^2}{x^3}-\cdots\infty\right)=\frac{1}{x}$$ **(As desired)**

(e) Given function is, $f(x)=\cos x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}=-\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\times\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=-\sin\left(\frac{2x+0}{2}\right)\times 1=-\sin x$$ **(As desired)**

(f) Given function is, $f(x)=\sin ax$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin a(x+h)-\sin ax}{h}=\lim_{h\to 0}\frac{\sin(ax+ah)-\sin ax}{h}=\lim_{h\to 0}\frac{2\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{h}=\lim_{h\to 0}\frac{2\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{h}=\lim_{h\to 0}\frac{a\cos\left(\frac{2ax+ah}{2}\right)\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)}$$

$$=a\lim_{h\to 0}\cos\left(\frac{2ax+ah}{2}\right)\times\lim_{h\to 0}\frac{\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)}=a\lim_{h\to 0}\cos\left(\frac{2ax+ah}{2}\right)=a\cos ax$$ **(As desired)**

(g) Given function is, $f(x)=\tan x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}=\lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}=\lim_{h\to 0}\frac{\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{h\cos(x+h)\cos x}=\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}=\lim_{h\to 0}\frac{\sin(h)}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{\sin(h)}{h}\times\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}=\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}=\frac{1}{\cos(x)\cos x}=\frac{1}{\cos^2 x}=\sec^2 x$$ **(As desired)**

(h) Given function is, $f(x)=x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{x+h-x}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$ **(As desired)**

(i) Given function is, $f(x)=c$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$$ **(As desired)**

(j) Given function is, $f(x)=c\,g(x)$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{cg(x+h)-cg(x)}{h}=c\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=c\frac{dg}{dx}$$ **(As desired)**

(k) Given function is, $f(x)=\sin^{-1}x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}(x)}{h}\cdots(i)$$

Say $\sin^{-1}(x)=y\Rightarrow x=\sin y$ and $\sin^{-1}(x+h)=y+k\Rightarrow x+h=\sin(y+k)$.

From equation(i), we get

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{y+k-y}{h}=\lim_{k\to 0}\frac{k}{\sin(y+k)-\sin(y)} \qquad [x+h=\sin(y+k)\Rightarrow h=0\Rightarrow k=0]$$

$$=\lim_{k\to 0}\frac{k}{\sin(y+k)-\sin(y)}=\lim_{k\to 0}\frac{k}{2\cos\left(\frac{2y+k}{2}\right)\sin\left(\frac{k}{2}\right)}=\lim_{k\to 0}\frac{\left(\frac{k}{2}\right)}{\cos\left(\frac{2y+k}{2}\right)\sin\left(\frac{k}{2}\right)}$$

$$=\lim_{k\to 0}\frac{1}{\cos\left(\frac{2y+k}{2}\right)}\times\lim_{k\to 0}\frac{\left(\frac{k}{2}\right)}{\sin\left(\frac{k}{2}\right)}=\frac{1}{\cos(y)}=\frac{1}{\sqrt{\cos^2 y}}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$ **(As desired)**

(i) Given function is, $f(x)=\tan^{-1}x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\tan^{-1}(x+h)-\tan^{-1}(x)}{h}\cdots(i)$$

Say $\tan^{-1}(x)=y\Rightarrow x=\tan y$ and $\tan^{-1}(x+h)=y+k\Rightarrow x+h=\tan(y+k)$.

From equation(i), we get

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{y+k-y}{h}=\lim_{k\to 0}\frac{k}{\tan(y+k)-\tan(y)} \qquad [x+h=\sin(y+k)\Rightarrow h=0\Rightarrow k=0]$$

$$=\lim_{k\to 0}\frac{k}{\frac{\sin(y+k)}{\cos(y+k)}-\frac{\sin(y)}{\cos(y)}}=\lim_{k\to 0}\frac{k}{\frac{\sin(y+k)\cos(y)-\sin(y)\cos(y+k)}{\cos(y+k)\cos(y)}}=\lim_{k\to 0}\frac{k\cos(y+k)\cos(y)}{\sin(y+k)\cos(y)-\sin(y)\cos(y+k)}$$

$$=\lim_{k\to 0}\frac{k\cos(y+k)\cos(y)}{\sin(y+k-y)}=\lim_{k\to 0}\frac{k\cos(y+k)\cos(y)}{\sin(k)}=\lim_{k\to 0}\frac{k}{\sin k}\times\lim_{k\to 0}\cos(y+k)\cos(y)$$

$$=\cos(y)\cos(y)=\cos^2 y=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$ **(As desired)**

Problem 02: Find $\frac{dy}{dx}$ for $f(x)=\sin(x^2)$ using first principle rule.

Solution: Given function is $f(x)=\sin(x^2)$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)^2-\sin(x)^2}{h}=\lim_{h\to 0}\frac{\sin\left(x^2+2xh+h^2\right)-\sin(x)^2}{h}$$

$$=\lim_{h\to 0}\frac{\sin\left(x^2+2xh+h^2\right)-\sin\left(x^2\right)}{h}=\lim_{h\to 0}\frac{2\cos\left(\frac{x^2+2xh+h^2+x^2}{2}\right)\sin\left(\frac{x^2+2xh+h^2-x^2}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{2x^2+2xh+h^2}{2}\right)\sin\left(\frac{2xh+h^2}{2}\right)}{h}=\lim_{h\to 0}\frac{\cos\left(\frac{2x^2+2xh+h^2}{2}\right)\sin\left(\frac{2xh+h^2}{2}\right)}{\frac{h(2x+h)}{2}}\times(2x+h)$$

$$=\lim_{h\to 0}\cos\left(\frac{2x^2+2xh+h^2}{2}\right)\times\lim_{h\to 0}\frac{\sin\left(\frac{2xh+h^2}{2}\right)}{\frac{h(2x+h)}{2}}\times\lim_{h\to 0}(2x+h) \quad =\cos\left(x^2\right)\times 1\times 2x=2x\cos\left(x^2\right)$$ **(As desired)**

Note: $\frac{d}{dx}\left(\sin x^2\right)=\cos x^2\times\frac{d}{dx}\left(x^2\right)=\cos x^2\times 2x=2x\cos x^2$

Problem 03: Find $\frac{dy}{dx}$ for $y=\sqrt{x}$ using first principle rule.

Solution: Given function is $f(x)=\sin(x^2)$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$ **(As desired)**

Problem 04: Find $\frac{dy}{dx}$ for the following functions:

(a) $y=e^x+2\sin x-\frac{1}{2}\log x$	(b) $y=x^2\tan^{-1}x$	(c) $y=a^x x^3\sin^{-1}x$	(d) $y=\frac{e^x}{1+x}$
(e) $y=e^{\tan^{-1}x}$	(f) $x^y+y^x=1$	(g) $x=a\cos\theta+b\sin\theta, y=b\sin\theta$	

(a) Given function is $y=e^x+2\sin x-\frac{1}{2}\log x$

Taking derivative with respect to x on both sides we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x+2\sin x-\frac{1}{2}\log x\right)$$

$$=\frac{d}{dx}\left(e^x\right)+2\frac{d}{dx}(\sin x)-\frac{1}{2}\frac{d}{dx}(\log x)=e^x+2\cos x-\frac{1}{2x}$$ **(As desired)**

(b) Given function is $y=x^2\tan^{-1}x$

Taking derivative with respect to x on both sides we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\tan^{-1}x\right)=x^2\frac{d}{dx}\left(\tan^{-1}x\right)+\tan^{-1}x\,\frac{d}{dx}\left(x^2\right)=x^2\times\frac{1}{1+x^2}+\tan^{-1}x\,(2x)=2x\tan^{-1}x+\frac{x^2}{1+x^2}$$ **(As desired)**

(c) Given function is $y=a^x x^3\sin^{-1}x$.

Taking derivative with respect to x on both sides we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(a^x x^3\sin^{-1}x\right)=a^x x^3\frac{d}{dx}\left(\sin^{-1}x\right)+a^x\sin^{-1}x\frac{d}{dx}\left(x^3\right)+x^3\sin^{-1}x\frac{d}{dx}\left(a^x\right)$$

$$=a^x x^3\times\left(-\frac{1}{\sqrt{1-x^2}}\right)+3x^2a^x\sin^{-1}x+a^x x^3\sin^{-1}x\ln a=3x^2a^x\sin^{-1}x+a^x x^3\sin^{-1}x\ln a-\frac{a^x x^3}{\sqrt{1-x^2}}$$ **(As desired)**

(d) Given function is $y=\frac{e^x}{1+x}$.

Taking derivative with respect to x on both sides we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^x}{1+x}\right)=\frac{(1+x)\times\frac{d}{dx}\left(e^x\right)-e^x\times\frac{d}{dx}(1+x)}{(1+x)^2}=\frac{(1+x)\times e^x-e^x\times(0+1)}{(1+x)^2}=\frac{(1+x)e^x-e^x}{(1+x)^2}=\frac{xe^x}{(1+x)^2}$$ **(As desired)**

(e) Given function is $y=e^{\tan^{-1}x}$.

Taking derivative with respect to x on both sides we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\tan^{-1}x}\right)=e^{\tan^{-1}x}\times\frac{d}{dx}\left(\tan^{-1}x\right)=e^{\tan^{-1}x}\times\frac{1}{1+x^2}=\frac{e^{\tan^{-1}x}}{1+x^2}$$ **(As desired)**

(f) Given equation is $x^y+y^x=1$.

Taking derivative with respect to x on both sides we get

$$\frac{d}{dx}\left(x^y + y^x\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}\left(x^y + y^x\right) = 0$$

$$\frac{d}{dx}\left(x^y\right) + \frac{d}{dx}\left(y^x\right) = 0$$

$$\frac{d}{dx}\left(e^{\ln x^y}\right) + \frac{d}{dx}\left(e^{\ln y^x}\right) = 0$$

$$e^{\ln x^y}\frac{d}{dx}\left(\ln x^y\right) + e^{\ln y^x}\frac{d}{dx}\left(\ln y^x\right) = 0$$

$$x^y\frac{d}{dx}(\mathrm{y}\ln x) + y^x\frac{d}{dx}(\mathrm{x}\ln y) = 0$$

$$x^y\left(y.\frac{1}{x} + \ln x.\frac{dy}{dx}\right) + y^x\left(x.\frac{1}{y}.\frac{dy}{dx} + \ln y.1\right) = 0$$

$$x^y\left(\frac{y}{x} + \ln x.\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}.\frac{dy}{dx} + \ln y\right) = 0$$

$$x^{y-1}y + x^y\ln x.\frac{dy}{dx} + y^{x-1}x\frac{dy}{dx} + y^x\ln y = 0$$

$$x^y\ln x.\frac{dy}{dx} + y^{x-1}x\frac{dy}{dx} = -\left(x^{y-1}y + y^x\ln y\right)$$

$$\left(x^y\ln x + y^{x-1}x\right)\frac{dy}{dx} = -\left(x^{y-1}y + y^x\ln y\right)$$

$$\frac{dy}{dx} = -\frac{\left(x^{y-1}y + y^x\ln y\right)}{\left(x^y\ln x + y^{x-1}x\right)}$$ **(As desired)**

(g) Given parametric equations are,

$$x = a\cos\theta + b\sin\theta$$

Taking derivative with respect to θ on both sides we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta) + \frac{d}{d\theta}(b\sin\theta) = a\frac{d}{d\theta}(\cos\theta) + b\frac{d}{d\theta}(\sin\theta) = a(-\sin\theta) + b(\cos\theta) = b\cos\theta - a\sin\theta$$

And

$$y = b\sin\theta$$

Taking derivative with respect to θ on both sides we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin\theta) = b\frac{d}{d\theta}(\sin\theta) = b\cos\theta$$

By chain rule: $\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = b\cos\theta.\frac{1}{b\cos\theta - a\sin\theta} = \frac{b\cos\theta}{b\cos\theta - a\sin\theta}$ **(As desired)**