

Beta & Gamma Functions

Beta Function or First Eulerian Integral: A function of the form,

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx \quad ; m, n > 0$$

is called Beta function or first Eulerian integral and it is denoted by, $\beta(m, n)$.

$$i.e, \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad ; m, n > 0.$$

Gamma Function or Second Eulerian Integral: A function of the form,

$$\int_0^{\infty} e^{-x} x^{n-1} dx \quad ; n > 0$$

is called Gamma function or second Eulerian integral and it is denoted by, $\Gamma(n)$.

$$i.e, \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad ; n > 0.$$

Properties of Beta and Gamma functions: The properties are given below:

1. $\beta(m, n) = \beta(n, m)$
2. $\Gamma(1) = 1$
3. $\Gamma(n+1) = n\Gamma(n) \quad ; n > 0$
4. $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
5. $\int_0^{\infty} e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n} \quad ; k, n > 0$
6. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi} \quad ; 0 < m < 1$
7. $\Gamma(1/2) = \sqrt{\pi}$
8. $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx.$
9. $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx = \frac{\left| \frac{p+1}{2} \right| \left| \frac{q+1}{2} \right|}{2 \left| \frac{p+q+2}{2} \right|}.$

Theorem-01: Prove that $\beta(m, n) = \beta(n, m)$.

Proof: We know, the beta function is

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad ; m, n > 0 \quad \dots\dots\dots(1)$$

Let $x=1-t \quad \therefore dx = -dt$

when $x=0 \quad$ then $t=1$

when $x=1 \quad$ then $t=0$

From (1) we get,

$$\begin{aligned} \beta(m, n) &= \int_1^0 (1-t)^{m-1} t^{n-1} (-dt) \quad ; m, n > 0 \\ &= \int_0^1 t^{n-1} (1-t)^{m-1} dt \\ &= \beta(n, m) \quad \text{(Proved)} \end{aligned}$$

Theorem-02: Prove that $\Gamma(1/2) = \sqrt{\pi}$.

Proof: We know, the beta function is

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad ; m, n > 0$$

If $m = n = \frac{1}{2}$ then,

$$\begin{aligned} \beta\left(\frac{1}{2}, \frac{1}{2}\right) &= \int_0^1 x^{\frac{1}{2}-1} (1-x)^{\frac{1}{2}-1} dx \\ \Rightarrow \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)} &= \int_0^1 \frac{1}{\sqrt{x}\cdot\sqrt{1-x}} dx \\ \Rightarrow \frac{\{\Gamma(1/2)\}^2}{\Gamma(1)} &= \int_0^1 \frac{dx}{\sqrt{x-x^2}} \\ \Rightarrow \frac{\{\Gamma(1/2)\}^2}{\Gamma(1)} &= \int_0^1 \frac{dx}{\sqrt{-(x^2-x)}} \end{aligned}$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \int_0^1 \frac{dx}{\sqrt{-\left[x^2 - 2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \right]}}$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left[x^2 - 2x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \right]}}$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \int_0^1 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \left[\sin^{-1} \frac{\left(x - \frac{1}{2}\right)}{\frac{1}{2}} \right]_0^1$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \left[\sin^{-1} (2x - 1) \right]_0^1$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \left[\sin^{-1} (2 \cdot 1 - 1) - \sin^{-1} (2 \cdot 0 - 1) \right]$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \sin^{-1} (1) - \sin^{-1} (-1)$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \sin^{-1} (1) + \sin^{-1} (1)$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = 2 \sin^{-1} (1)$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = 2 \sin^{-1} \cdot \sin \left(\frac{\pi}{2} \right)$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = 2 \left(\frac{\pi}{2} \right)$$

$$\Rightarrow \left\{ \Gamma\left(\frac{1}{2}\right) \right\}^2 = \pi$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{(Proved)}$$

Theorem-03: Prove that i). $\Gamma(1)=1$; ii). $\Gamma(n+1)=n\Gamma(n)$.

Proof: We know, the Gamma function is

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad ; \quad n > 0 \quad \dots \dots \dots (1)$$

If $n=1$ then, from (1) we get,

$$\begin{aligned} \Gamma(1) &= \int_0^{\infty} e^{-x} x^{1-1} dx \\ &= \int_0^{\infty} e^{-x} dx \\ &= \left[-e^{-x} \right]_0^{\infty} \\ &= \left(-e^{-\infty} + e^0 \right) \\ &= (0+1) \\ &= 1 \end{aligned}$$

$\therefore \Gamma(1)=1$ **(Proved)**

Again, replacing n by $(n+1)$ in (1) we get,

$$\begin{aligned} \Gamma(n+1) &= \int_0^{\infty} e^{-x} x^n dx \\ &= \left[-x^n e^{-x} \right]_0^{\infty} + n \int_0^{\infty} e^{-x} x^{n-1} dx \quad \text{[Integrating by parts]} \\ &= 0 + n\Gamma(n) \\ &= n\Gamma(n) \end{aligned}$$

$\therefore \Gamma(n+1)=n\Gamma(n)$ **(Proved)**

Theorem-04: Establish the relation between Gamma and Beta function.

Or, Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Proof: From the definition of Gamma function we can write

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad ; n > 0$$

Assume $x = \lambda u \therefore dx = \lambda du$.

Limit: when $x = 0$, then $u = 0$ and when $x = \infty$, then $u = \infty$.
 From above relation we have

$$\begin{aligned}\Gamma(n) &= \int_0^{\infty} e^{-\lambda u} (\lambda u)^{n-1} \lambda du \\ &= \int_0^{\infty} e^{-\lambda u} u^{n-1} \lambda^{n-1} \lambda du \\ &= \int_0^{\infty} e^{-\lambda u} u^{n-1} \lambda^n du \dots\dots\dots(i)\end{aligned}$$

Again,

$$\Gamma(m) = \int_0^{\infty} e^{-\lambda} \lambda^{m-1} d\lambda \dots\dots\dots(ii)$$

Multiplying (i) and (ii) we get

$$\begin{aligned}\Gamma(n)\Gamma(m) &= \int_0^{\infty} e^{-\lambda u} u^{n-1} \lambda^n du \int_0^{\infty} e^{-\lambda} \lambda^{m-1} d\lambda \\ \Rightarrow \Gamma(m)\Gamma(n) &= \int_0^{\infty} \int_0^{\infty} e^{-\lambda u} u^{n-1} \lambda^n e^{-\lambda} \lambda^{m-1} d\lambda du \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-\lambda(1+u)} \lambda^{m+n-1} d\lambda \right] u^{n-1} du \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-\lambda(1+u)} \lambda^{m+n-1} d\lambda \right] u^{n-1} du \\ &= \int_0^{\infty} \left[\frac{\overline{m+n}}{(1+u)^{m+n}} \right] u^{n-1} du \quad \left[\because \frac{\overline{n}}{k^n} = \int_0^{\infty} e^{-kx} x^{n-1} dx \right] \\ &= \overline{m+n} \int_0^{\infty} \frac{u^{n-1}}{(1+u)^{m+n}} du \\ &= \overline{m+n} \times \beta(m, n) \quad \left[\because \beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx \right] \\ \therefore \beta(m, n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{(Proved)}\end{aligned}$$

Theorem-05: Prove that $\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x dx = \frac{\frac{p+1}{2} \frac{q+1}{2}}{2 \frac{p+q+2}{2}}$

Proof: We know that

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Let $x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$.

Limit: $x = 0 \Rightarrow \theta = 0$ and $x = \infty \Rightarrow \theta = \frac{\pi}{2}$.

Now,

$$\begin{aligned}
 \beta(m, n) &= \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} \times 2 \sin \theta \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin^{2m-2} \theta (\cos^2 \theta)^{n-1} \times 2 \sin \theta \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin^{2m-2} \theta \cos^{2n-2} \theta \times 2 \sin \theta \cos \theta d\theta \\
 \therefore \beta(m, n) &= 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta
 \end{aligned}$$

Assume $2m-1 = p$ and $2n-1 = q \Rightarrow m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$.

Now from above equation we get

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2 \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta$$

Using the relation between beta and gamma function $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$, we have

$$\begin{aligned}
 \frac{\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}}{2} &= 2 \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta \\
 \therefore \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta &= \frac{\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2}}{2 \frac{\Gamma\left(\frac{p+q+2}{2}\right)}{2}} \quad \text{(Proved)}
 \end{aligned}$$

Problem-01: Evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x dx$

Solution: Let, $I = \int_0^{\frac{\pi}{2}} \cos^7 x dx$

$$= \int_0^{\frac{\pi}{2}} \sin^0 x \cos^7 x dx$$

$$= \frac{\frac{\Gamma(0+1)}{2} \frac{\Gamma(7+1)}{2}}{2 \frac{\Gamma(0+7+2)}{2}}$$

Exer.-01: $\int_0^{\frac{\pi}{2}} \cos^4 x dx$

Ans: $\frac{3\pi}{16}$

Exer.-02: $\int_0^{\frac{\pi}{2}} \sin^5 x dx$

Ans: $\frac{8}{15}$

$$\begin{aligned}
&= \frac{\left| \frac{1}{2} \right| \left| \frac{8}{2} \right|}{2 \left| \frac{9}{2} \right|} \\
&= \frac{\left| \frac{1}{2} \right| \left| 4 \right|}{2 \left| \frac{9}{2} \right|} \\
&= \frac{\left| \frac{1}{2} \right| \left| 3+1 \right|}{2 \left| \frac{7}{2} + 1 \right|} \\
&= \frac{\left| \frac{1}{2} \right| 3!}{2 \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \left| \frac{1}{2} \right|} \\
&= \frac{3 \cdot 2 \cdot 1}{2 \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}} \\
&= \frac{16}{35} \quad \text{(Ans.)}
\end{aligned}$$

Exer.-03: $\int_0^{\frac{\pi}{2}} \sin^8 x \, dx$

Ans: $\frac{35\pi}{256}$

Problem-02: Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

Solution: Solution: Let, $I = \int_0^{\frac{\pi}{2}} \sin^6 x \, dx$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \sin^6 x \cos^0 x \, dx \\
&= \frac{\left| \frac{6+1}{2} \right| \left| \frac{0+1}{2} \right|}{2 \left| \frac{6+0+2}{2} \right|} \\
&= \frac{\left| \frac{7}{2} \right| \left| \frac{1}{2} \right|}{2 \left| \frac{8}{2} \right|} \\
&= \frac{\left| \frac{7}{2} \right| \left| \frac{1}{2} \right|}{2 \left| 4 \right|}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\frac{5}{2}+1}{2} \frac{\frac{1}{2}}{\sqrt{3+1}} \\
&= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{2 \cdot 3!} \\
&= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \sqrt{\pi}}{2 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{5 \cdot \pi}{32} \quad \text{(Ans.)}
\end{aligned}$$

Problem-03: Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x \, dx$

Solution: Let, $I = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x \, dx$

$$\begin{aligned}
&= \frac{\frac{4+1}{2} \frac{3+1}{2}}{2 \frac{4+3+2}{2}} \\
&= \frac{\frac{5}{2} \frac{4}{2}}{2 \frac{9}{2}} \\
&= \frac{\frac{3}{2}+1}{2} \frac{\sqrt{2}}{\frac{7}{2}+1} \\
&= \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{2 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} \\
&= \frac{2}{35} \quad \text{(Ans.)}
\end{aligned}$$

Problem-04: Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \, dx$

Solution: Let, $I = \int_0^{\frac{\pi}{2}} \cos^3 x \cos 2x \, dx$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x \, dx$

Ans: $\frac{8}{693}$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x \, dx$

Ans: $\frac{7\pi}{2048}$

Exer.-06: $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^3 x \, dx$

Ans: $\frac{2}{63}$

Exer.-04: $\int_0^{\frac{\pi}{2}} \sin 2x \cos^4 x \, dx$

Ans: $\frac{1}{3}$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \cos^3 x (\cos^2 x - \sin^2 x) dx \\
&= \int_0^{\frac{\pi}{2}} (\cos^5 x - \cos^3 x \sin^2 x) dx \\
&= \int_0^{\frac{\pi}{2}} \cos^5 x dx - \int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx \\
&= \frac{\frac{0+1}{2} \left| \frac{5+1}{2} \right|}{2 \left| \frac{0+5+2}{2} \right|} - \frac{\frac{2+1}{2} \left| \frac{3+1}{2} \right|}{2 \left| \frac{2+3+2}{2} \right|} \\
&= \frac{\frac{1}{2} \sqrt{3}}{2 \sqrt{\frac{7}{2}}} - \frac{\frac{3}{2} \sqrt{2}}{2 \sqrt{\frac{7}{2}}} \\
&= \frac{\frac{1}{2} \sqrt{2+1}}{2 \sqrt{\frac{5}{2}+1}} - \frac{\frac{1}{2} + 1 \sqrt{1+1}}{2 \sqrt{\frac{5}{2}+1}} \\
&= \frac{\frac{1}{2} \cdot 2 \sqrt{1}}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}} - \frac{1}{2} \cdot \frac{1}{2} \cdot 1 \sqrt{\frac{1}{2}}} \\
&= \frac{8}{15} - \frac{2}{15} \\
&= \frac{8-2}{15} \\
&= \frac{6}{15} \\
&= \frac{2}{5} \quad \text{(Ans.)}
\end{aligned}$$

Problem-05: Evaluate $\int_0^{2\pi} \sin^4 x \cos^6 x dx$

Solution: Let, $I = \int_0^{2\pi} \sin^4 x \cos^6 x dx$

$$\begin{aligned}
&= 2 \int_0^{\pi} \sin^4 x \cos^6 x dx \\
&= 4 \int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x dx
\end{aligned}$$

Exer.-05: $\int_0^{\frac{\pi}{2}} \sin 2x \sin^2 x \cos^5 x dx$

Ans: $\frac{4}{63}$

Exer.-06: $\int_0^{\pi} \sin^2 x \cos^4 x dx$

Ans: $\frac{\pi}{16}$

$$\begin{aligned}
&= 4 \cdot \frac{\sqrt{\frac{4+1}{2}} \sqrt{\frac{6+1}{2}}}{2 \sqrt{\frac{4+6+2}{2}}} \\
&= 2 \cdot \frac{\sqrt{\frac{5}{2}} \sqrt{\frac{7}{2}}}{\sqrt{6}} \\
&= 2 \cdot \frac{\sqrt{\frac{3}{2}+1} \sqrt{\frac{5}{2}+1}}{\sqrt{5+1}} \\
&= 2 \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{5 \cdot 4 \cdot 3} \\
&= \frac{1}{60} \times \frac{45\pi}{32} \\
&= \frac{3\pi}{128} \quad (\text{Ans.})
\end{aligned}$$

Problem-06: Evaluate $\int_0^{\pi} x \sin^6 x \cos^4 x \, dx$

Solution: Let, $I = \int_0^{\pi} x \sin^6 x \cos^4 x \, dx$

$$= \int_0^{\pi} (\pi - x) \sin^6 (\pi - x) \cos^4 (\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \sin^6 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx - \int_0^{\pi} x \sin^6 x \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx - I$$

$$\therefore 2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx$$

$$= 2\pi \cdot \frac{\sqrt{\frac{6+1}{2}} \sqrt{\frac{4+1}{2}}}{2 \sqrt{\frac{6+4+2}{2}}}$$

Exer.-07: $\int_0^{\pi} x \sin^2 x \cos^4 x \, dx$

Ans: $\frac{\pi^2}{32}$

Exer.-08: $\int_0^{\pi} x \sin x \cos^2 x \, dx$

Ans: $\frac{\pi}{3}$

$$\begin{aligned}
&= \pi \cdot \frac{\sqrt{\frac{7}{2}} \sqrt{\frac{5}{2}}}{\sqrt{6}} \\
&= \pi \cdot \frac{\sqrt{\frac{5}{2}+1} \sqrt{\frac{3}{2}+1}}{\sqrt{5+1}} \\
&= \pi \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \pi \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{120} \\
&= \frac{1}{120} \times \frac{45\pi^2}{32} \\
&= \frac{3\pi^2}{256} \\
\therefore I &= \frac{3\pi^2}{512} \quad \text{(Ans.)}
\end{aligned}$$

Problem-08: Evaluate $\int_0^1 x^3 (1-x^2)^{\frac{5}{2}} dx$

Solution: Let, $I = \int_0^1 x^3 (1-x^2)^{\frac{5}{2}} dx$

Put $x = \sin \theta \quad \therefore dx = \cos \theta d\theta$

Limit : when $x = 0$ then $\theta = 0$

when $x = 1$ then $\theta = \frac{\pi}{2}$

$$\begin{aligned}
\text{Now, } I &= \int_0^{\frac{\pi}{2}} \sin^3 \theta (1 - \sin^2 \theta)^{\frac{5}{2}} \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^3 \theta (\cos^2 \theta)^{\frac{5}{2}} \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^5 \theta \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^6 \theta d\theta \\
&= \frac{\sqrt{\frac{3+1}{2}} \sqrt{\frac{6+1}{2}}}{2 \sqrt{\frac{3+6+2}{2}}}
\end{aligned}$$

Exer.-09: $\int_0^1 x^2 (1-x^2)^{\frac{1}{2}} dx$

Ans: $\frac{\pi}{16}$

Exer.-10: $\int_0^1 x^4 (1-x^2)^{\frac{3}{2}} dx$

Ans: $\frac{3\pi}{256}$

Exer.-11: $\int_0^1 x^6 (1-x^2)^{\frac{1}{2}} dx$

Ans: $\frac{5\pi}{256}$

$$\begin{aligned}
&= \frac{\sqrt{2} \sqrt{\frac{7}{2}}}{2 \sqrt{\frac{11}{2}}} \\
&= \frac{\sqrt{1+1} \sqrt{\frac{5}{2}+1}}{2 \sqrt{\frac{9}{2}+1}} \\
&= \frac{1 \sqrt{1} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{2 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}} \\
&= \frac{2}{63} \quad \text{(Ans.)}
\end{aligned}$$

Problem-09: Evaluate $\int_0^1 \frac{x^5}{\sqrt{(1-x^2)}} dx$

Solution: Let, $I = \int_0^1 \frac{x^3}{\sqrt{(1-x^2)}} dx$

Put $x = \sin \theta \quad \therefore dx = \cos \theta d\theta$

Limit : when $x = 0$ then $\theta = 0$

when $x = 1$ then $\theta = \frac{\pi}{2}$

$$\begin{aligned}
\text{Now, } I &= \int_0^{\frac{\pi}{2}} \frac{\sin^5 \theta}{\sqrt{(1-\sin^2 \theta)}} \cdot \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^5 \theta}{\sqrt{\cos^2 \theta}} \cdot \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin^5 \theta}{\cos \theta} \cdot \cos \theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^5 \theta d\theta \\
&= \frac{\sqrt{\frac{5+1}{2}} \sqrt{\frac{0+1}{2}}}{2 \sqrt{\frac{5+0+2}{2}}} \\
&= \frac{\sqrt{3} \sqrt{\frac{1}{2}}}{2 \sqrt{\frac{7}{2}}}
\end{aligned}$$

Exer.-12: $\int_0^1 \frac{x^6}{\sqrt{(1-x^2)}} dx$

Ans: $\frac{5\pi}{32}$

$$\begin{aligned}
&= \frac{\sqrt{2+1} \sqrt{\frac{1}{2}}}{2 \sqrt{\frac{5}{2}+1}} \\
&= \frac{2.1 \sqrt{1} \cdot \sqrt{\frac{1}{2}}}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}} \\
&= \frac{8}{15} \quad \text{(Ans.)}
\end{aligned}$$

Problem-10: Evaluate $\int_0^{\infty} \frac{x^3}{(1+x^2)^{\frac{9}{2}}} dx$

Solution: Let, $I = \int_0^{\infty} \frac{x^3}{(1+x^2)^{\frac{9}{2}}} dx$

Put $x = \tan \theta \quad \therefore dx = \sec^2 \theta d\theta$

Limit : when $x=0$ then $\theta=0$

when $x = \infty$ then $\theta = \frac{\pi}{2}$

$$\text{Now, } I = \int_0^{\frac{\pi}{2}} \frac{\tan^3 \theta}{(1+\tan^2 \theta)^{\frac{9}{2}}} \cdot \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3 \theta}{(\sec^2 \theta)^{\frac{9}{2}}} \cdot \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3 \theta}{\sec^9 \theta} \cdot \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^3 \theta}{\sec^7 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3 \theta}{\cos^3 \theta} \cdot \cos^7 \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta d\theta$$

$$= \frac{\sqrt{\frac{3+1}{2}} \sqrt{\frac{4+1}{2}}}{2 \sqrt{\frac{3+4+2}{2}}}$$

$$\begin{aligned}
&= \frac{\sqrt{2} \sqrt{\frac{5}{2}}}{2 \sqrt{\frac{9}{2}}} \\
&= \frac{\sqrt{1+1} \sqrt{\frac{3}{2}+1}}{2 \sqrt{\frac{7}{2}+1}} \\
&= \frac{1 \sqrt{1} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{2 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}} \\
&= \frac{2}{35} \quad \text{(Ans.)}
\end{aligned}$$

Problem-11: Evaluate $\int_0^1 x^4 (1-x)^{\frac{3}{2}} dx$

Solution: Let, $I = \int_0^1 x^4 (1-x)^{\frac{3}{2}} dx$

$$\begin{aligned}
&= \int_0^1 x^{5-1} (1-x)^{\frac{5}{2}-1} dx \\
&= \beta\left(5, \frac{5}{2}\right) \\
&= \frac{\sqrt{5} \sqrt{\frac{5}{2}}}{\sqrt{5 + \frac{5}{2}}} \\
&= \frac{\sqrt{4+1} \sqrt{\frac{5}{2}}}{\sqrt{\frac{13}{2}+1}} \\
&= \frac{4.3.2.1 \sqrt{\frac{5}{2}}}{\frac{13}{2} \cdot \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \sqrt{\frac{5}{2}}} \\
&= \frac{256}{15015} \quad \text{(Ans.)}
\end{aligned}$$

Problem-12: Show that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Solution: Let, $I = \int_0^{\infty} e^{-x^2} dx$

Put, $x^2 = z$

Exer.-13: $\int_0^1 x^3 (1-x)^3 dx$

Ans: $\frac{1}{140}$

Exer.-14: Show that $\int_0^{\infty} e^{-3x^2} dx = \frac{\sqrt{\pi}}{2\sqrt{3}}$

Exer.-15: Show that $\int_0^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a}$

$$\therefore 2x dx = dz$$

$$\Rightarrow dx = \frac{1}{2\sqrt{z}} dz$$

Limit : when $x=0$ then $z=0$

when $x=\infty$ then $z=\infty$

$$\text{Now, } I = \int_0^{\infty} e^{-z} \cdot \frac{1}{2\sqrt{z}} dz$$

$$= \frac{1}{2} \int_0^{\infty} e^{-z} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{\sqrt{1}}{2}$$

$$= \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (\text{Showed.})$$

Problem-13: Show that $\int_0^{\infty} e^{-x} x^{\frac{3}{2}} dx = \frac{3\sqrt{\pi}}{4}$

Exer.-16: Show that $\int_0^{\infty} \sqrt{x} e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$

Solution: Let, $I = \int_0^{\infty} e^{-x} x^{\frac{3}{2}} dx$

$$= \int_0^{\infty} e^{-x} x^{\frac{5}{2}-1} dx$$

$$= \frac{\sqrt{5}}{2}$$

$$= \frac{3}{2} + 1$$

$$= \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{3\sqrt{\pi}}{4}$$

$$\therefore \int_0^{\infty} e^{-x} x^{\frac{3}{2}} dx = \frac{3\sqrt{\pi}}{4} \quad (\text{Showed})$$

Problem-14: Show that $\int_0^{\infty} e^{-3x} x^{\frac{3}{2}} dx = \frac{\sqrt{3\pi}}{36}$

Exer.-17: Show that $\int_0^{\infty} e^{-4x^2} dx = \frac{3\sqrt{\pi}}{128}$

Solution: Let, $I = \int_0^{\infty} e^{-3x} x^{\frac{3}{2}} dx$

$$= \int_0^{\infty} e^{-3x} x^{\frac{5}{2}-1} dx$$

$$= \frac{\sqrt{\frac{5}{2}}}{3^{\frac{5}{2}}}$$

$$= \frac{\sqrt{\frac{3}{2}+1}}{\sqrt{3^5}}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\frac{1}{2}}}{9\sqrt{3}}$$

$$= \frac{\sqrt{3\pi}}{36}$$

$\therefore \int_0^{\infty} e^{-3x} x^{\frac{3}{2}} dx = \frac{\sqrt{3\pi}}{36}$ (Showed.)

Multiple choice questions:

1. Which one of the following is answer of $\frac{\Gamma(6)}{2\Gamma(3)}$?

- a). 35
- b). 30
- c). 40
- d). 25

2. The value of the integral $\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}$ is :

- a). $\frac{1}{3}$
- b). $\frac{3\pi}{4}$
- c). π
- d). $\frac{3}{4}$

3. Calculate $\int_0^{\infty} e^{-\sqrt{x}} \sqrt[4]{x} dx$.

a). $\frac{3\sqrt{\pi}}{2}$

b). $\frac{\sqrt{\pi}}{2}$

c). $\frac{2\sqrt{\pi}}{3}$

d). None of them.

4. Evaluate $\int_0^{\infty} e^{-x^2} dx$.

a). $\sqrt{\pi}$

b). $\frac{3\sqrt{\pi}}{2}$

c). $\frac{\sqrt{\pi}}{2}$

d). 0

5. Find the indefinite integral $\int_0^{\frac{\pi}{2}} \sin^6 x dx$.

a). $\frac{5\pi}{32}$

b). $\frac{\pi}{32}$

c). $\frac{5\pi}{3}$

d). $\frac{3\pi}{32}$

6. What is the value of $\int_0^{\pi} \cos^4 x dx$?

a). $\frac{\pi}{2}$

b). $\frac{3\pi}{8}$

c). $\frac{5\pi}{32}$

d). $\frac{5\pi}{3}$

7. Which one of the following is correct for $B(m, n)$?

a). $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

b). $\frac{\Gamma(m)\Gamma(n)}{\Gamma(mn)}$

c). $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m-n)}$

d). $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m \div n)}$

8. Which one of the following is correct for $\Gamma(n+1)$?

a). $(n+1)!$

b). $n!$

c). $(n-1)!$

d). n

9. What is the value of $\Gamma(2)$?

a). 1

b). 2

c). 0

d). -1

10. Evaluate the value of $\int_0^1 x^4(1-x)^3 dx$.

a). 580

b). 120

c). 2320

d). $\frac{1}{280}$