Transformations to Kinematics (with background)

CSE444: Introduction to Robotics

Lessen 3-5

Fall 2019

Definitions

- **Velocity:** The derivative of position with respect to time.
- **Acceleration:** The derivative of velocity with respect to time.
- **Jerk:** The derivative of acceleration with respect to time.
- **Link:** Nearly rigid structure between joints.
- **Joint:** Allow relative motion between links.
- **Joint Angle**: Measurement of the relative position of two links

$$x = x(t)$$

$$v = \dot{x} = \frac{\partial x}{\partial t}$$

$$a = \dot{v} = \ddot{x} = \frac{\partial v}{\partial t}$$

$$j = \dot{a} = \ddot{v} = \dddot{x} = \frac{\partial a}{\partial t}$$

Definitions ..

- **Joint Space**: Relative coordinates that are referenced to coordinate frames at the robot joints.
- **Cartesian Space or Task Space**. Global or base coordinate frame
- **Jacobian**: Specifies a mapping of Velocities in joint space to velocity in Cartesian or Task Space.
- **Singularity**: Region or point at which the Jacobian is singular.

Mathematical Background

$\sin(\Delta) = \Delta$ $\quad \sin(0) = 0$ $\quad \cos(0) = 1$ $\quad \sin(45 \cdot deg) = 0.707$

$90 \cdot deg = 1.571 \cdot rad$ $\quad \pi = 3.142 \cdot rad$ $\quad 180 \cdot deg = 3.142 \cdot rad$

$\frac{d}{dx}\sin(x) = \cos(x)$ $\quad \frac{d}{dx}\cos(x) = -\sin(x)$

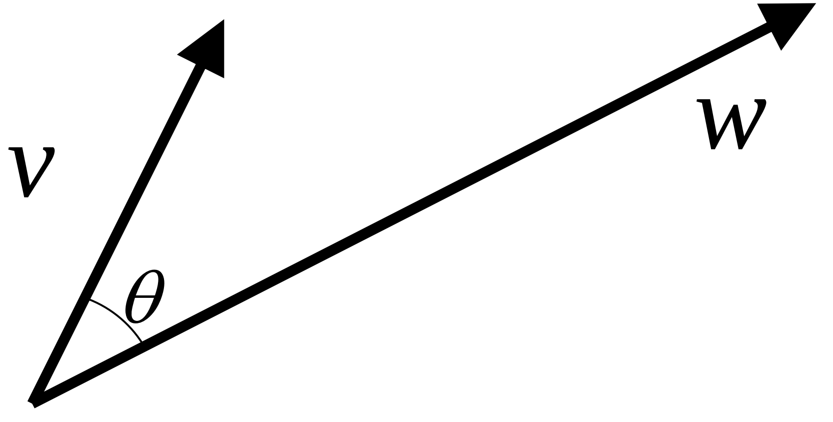
Chain Rule $\quad \frac{d}{dt}F(u) = \frac{\delta F}{\delta u} \cdot \frac{\delta u}{\delta t}$

Example

Let $\quad F(u) = \sin(u)$ $\quad u = x(t)$ $\quad$ Then $\quad \frac{d}{dt}F = \cos(x) \cdot \frac{\delta x}{\delta t}$

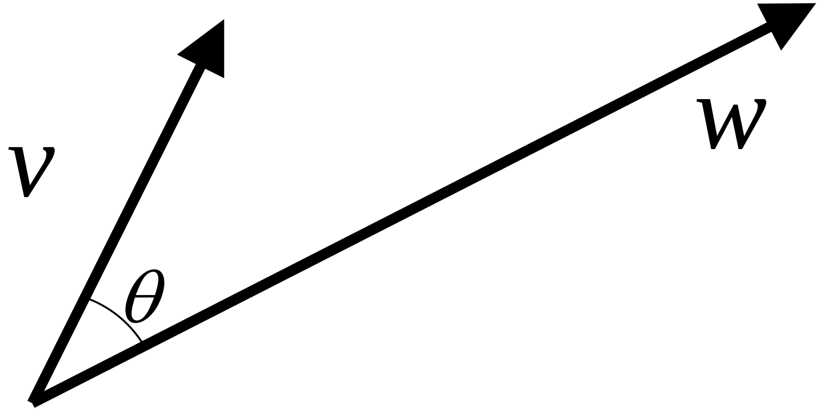
Review – Vector Operations

- Dot Product



Review – Vector Operations

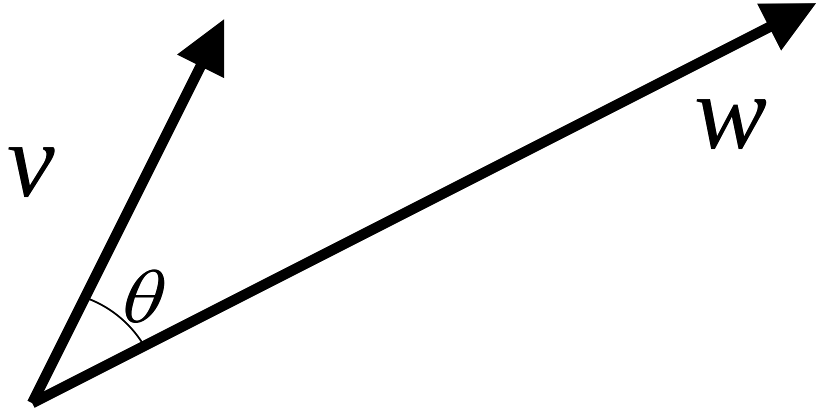
- Dot Product: measuring similarity between two vectors



$$v \cdot w = |v||w|\cos(\theta)$$

Review – Vector Operations

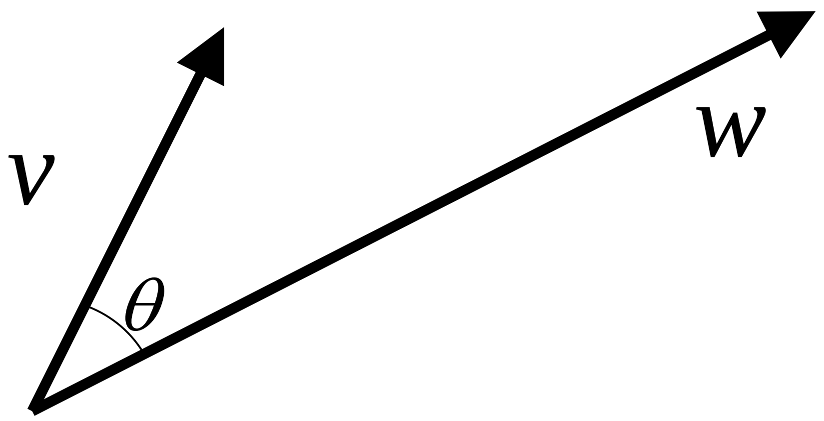
- Dot Product: measuring similarity between two vectors



$$w \cdot v = |v||w|\cos(\theta)$$

Review – Vector Operations

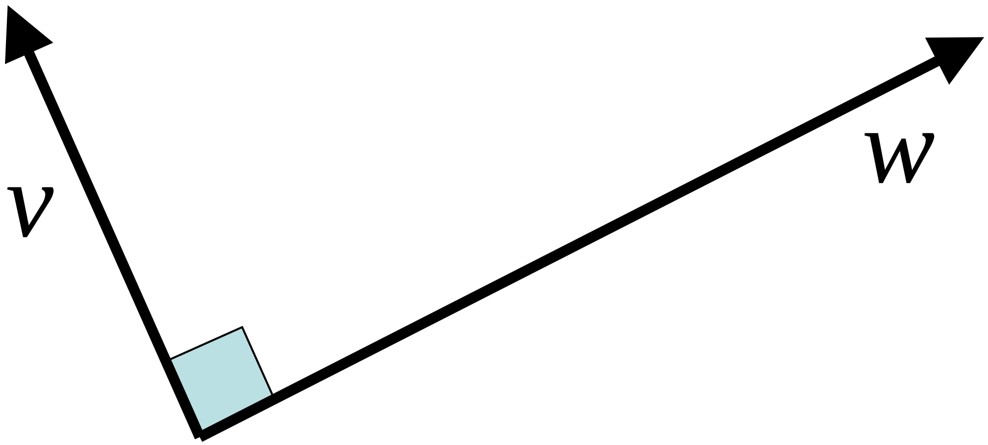
- Dot Product: measuring similarity between two vectors



Unit vector: $v \cdot v = 1$

Review – Vector Operations

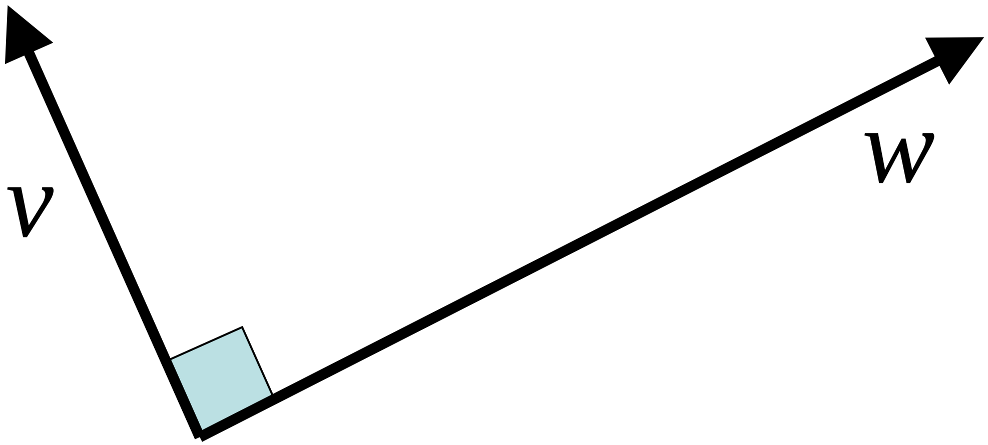
- Dot Product: measuring similarity between two vectors



$$v \cdot w = |v||w|\cos(\theta)$$

Review – Vector Operations

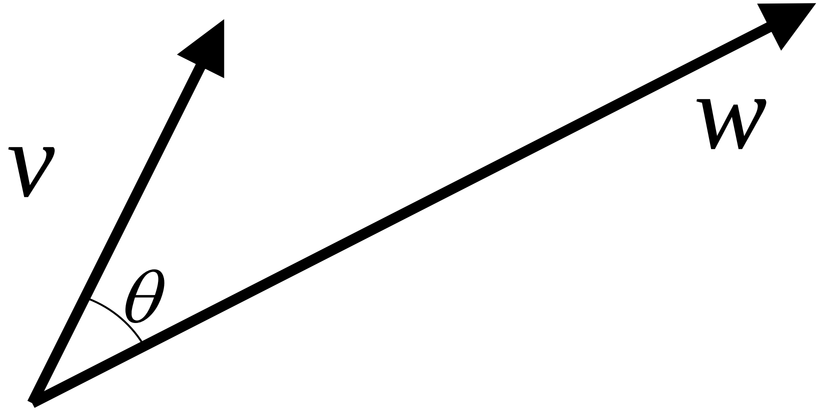
- Dot Product: measuring similarity between two vectors



$$v \cdot w = 0$$

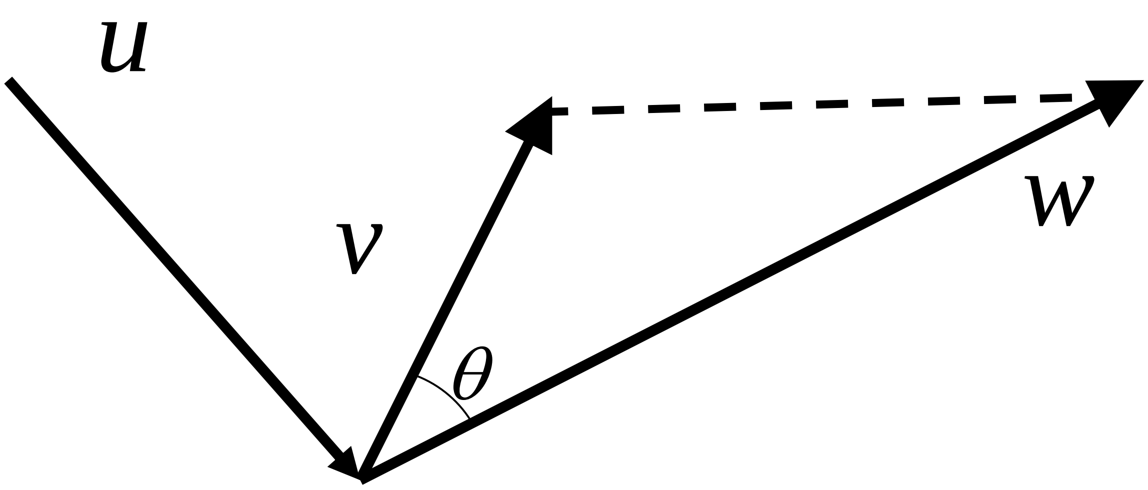
Review – Vector Operations

- Cross Product: measuring the area determined by two vectors



Review – Vector Operations

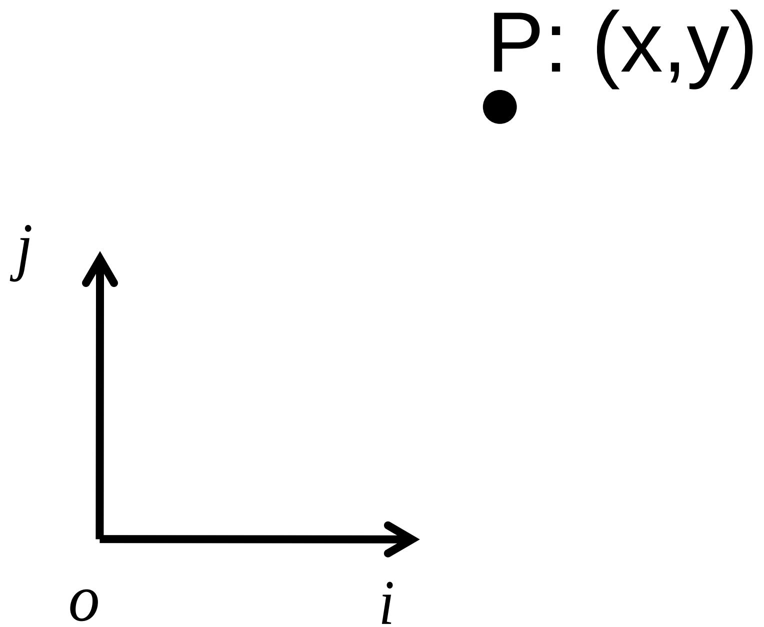
- Cross Product: measuring the area determined by two vectors



$$v \times w = |v||w| \sin\theta \vec{u} = 2 * area \cdot \vec{u}$$

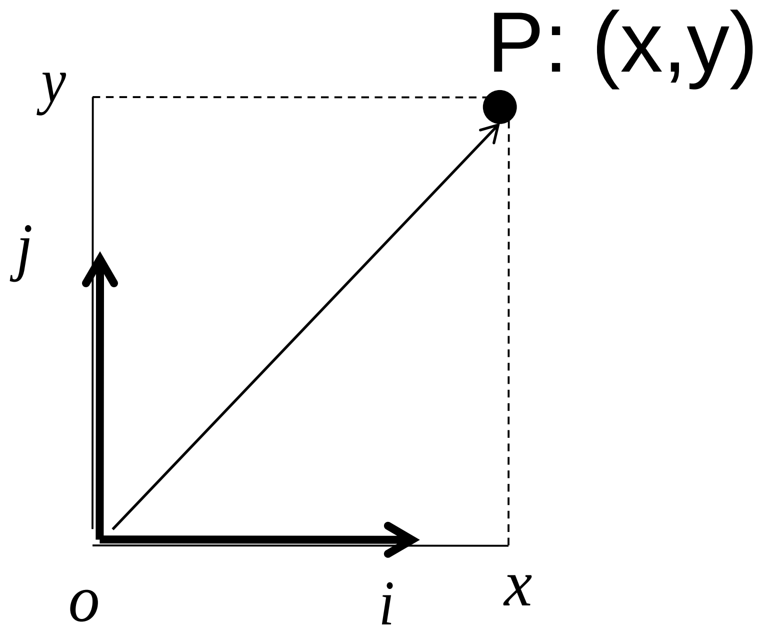
2D Coordinates

- 2D Cartesian coordinate system:



2D Coordinate Transformation

- 2D Cartesian coordinate system:



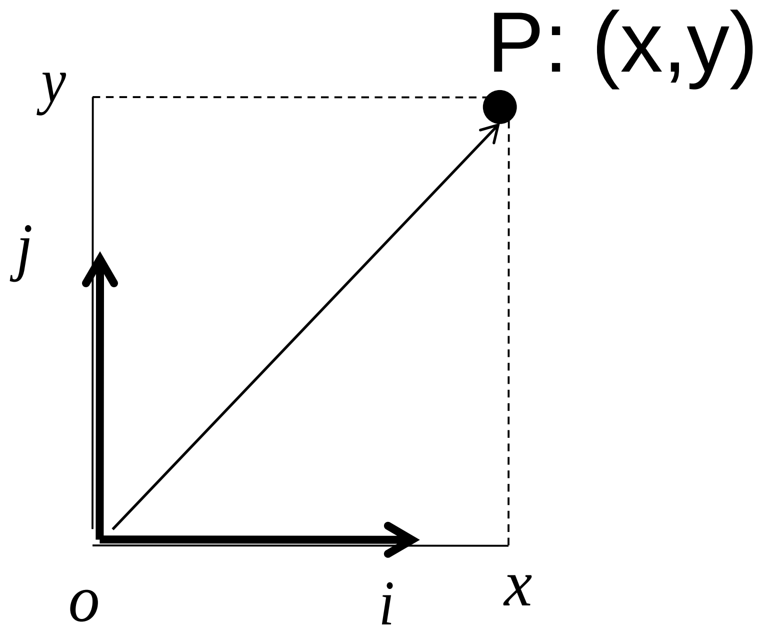
$$\vec{i} \bullet \vec{i} = 1$$

$$\vec{j} \bullet \vec{j} = 1$$

$$\vec{i} \bullet \vec{j} = 0$$

2D Coordinate Transformation

- 2D Cartesian coordinate system:

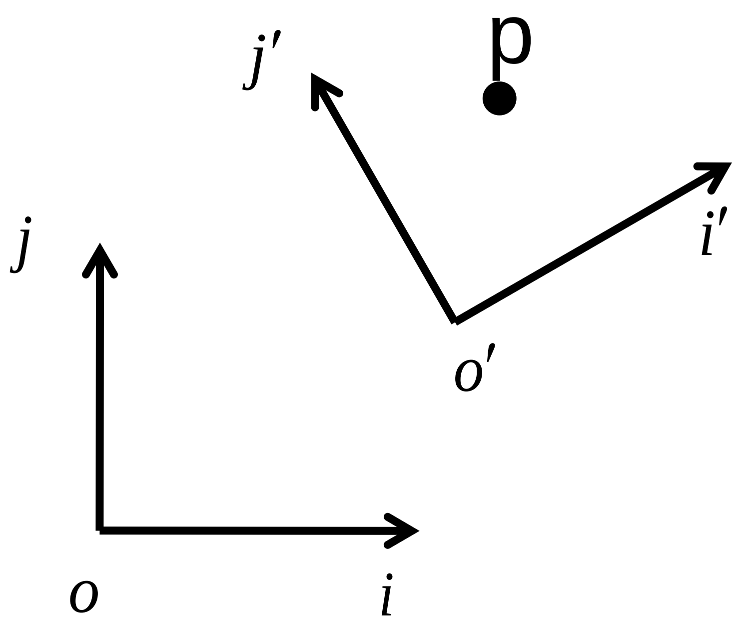


$$\overrightarrow{op} = x\vec{i} + y\vec{j}$$

$$\vec{i} \bullet \vec{i} = 1$$
$$\vec{j} \bullet \vec{j} = 1$$
$$\vec{i} \bullet \vec{j} = 0$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



2D Coordinate Transformation

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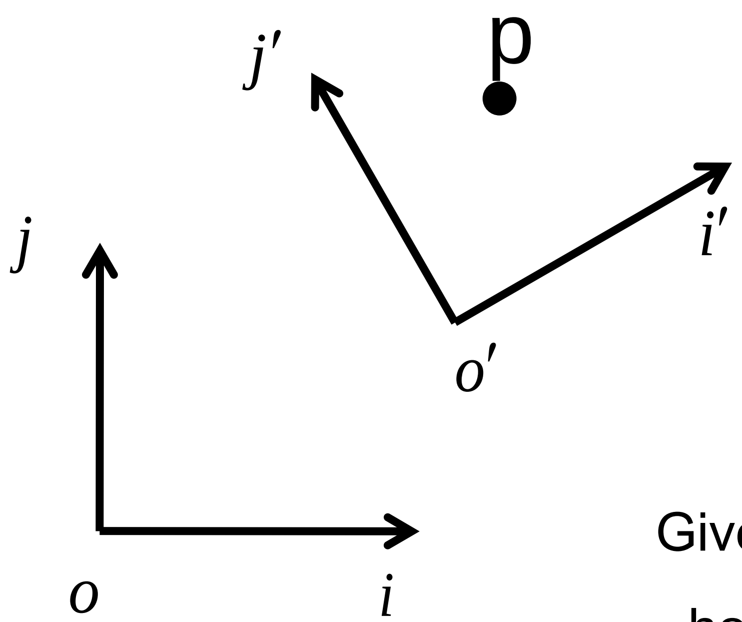
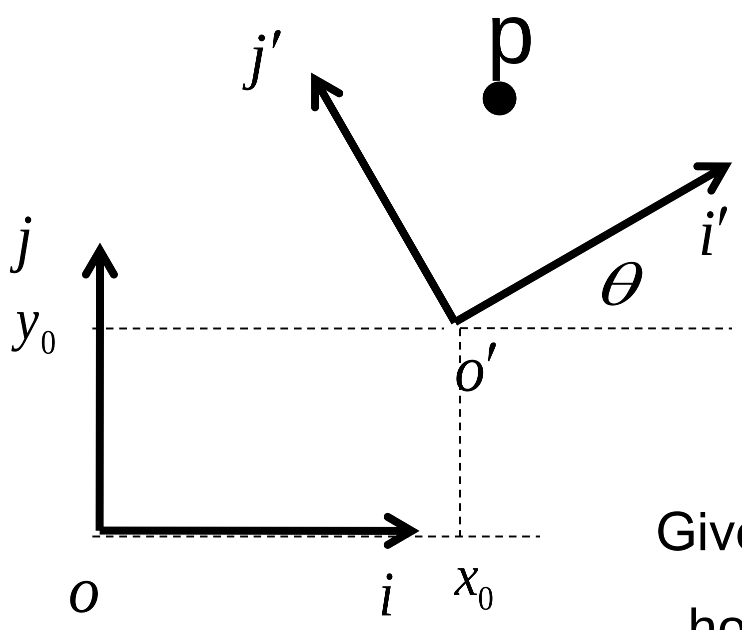


Given the coordinates (x’,y’) in i’j’

- how to compute the coordinates (x,y) in ij?

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



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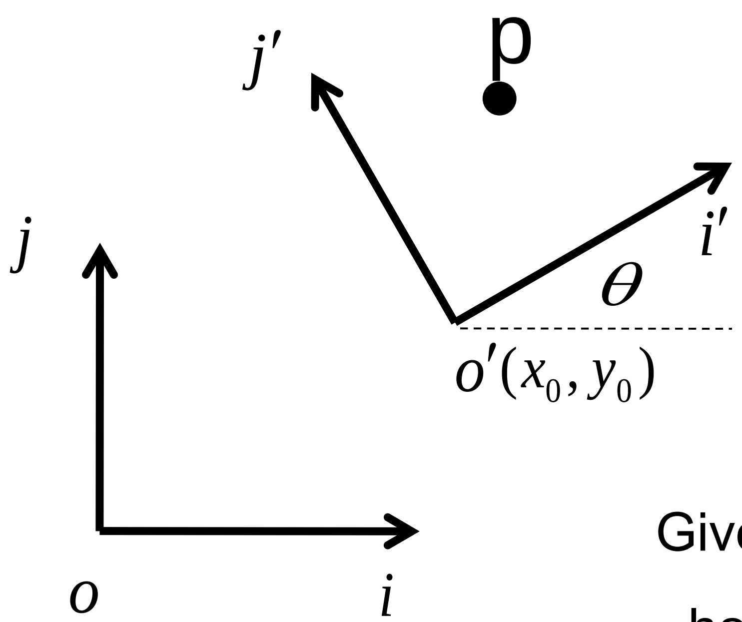


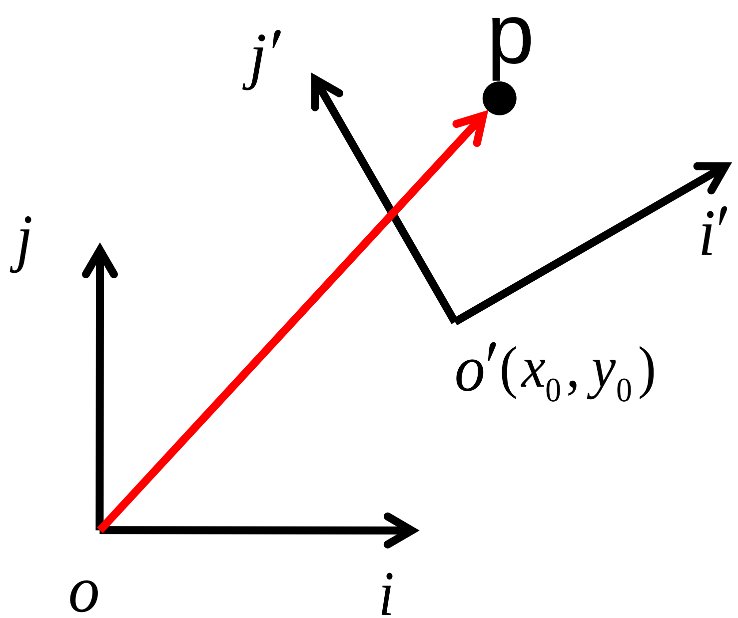
$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \cos\theta & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

Given the coordinates (x’,y’) in i’j’

- how to compute the coordinates (x,y) in ij?

2D Coordinate Transformation

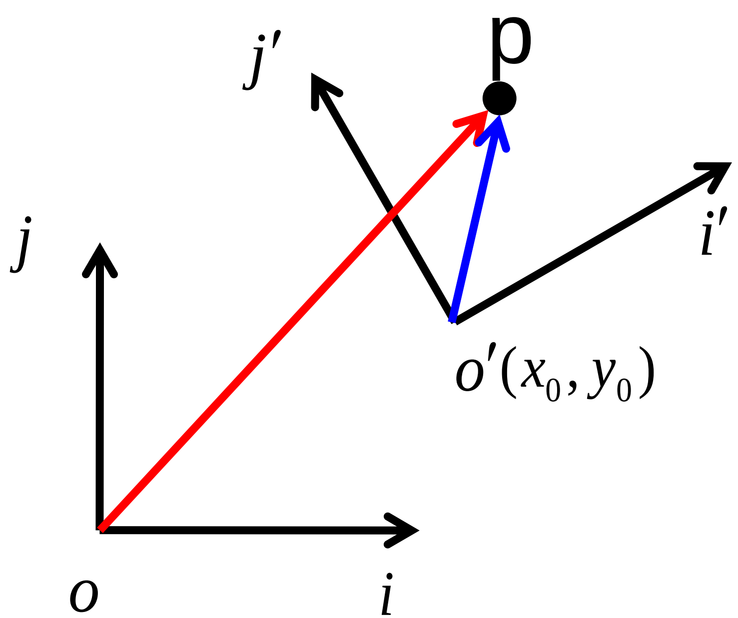
- Transform object description from $i'j'$ to ij



$$\overrightarrow{op} = x\vec{i} + y\vec{j}$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij

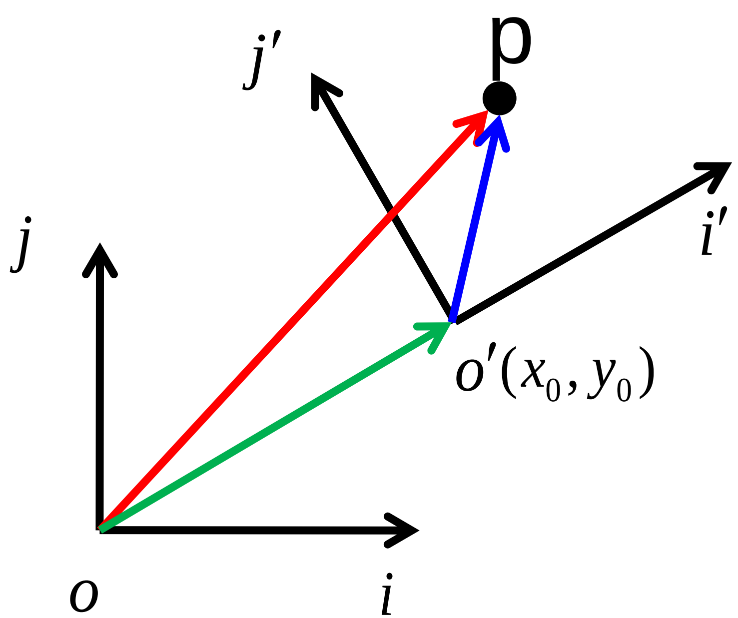


$$\overrightarrow{op} = x\vec{i} + y\vec{j}$$

$$\overrightarrow{o'p} = x'\vec{i}\,' + y'\vec{j}\,'$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



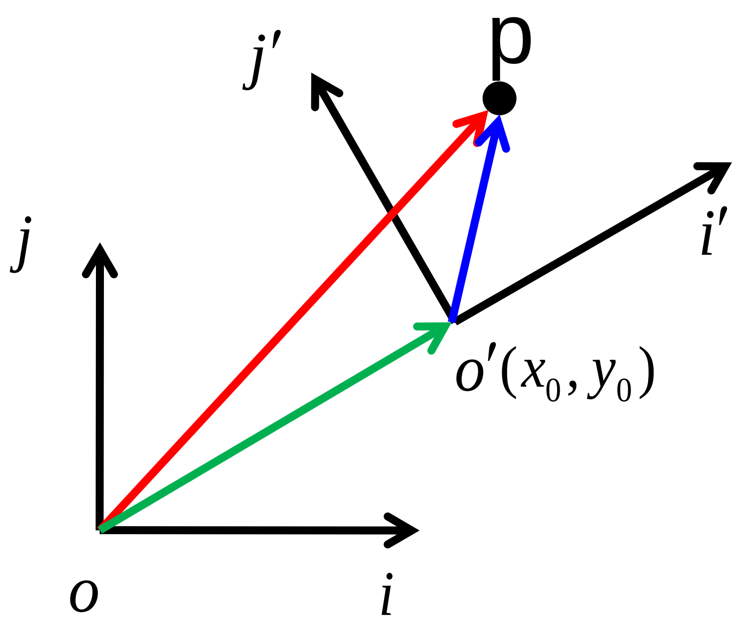
$$\overrightarrow{op} = x\vec{i} + y\vec{j}$$

$$\overrightarrow{o'p} = x'\vec{i}' + y'\vec{j}'$$

$$\overrightarrow{oo'} = x_0\vec{i}' + y_0\vec{j}'$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

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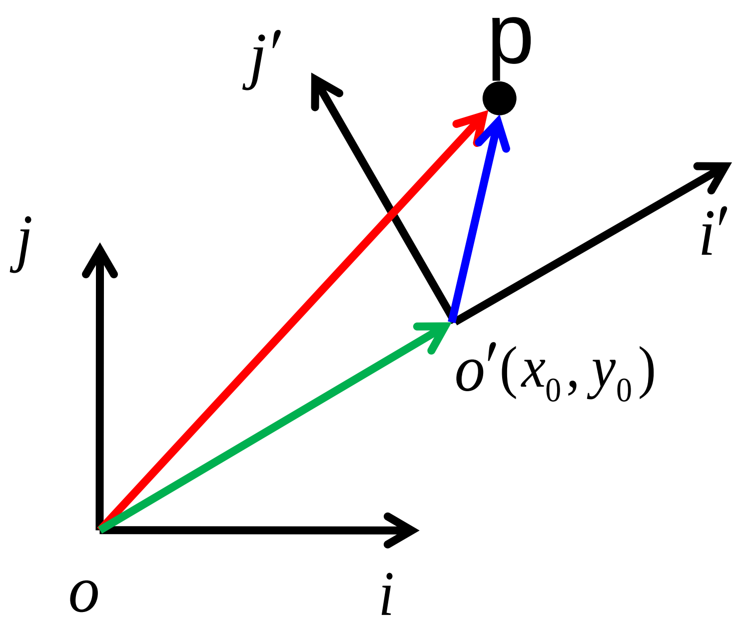

$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

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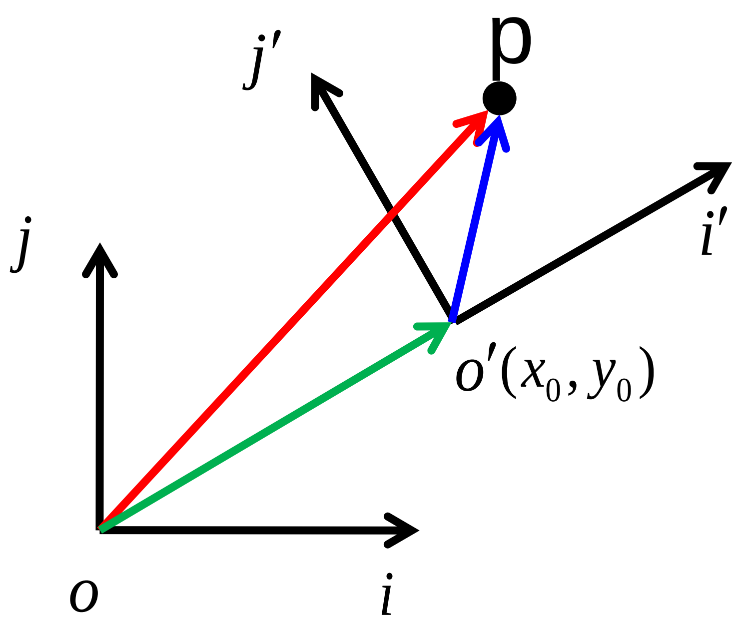
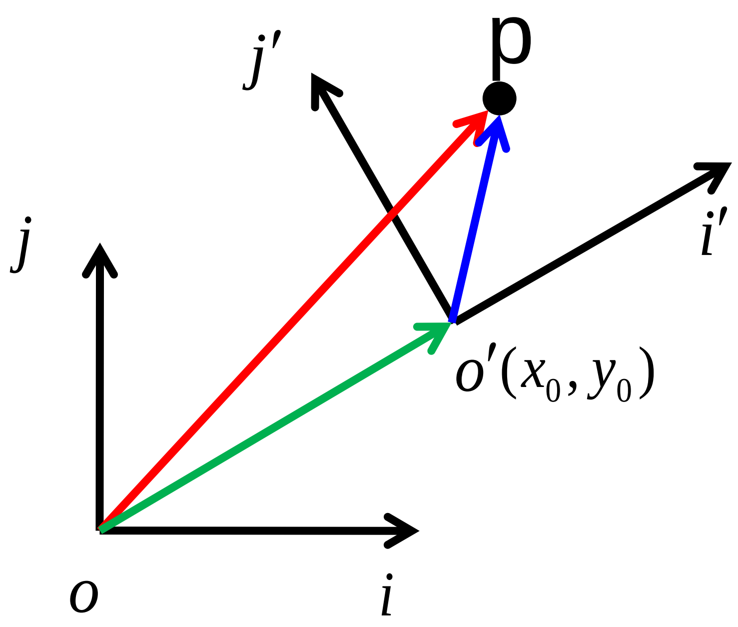


$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i}' + y'\vec{j}'$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



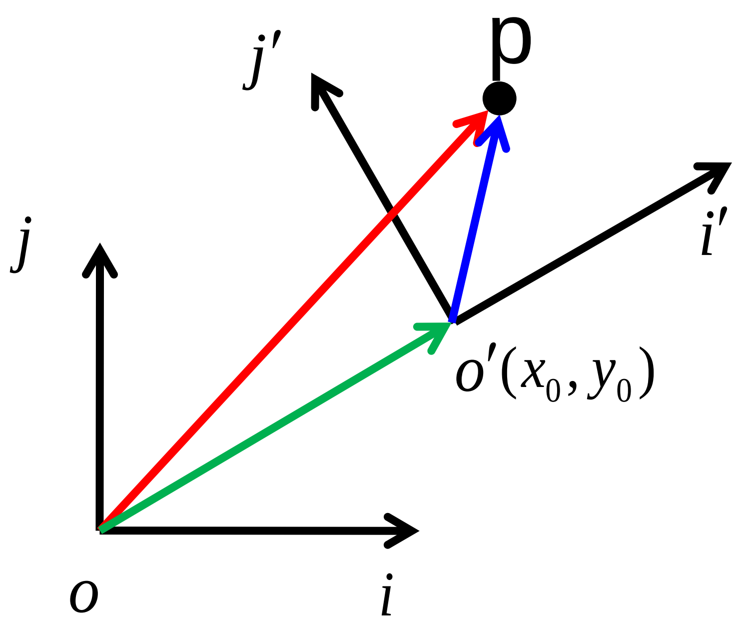
$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i}' + y'\vec{j}'$$

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} = x'\vec{i}' + y'\vec{j}'$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\overrightarrow{op} = \overrightarrow{oo'} + \overrightarrow{o'p}$$

$$x\vec{i} + y\vec{j} = x_0\vec{i} + y_0\vec{j} + x'\vec{i}' + y'\vec{j}'$$

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} = x'\vec{i}' + y'\vec{j}'$$

$$\begin{pmatrix}\vec{i} & \vec{j}\end{pmatrix}\begin{pmatrix}x - x_0 \\ y - y_0\end{pmatrix} = \begin{pmatrix}\vec{i}' & \vec{j}'\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix}$$

2D Coordinate Transformation

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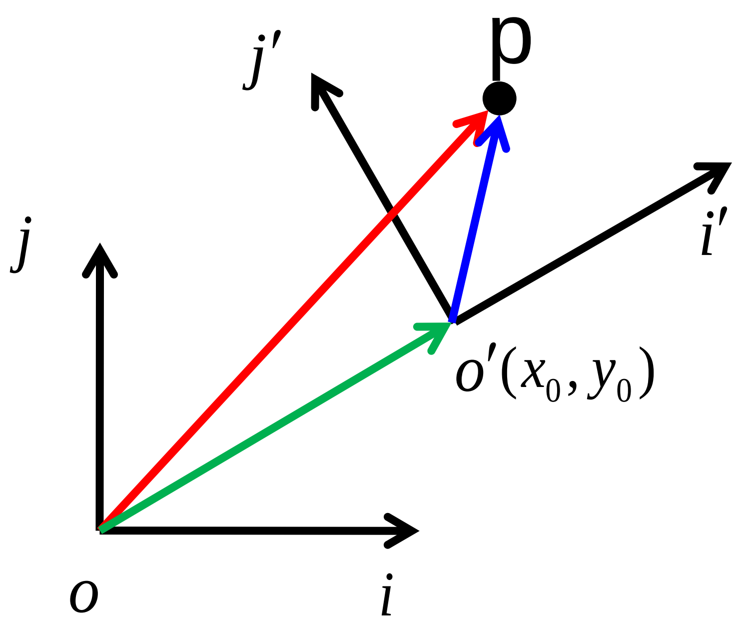

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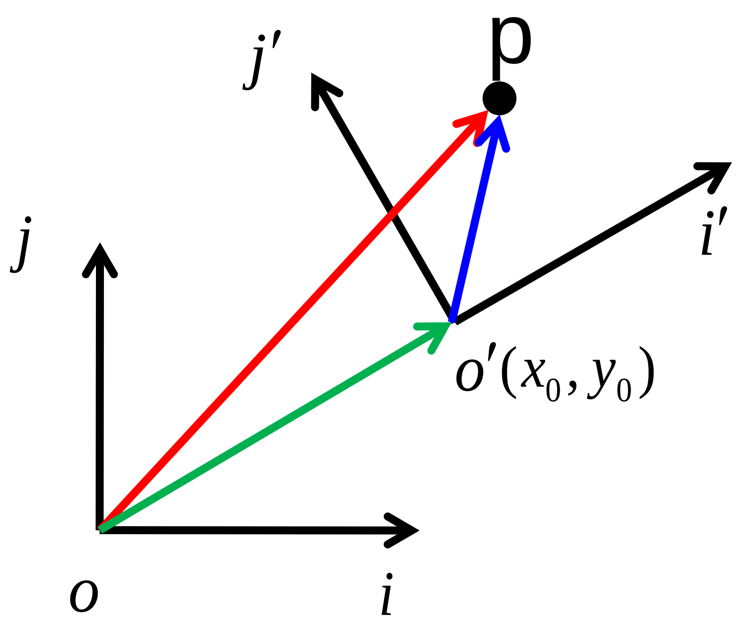

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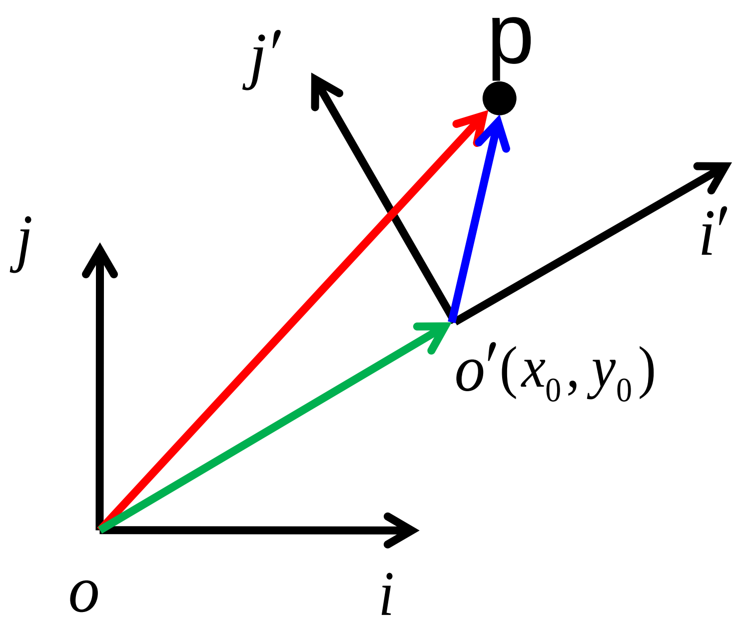
$$\begin{pmatrix}x - x_0 \\ y - y_0\end{pmatrix} = \begin{pmatrix}\vec{i} & \vec{j}\end{pmatrix}^{-1}\begin{pmatrix}\vec{i}' & \vec{j}'\end{pmatrix}\begin{pmatrix}x' \\ y'\end{pmatrix}$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$(\vec{i}\quad \vec{j})\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = (\vec{i}'\quad \vec{j}')\begin{pmatrix}x'\\ y'\end{pmatrix}$$

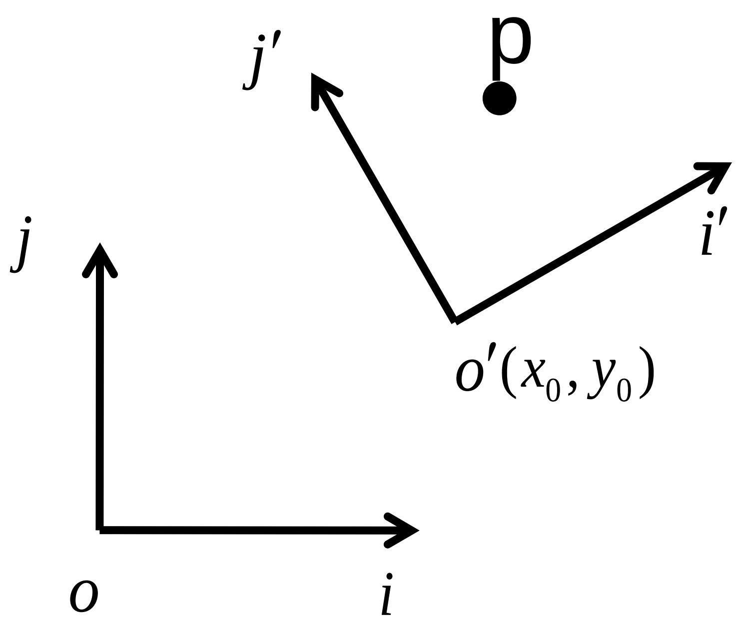
$$\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = (\vec{i}\quad \vec{j})^{-1}(\vec{i}'\quad \vec{j}')\begin{pmatrix}x'\\ y'\end{pmatrix}$$

$$\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = \begin{pmatrix}\vec{i}^T\\ \vec{j}^T\end{pmatrix}(\vec{i}'\quad \vec{j}')\begin{pmatrix}x'\\ y'\end{pmatrix}$$

$$(\vec{i}\quad \vec{j})\begin{pmatrix}\vec{i}^T\\ \vec{j}^T\end{pmatrix} = \begin{pmatrix}\vec{i}\bullet\vec{i} & \vec{i}\bullet\vec{j}\\ \vec{j}\bullet\vec{i} & \vec{j}\bullet\vec{j}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij

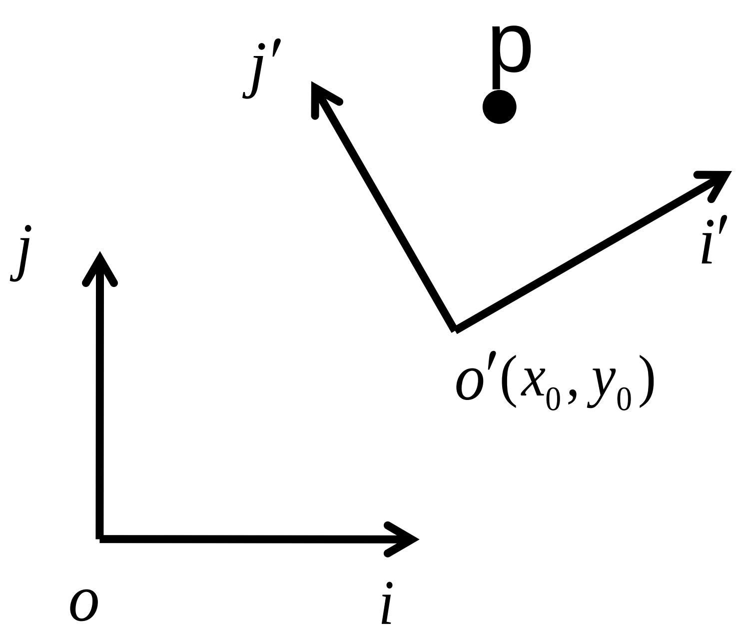


$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^{\,T} \\ \vec{j}^{\,T} \end{pmatrix} \begin{pmatrix} \vec{i}\,' & \vec{j}\,' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{i}^{\,T}\vec{i}\,' & \vec{i}^{\,T}\vec{j}\,' & x_0 \\ \vec{j}^{\,T}\vec{i}\,' & \vec{j}^{\,T}\vec{j}\,' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \\ \vec{j}^T \end{pmatrix} \begin{pmatrix} \vec{i}' & \vec{j}' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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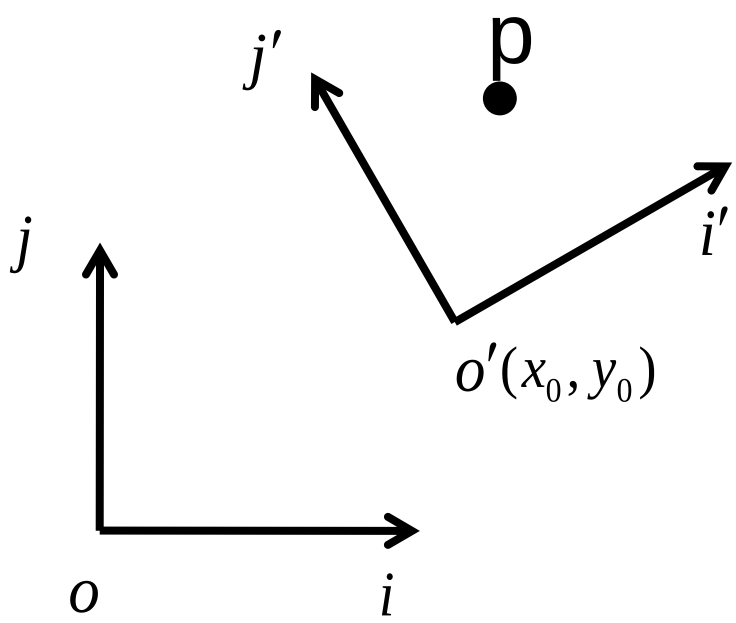
2D Coordinate Transformation

$i'j'$ ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^{\,T} \\ \vec{j}^{\,T} \end{pmatrix} \begin{pmatrix} \vec{i}\,' & \vec{j}\,' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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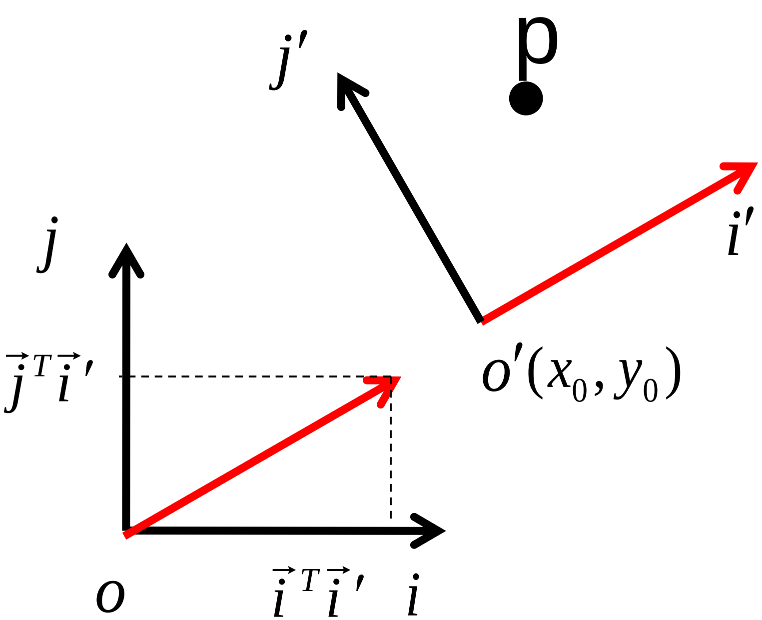
What does this column vector mean?

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \\ \vec{j}^T \end{pmatrix} \begin{pmatrix} \vec{i}' & \vec{j}' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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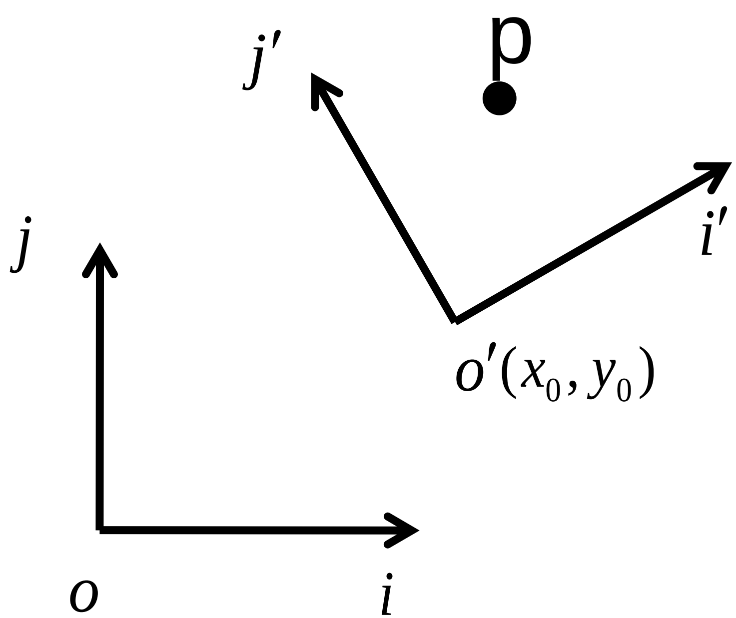
What does this column vector mean? Vector i’ in the new reference system

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^{\,T} \\ \vec{j}^{\,T} \end{pmatrix} \begin{pmatrix} \vec{i}\,' & \vec{j}\,' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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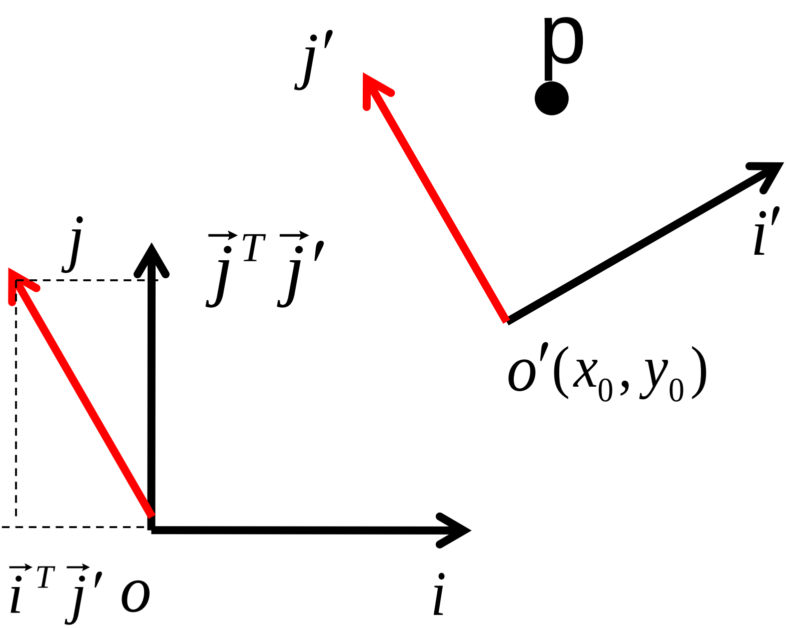
What does this column vector mean?

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \\ \vec{j}^T \end{pmatrix} \begin{pmatrix} \vec{i}' & \vec{j}' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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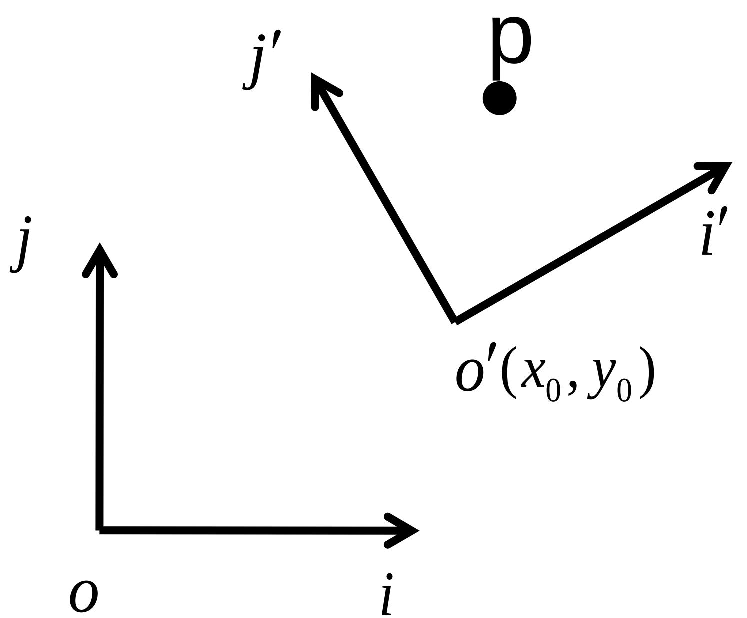
What does this column vector mean? Vector j' in the new reference system

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \\ \vec{j}^T \end{pmatrix} \begin{pmatrix} \vec{i}' & \vec{j}' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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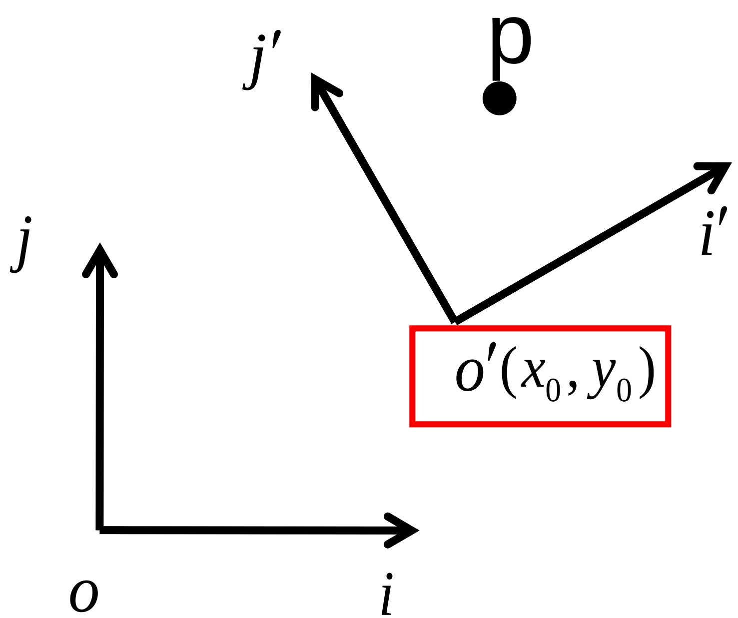
What does this column vector mean?

2D Coordinate Transformation

- Transform object description from $i'j'$ to ij



$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \vec{i}^{\,T} \\ \vec{j}^{\,T} \end{pmatrix} \begin{pmatrix} \vec{i}\,' & \vec{j}\,' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

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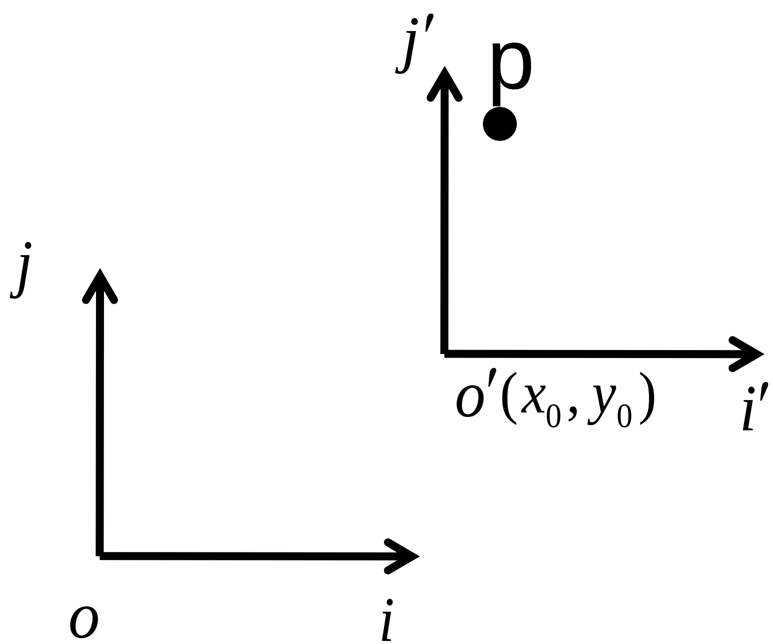
What does this column vector mean? The old origin in the new reference system

2D Coordinate Transformation

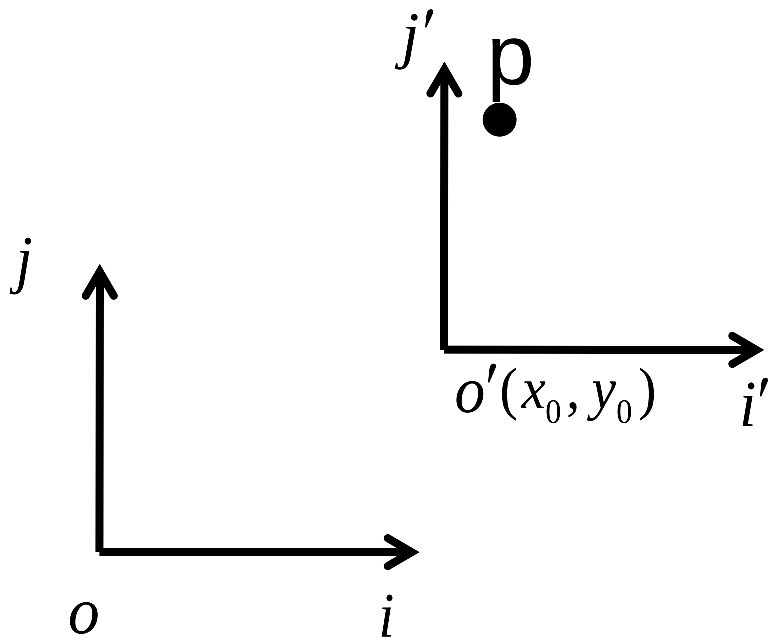
- 2D translation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{i}^T\vec{i}' & \vec{i}^T\vec{j}' & x_0 \\ \vec{j}^T\vec{i}' & \vec{j}^T\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- 2D translation

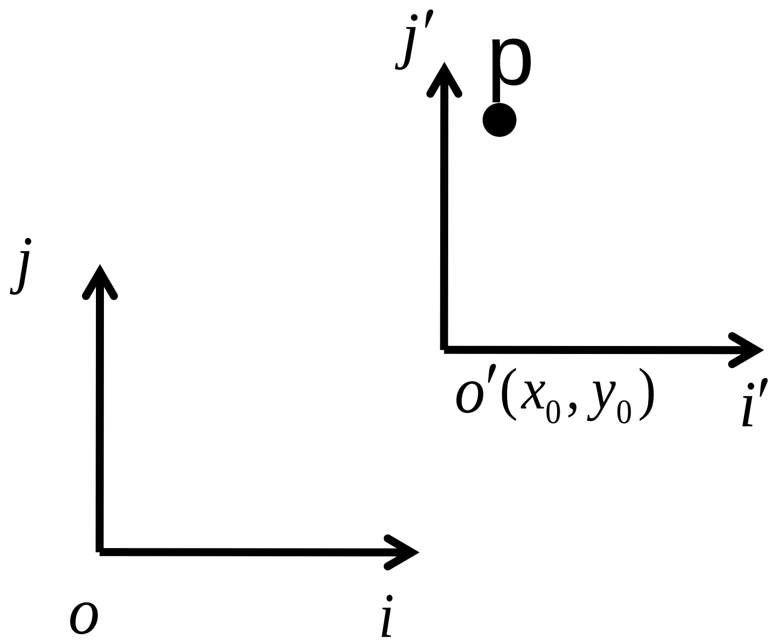


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \vec{i}' & \vec{i}^T \vec{j}' & x_0 \\ \vec{j}^T \vec{i}' & \vec{j}^T \vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

?
?
?
?

2D Coordinate Transformation

- 2D translation

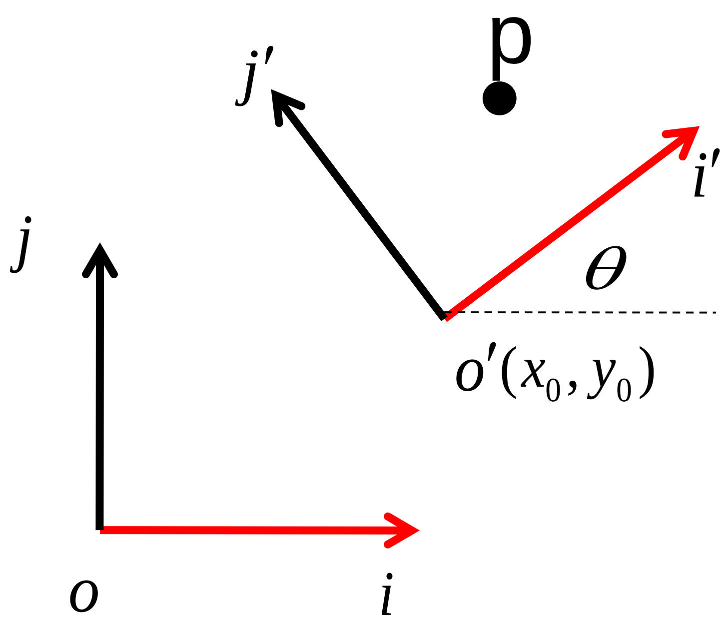


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{i}\cdot\vec{i}' & \vec{i}\cdot\vec{j}' & x_0 \\ \vec{j}\cdot\vec{i}' & \vec{j}\cdot\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2D Coordinate Transformation

- 2D translation&rotation

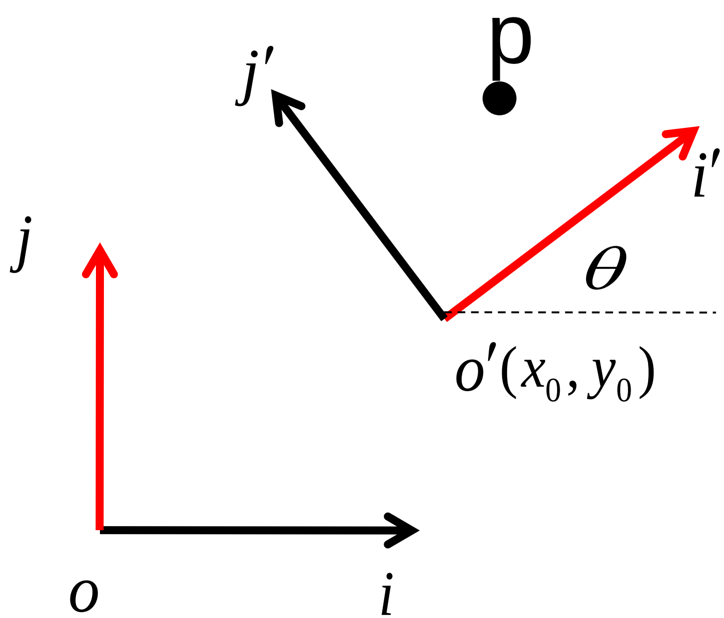


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{i}^T \vec{i}' & \vec{i}^T \vec{j}' & x_0 \\ \vec{j}^T \vec{i}' & \vec{j}^T \vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

?

2D Coordinate Transformation

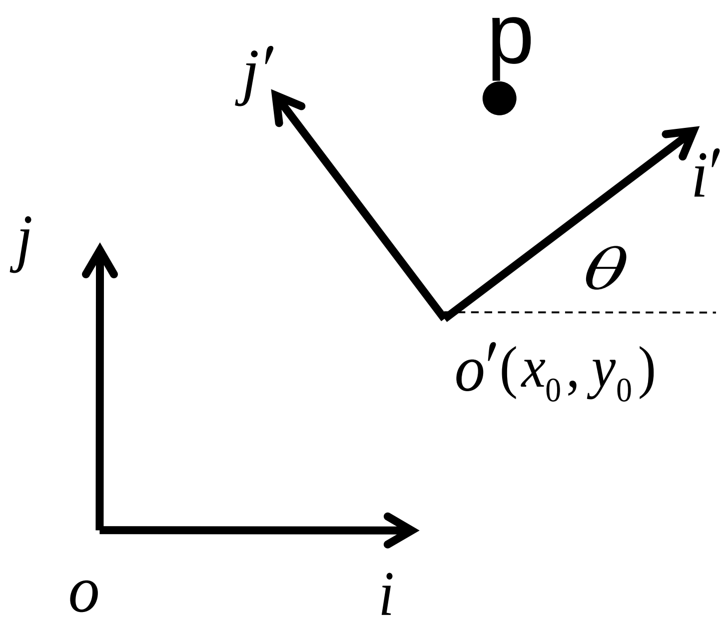
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \vec{i}^T\vec{j}' & x_0 \\ \vec{j}^T\vec{i}' \; \textbf{?} & \vec{j}^T\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

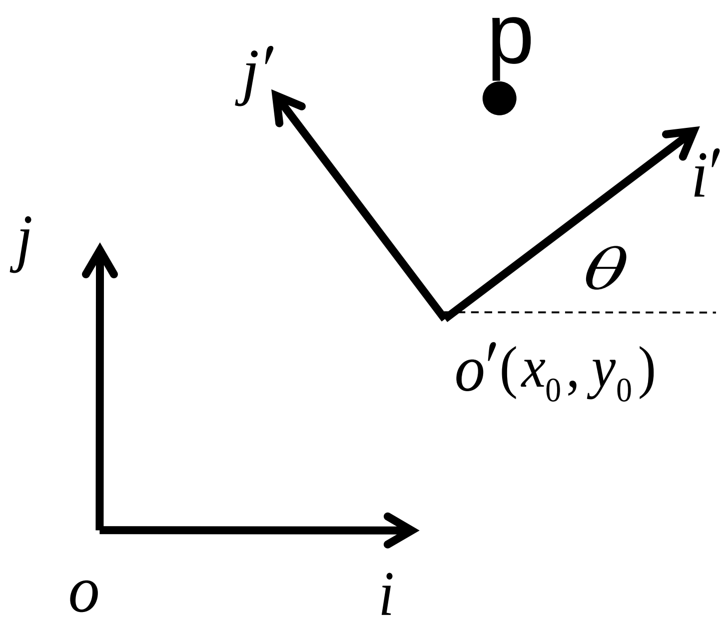
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \vec{i}^{\,T}\vec{j}' & x_0 \\ \sin\theta & \vec{j}^{\,T}\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

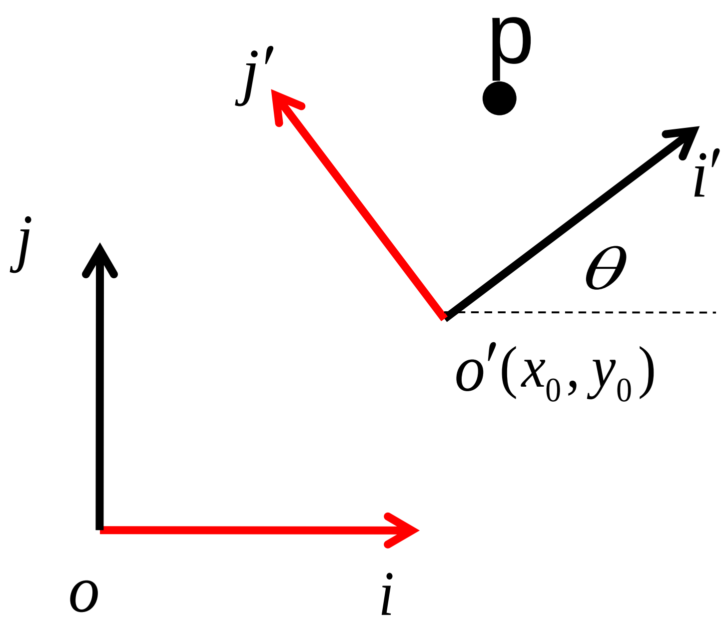
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \vec{i}^T\vec{j}' \; ? & x_0 \\ \sin\theta & \vec{j}^T\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

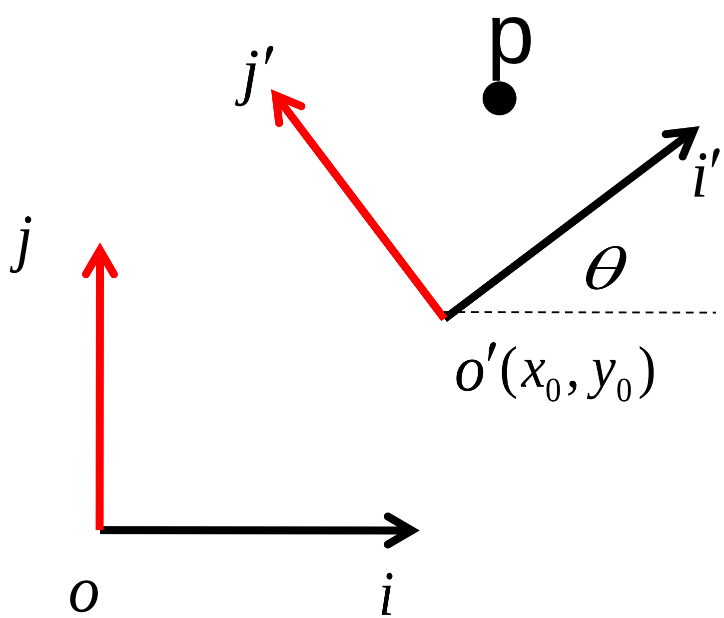
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \vec{j}^{T}\vec{j}' & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

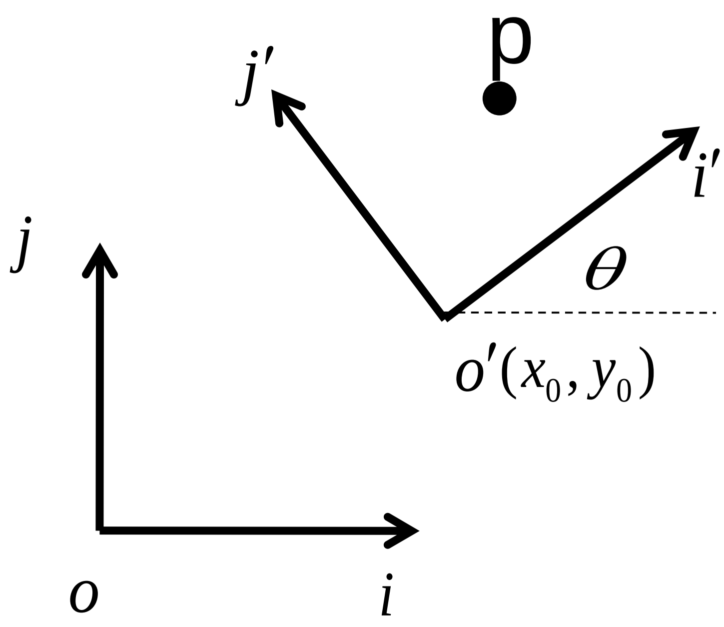
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \vec{j}\cdot\vec{j}' \; ? & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- 2D translation&rotation

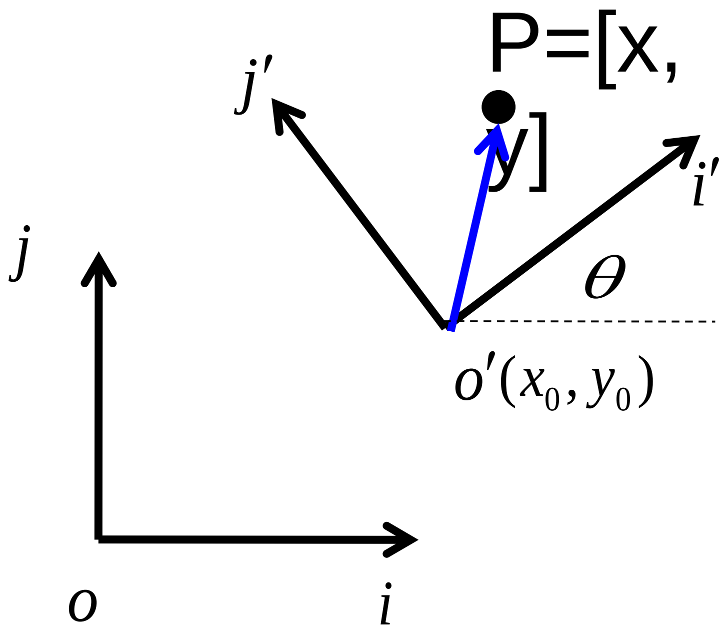


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \cos\theta & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- An alternative way to look at the problem

  - set up a transformation that superimposes the x'y' axes onto the xy axis

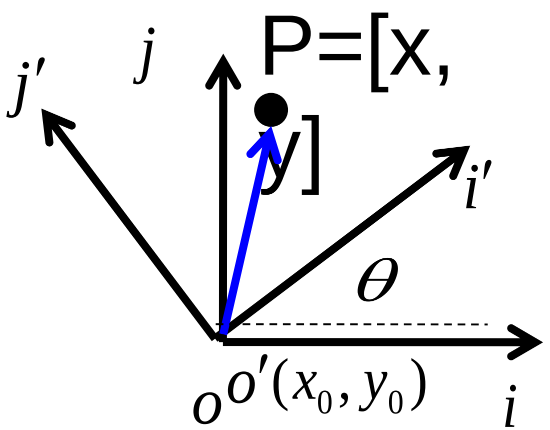


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- An alternative way to look at the problem

  - set up a transformation that superimposes the x'y' axes onto the xy axis



$$\begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

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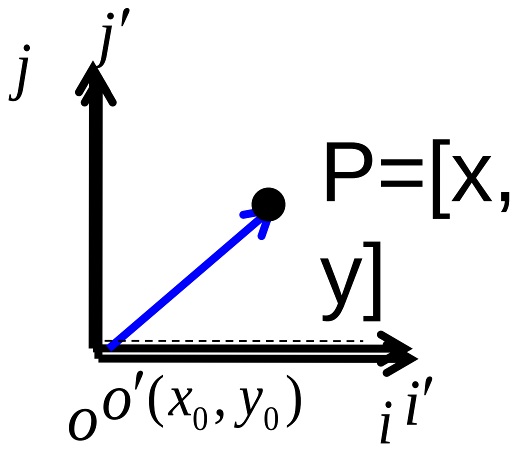
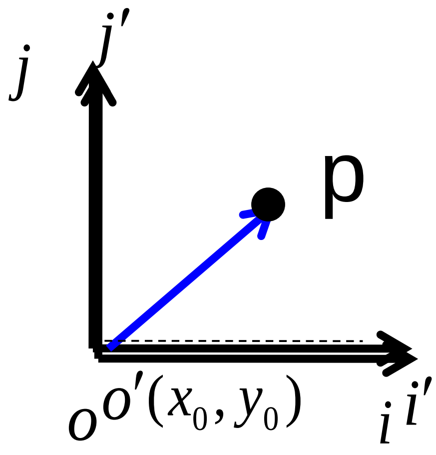


$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

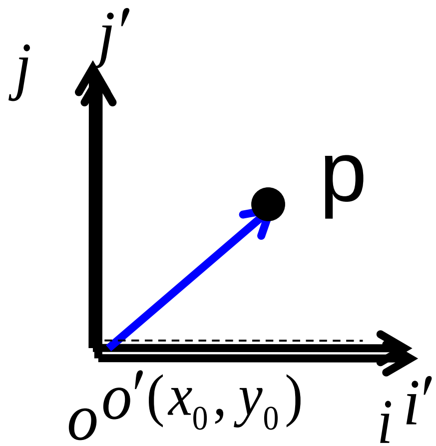
- An alternative way to look at the problem
- This transforms the point from (x,y) to (x',y')



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

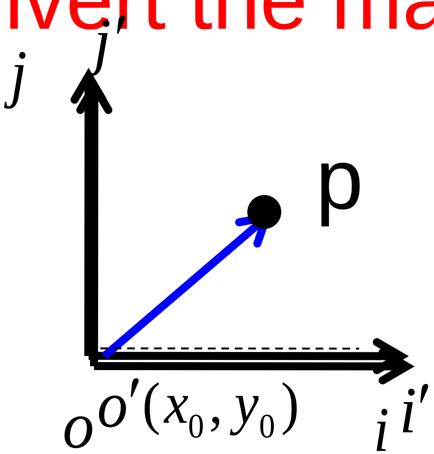
- An alternative way to look at the problem
- This transforms the point from (x,y) to (x’,y’)
- How to transform the point from (x’,y’) to (x,y)?



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- An alternative way to look at the problem
- This transforms the point from (x,y) to (x',y')
- How to transform the point from (x',y') to (x,y)? Invert the matrix!



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

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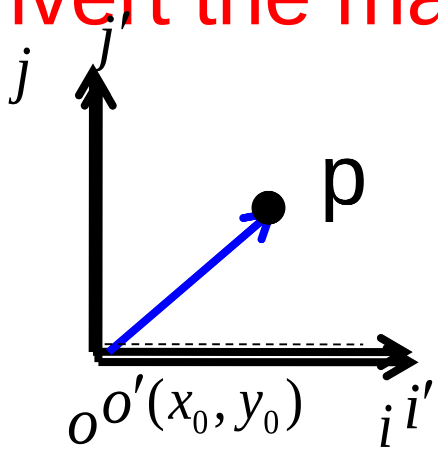

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

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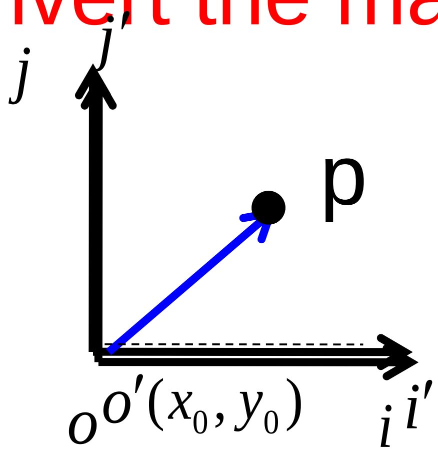

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

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- This transforms the point from (x,y) to (x',y')
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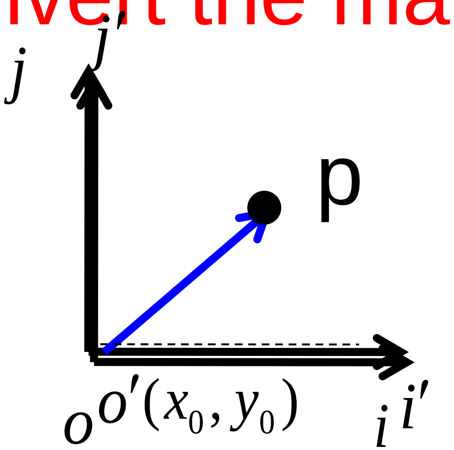

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

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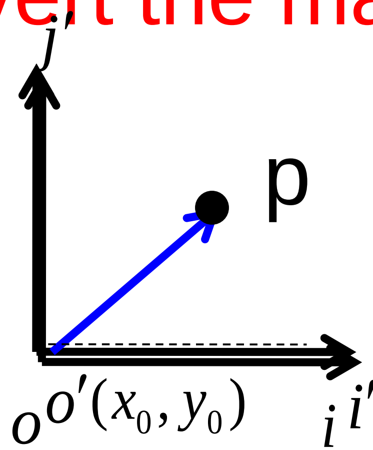
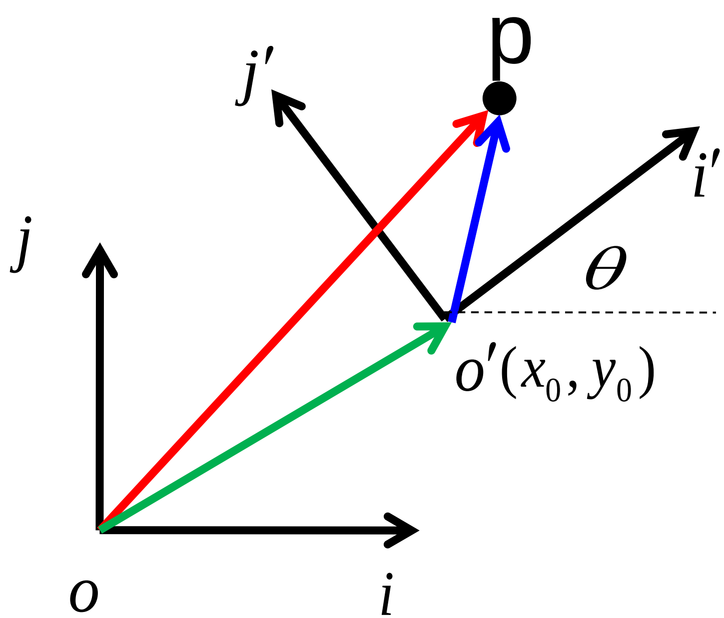


$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2D Coordinate Transformation

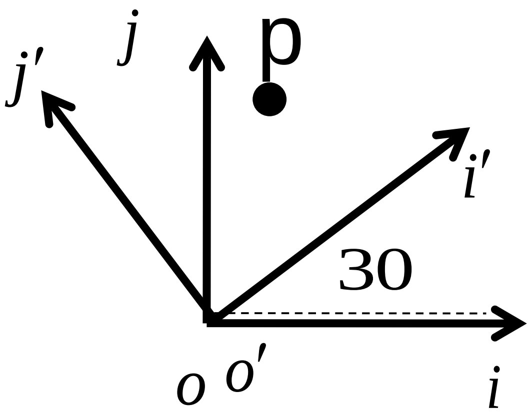
- Same results!



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \cos\theta & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

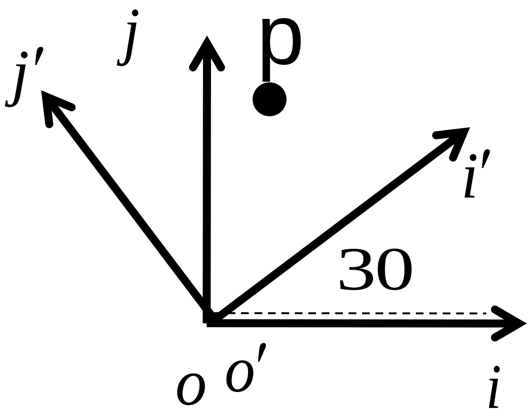
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

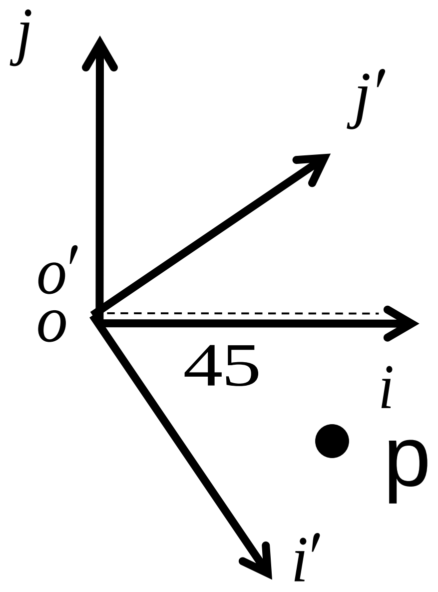
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

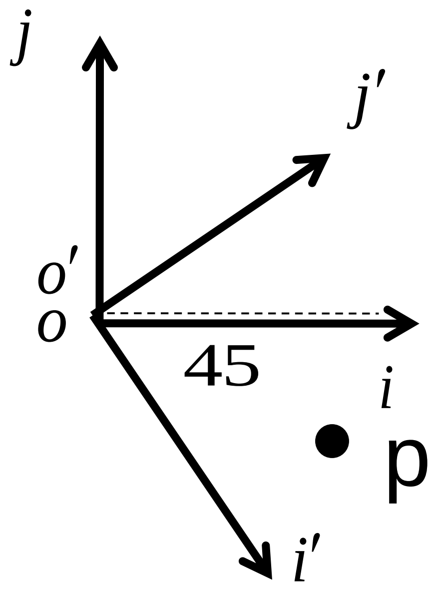
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

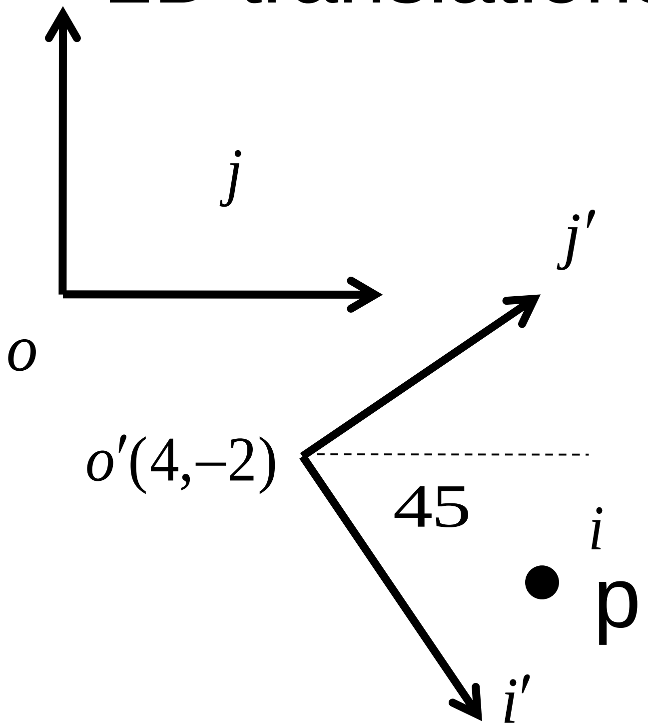
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

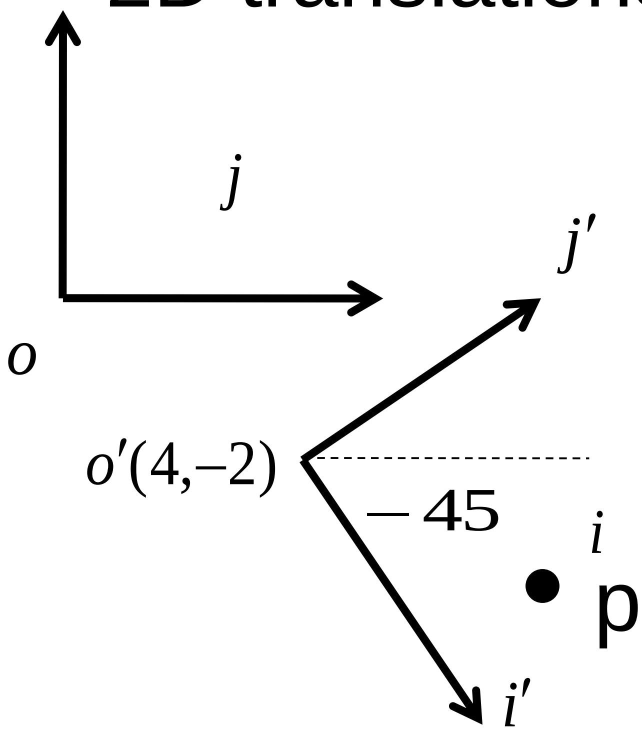
- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

2D Coordinate Transformation

- 2D translation&rotation



$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 45 & \sin 45 & 4 \\ -\sin 45 & \cos 45 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

Notation

- Coordinate systems are represented with brackets {B}, {0}, etc.
- Vectors
  – Lets Look at a Vector P Described in Frame A

Frame of Reference

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

  – Leading Subscript describes the frame in which the Vector is described or Referenced
  – Individual Elements of a vector are described by a trailing subscript

Matrix Notation

Homogenous Transformations Represent 3 Things

- Describe a Frame
- Map from one Frame to another
- Act as an Operator to move within a Frame

$$ {}^{A}_{B}T $$

Transforms Describe Frames

- Frames can be described by A Homogenous Transformation Matrices

$$ {}^A_BT = \left[\begin{array}{ccc|c} & {}^A_BR & & \begin{array}{c} {}^AP_{BorgX} \\ {}^AP_{BorgY} \\ {}^AP_{BorgZ} \end{array} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

- Description of Frame
  - Columns of ${}^A_BR$ are the Unit Vectors defining the directions of the principle axes of {B} in terms of {A}

$$ {}^A_BR = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} {}^B\hat{X}_A \\ {}^B\hat{Y}_A \\ {}^B\hat{Z}_A \end{bmatrix} $$

  - Rows of ${}^A_BR$ are the Unit Vectors defining the directions of the principle axes of {A} in terms of {B}
  - ${}^AP_{Borg}$ is the location of the origin of {B} in terms or {A}

Mapping Between Frames

- Maps vector from Frame {B} to Frame {A}
- ${}^{A}_{B}R$ will rotate a vector to project its components originally described in {B} in the {A} of Frame
- ${}^{A}P_{Borg}$ will translate the vector to adjust its origin from frame {B} to its new origin in {A}

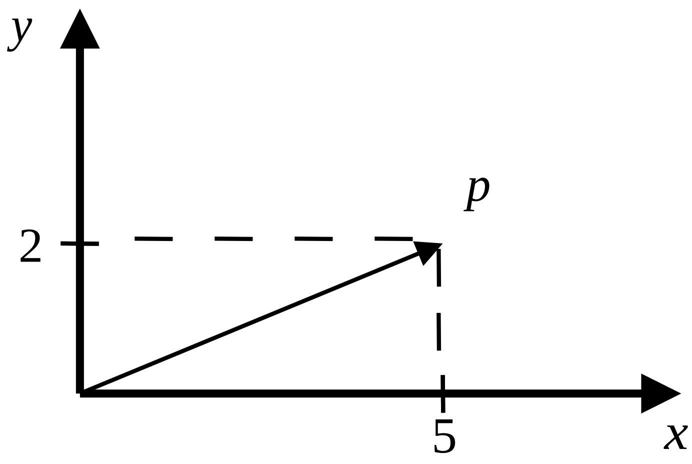
$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & \begin{matrix} {}^{A}P_{BorgX} \\ {}^{A}P_{BorgY} \\ {}^{A}P_{BorgZ} \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{B}P \mapsto {}^{A}P $$

Representing Position (2D)

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

("column" vector)



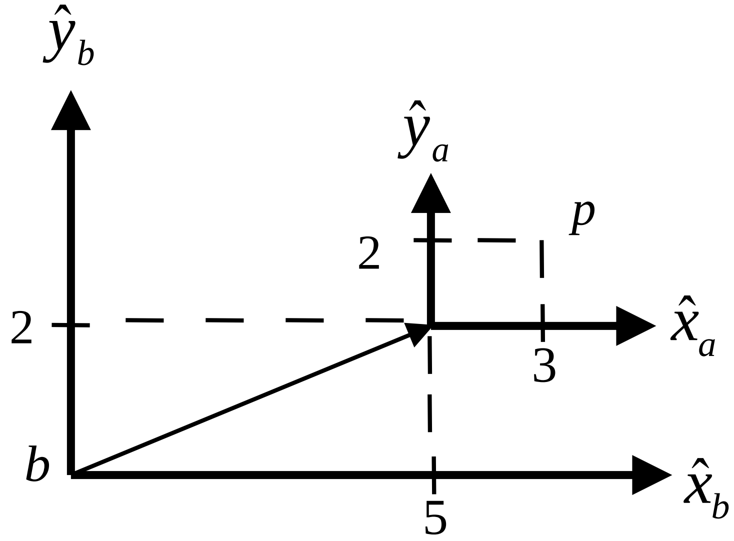
$$p = 5\hat{x} + 2\hat{y}$$

$\hat{x}$ ← A vector of length one pointing in the direction of the base frame *x* axis

$\hat{y}$ ← A vector of length one pointing in the direction of the base frame *y* axis

Representing Position: vectors

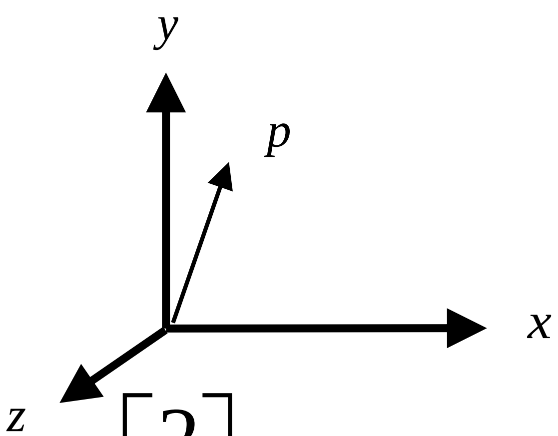
- The prefix superscript denotes the reference frame in which the vector should be understood

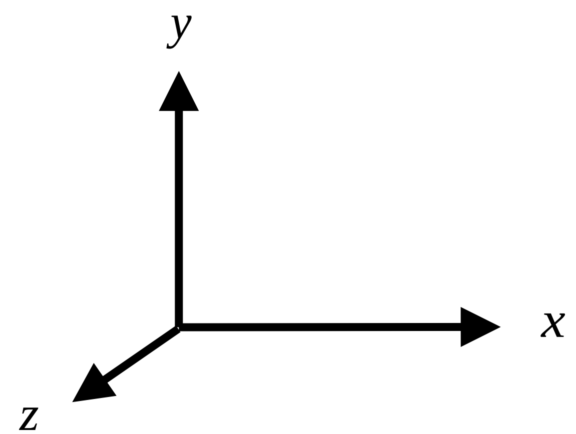


$$ {}^{b}p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad {}^{a}p = \begin{bmatrix} 3 \\ 2 \end{bmatrix} $$

Same point, two different reference frames

Representing Position: vectors (3D)





right-handed coordinate frame

$$p = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

$$p = 2\hat{x} + 5\hat{y} + 2\hat{z}$$

$\hat{x}$ ← A vector of length one pointing in the direction of the base frame *x* axis

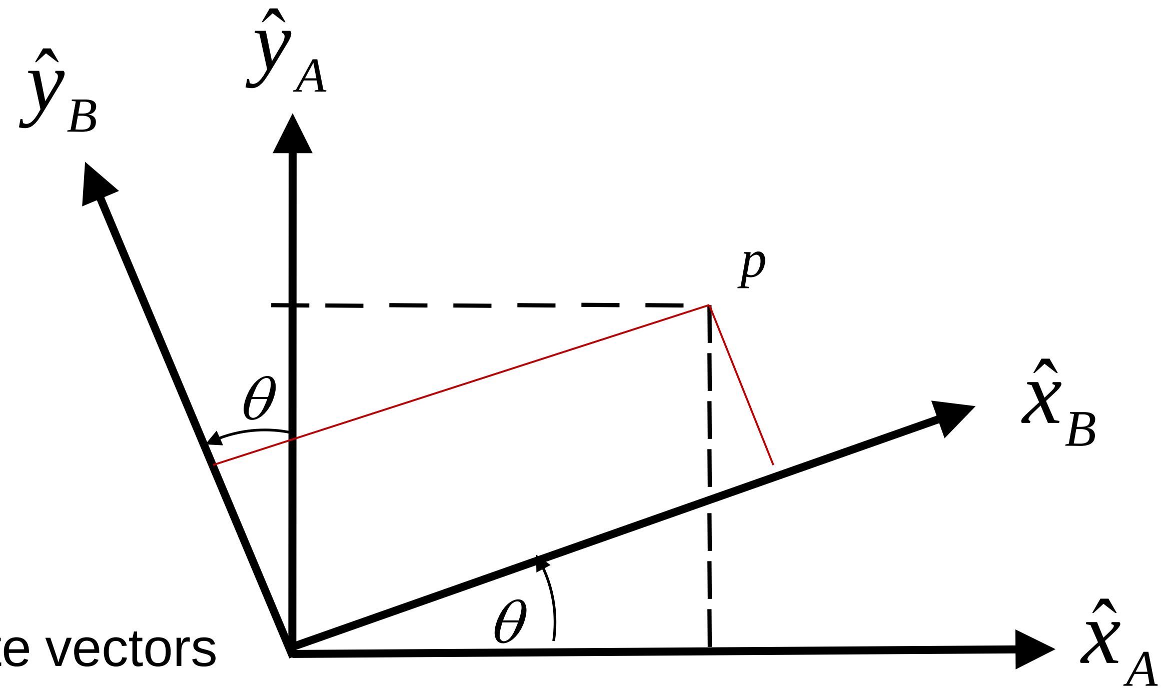
$\hat{y}$ ← A vector of length one pointing in the direction of the base frame *y* axis

$\hat{z}$ ← A vector of length one pointing in the direction of the base frame *z* axis

The Rotation Matrix

$$ {}^{A}R_{B} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} $$

$$ {}^{A}p = {}^{A}R_{B} \, {}^{B}p $$



${}^{A}R_{B}$:To specify the coordinate vectors for the fame B with respect to frame A

$$ {}^{B}R_{A} = {}^{A}R_{B}^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} $$

$$ {}^{B}p = {}^{B}R_{A} \, {}^{A}p $$

θ: The angle between $\hat{x}_A$ and $\hat{x}_B$

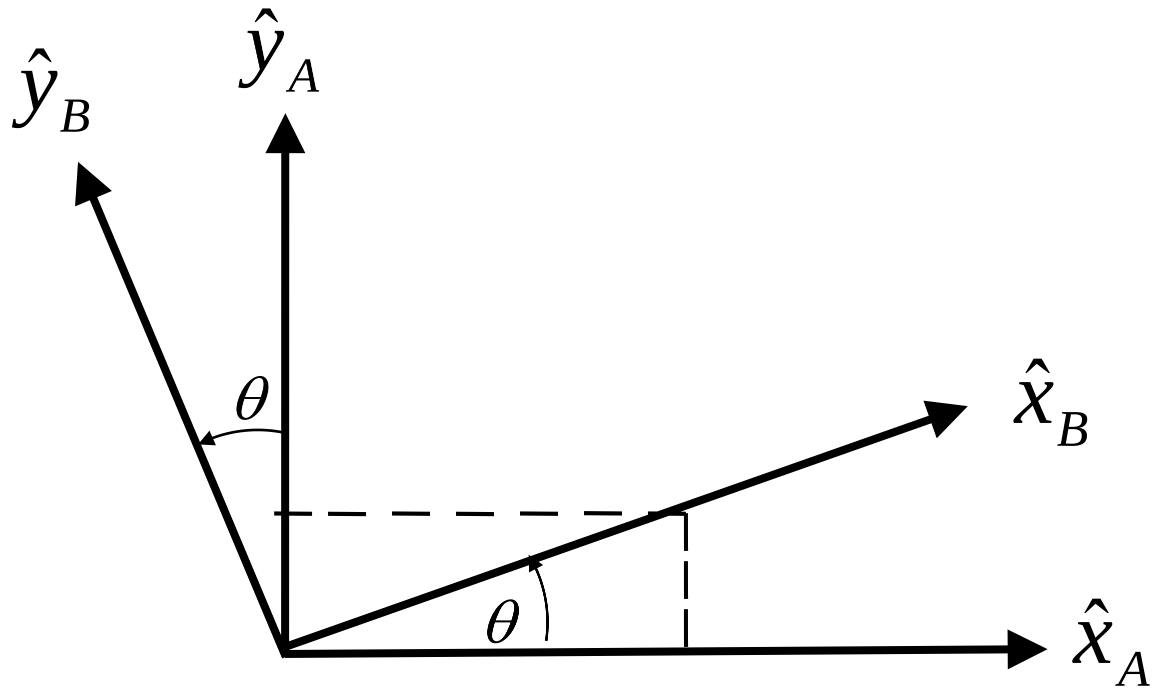
in anti clockwise direction

The Rotation Matrix

$$^{A}R_{B} = \left[{}^{A}x_{B2x1} \quad {}^{A}y_{B2x1} \right]$$

$$^{A}x_{B2x1} = \cos\theta \hat{x}_{A} + \sin\theta . \hat{y}_{A}$$

$$^{A}y_{B2x1} = -\sin\theta \hat{x}_{A} + \cos\theta \hat{y}_{A}$$



Useful formulas

$${}^{B}_{A}R = ({}^{A}_{B}R)^{-1} = ({}^{A}_{B}R)^{T}$$

$$R \cdot R^{-1} = I$$

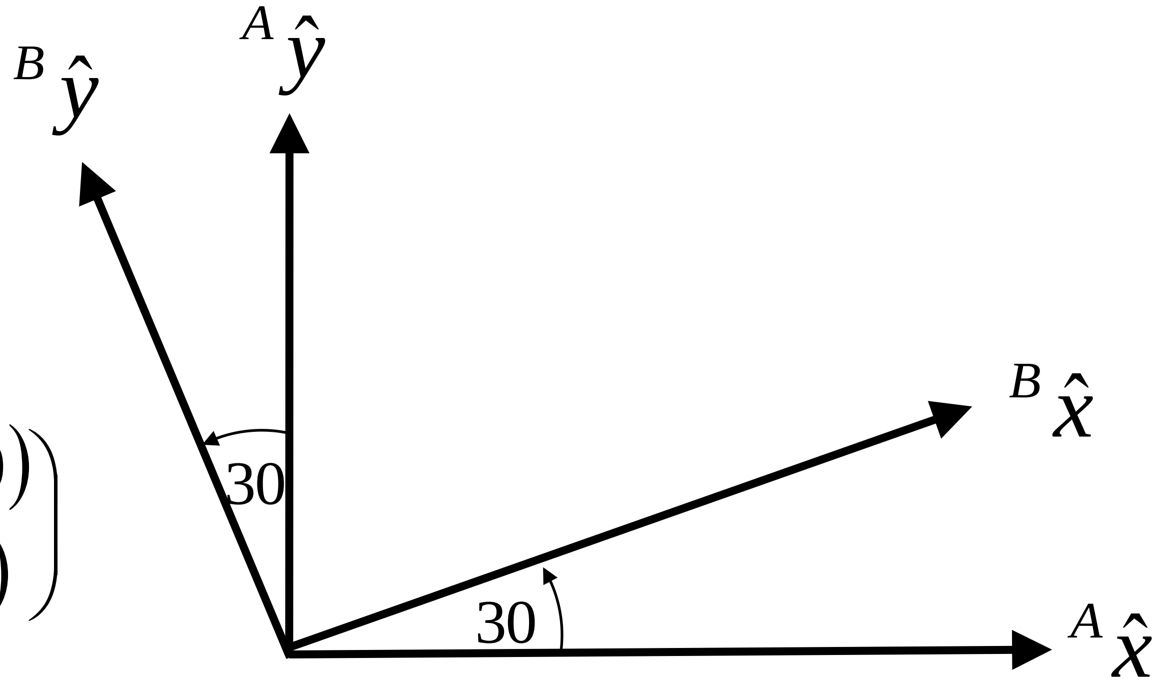
$${}^{B}_{A}R . {}^{A}_{B}R = I$$

$$Det(R) = 1$$

Example 1

$$^{B}p = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$find\ ^{A}p$$



$$^{A}R_{B} = \begin{pmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{pmatrix}$$

$$^{A}R_{B} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$^{A}p = {}^{A}R_{B}\ {}^{B}p$$

$$^{A}P = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 3.6603 \\ 13.6603 \end{pmatrix}$$

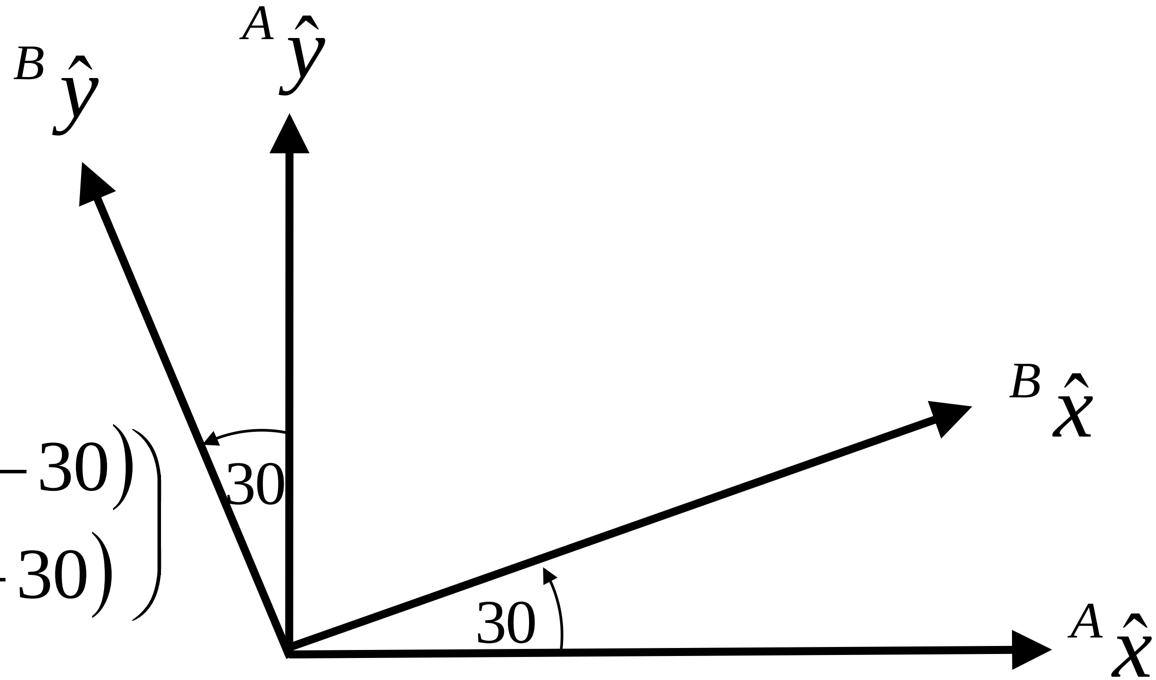
Example 1

$$^{A}p = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$find\ ^{B}p$$

$$^{B}R_{A} = \begin{pmatrix} \cos(-30) & -\sin(-30) \\ \sin(-30) & \cos(-30) \end{pmatrix}$$

$$^{B}R_{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$



$$^{B}p = {}^{B}R_{A}\, {}^{A}p$$

$$^{B}P = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 13.6603 \\ 3.6603 \end{pmatrix}$$

Example 1

Another Solution

$$^{A}R_{B} = \begin{pmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{pmatrix}$$

$$^{B}R_{A} = {^{A}R_{B}}^{-1} \qquad OR \qquad ^{B}R_{A} = {^{A}R_{B}}^{T}$$

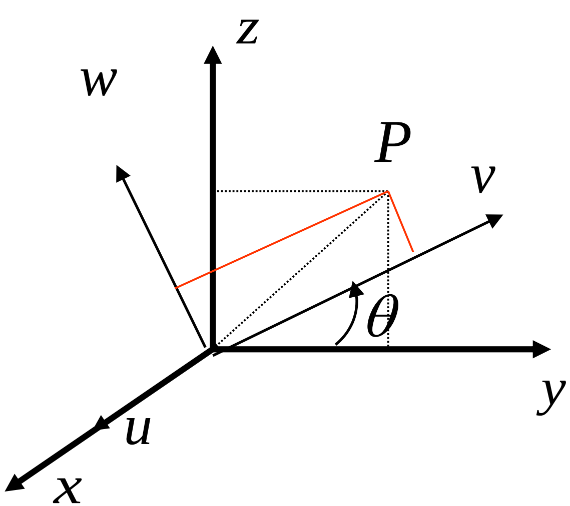
$$^{B}R_{A} = \begin{pmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{pmatrix}$$

Basic Rotation Matrix

– Rotation about x-axis with

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = R(x,\theta) \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$



$$p_x = p_u$$

$$p_y = p_v \cos\theta - p_w \sin\theta$$

$$p_z = p_v \sin\theta + p_w \cos\theta$$

Basic Rotation Matrices

– Rotation about x-axis with θ

$$R_{x,\theta} = Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

– Rotation about y-axis with θ

$$R_{y,\theta} = Rot(y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix}$$

– Rotation about z-axis with θ

$$R_{z,\theta} = Rot(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{xyz} = RP_{uvw}$$

Example 2

- A point $p_{uvw} = (4,3,2)$ is attached to a rotating frame, the frame rotates 60 degree about the OZ axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation.

$$p_{xyz} = Rot(z,60)\, p_{uvw}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

Example 3

- A point $a_{xyz} = (4,3,2)$ is the coordinate w.r.t. the reference coordinate system, find the corresponding point a_{uvw} w.r.t. the rotated OUVW coordinate system if it has been rotated 60 degree about OZ axis.

$$p_{uvw} = Rot(z,60)^T \, p_{xyz}$$

$$OR: p_{uvw} = Rot(z,60)^{-1} \, p_{xyz}$$

$$OR: p_{uvw} = Rot(z,-60) \, p_{xyz}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.598 \\ -1.964 \\ 2 \end{bmatrix}$$

Composite Rotation Matrix

- A sequence of finite rotations
  - matrix multiplications do not commute
  - rules:
    - if rotating coordinate OUVW is rotating about principal axis of OXYZ frame, then ***Pre-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix [rotation about fixed frame]
    - if rotating coordinate OUVW is rotating about its own principal axes, then ***post-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix [rotation about current frame]

Rotation with respect to Current Frame

$$^{A}P = {}^{A}R_{B} P^{B}$$

$$^{B}P = {}^{B}R_{C} P^{C}$$

$$^{C}P = {}^{C}R_{D} P^{D}$$

$$^{A}P = {}^{A}R_{D} P^{D} = {}^{A}R_{B} {}^{B}R_{C} {}^{C}R_{D} P^{D}$$

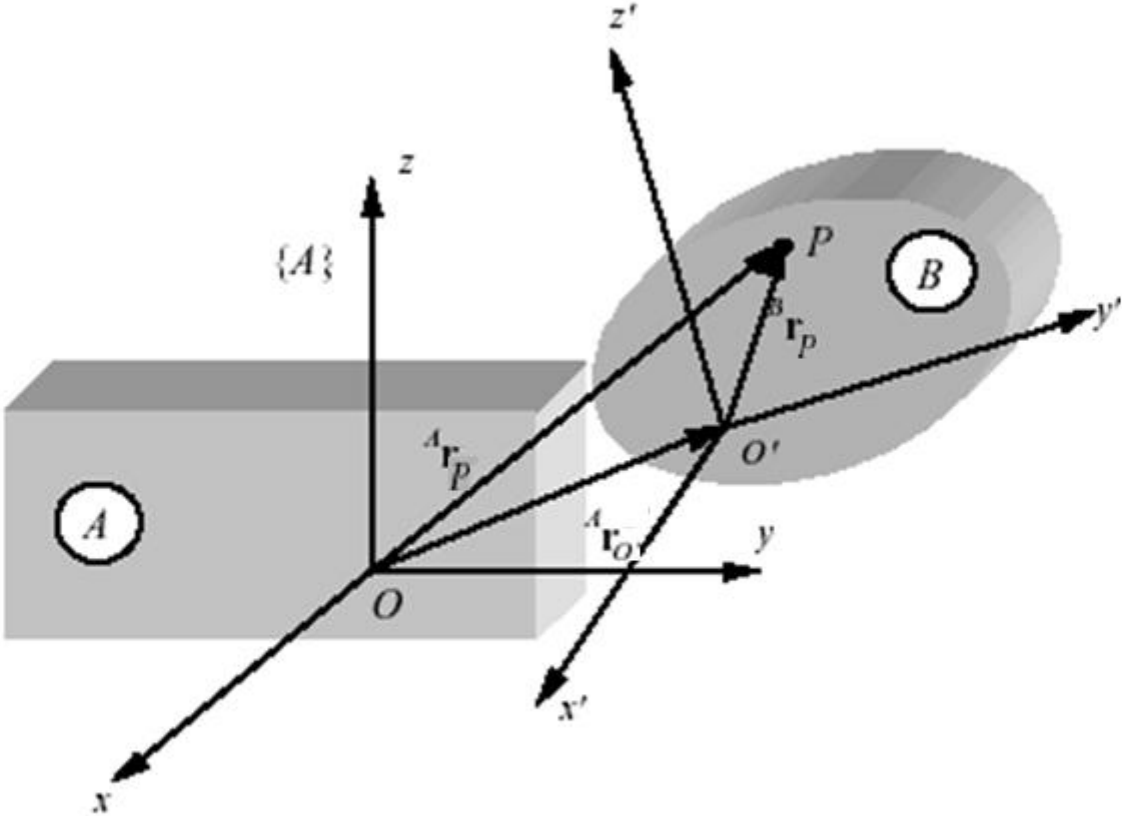
$$^{A}R_{D} = {}^{A}R_{B} {}^{B}R_{D}$$

$$^{A}R_{D} = {}^{A}R_{B} {}^{B}R_{C} {}^{C}R_{D}$$

Coordinate Transformations

- position vector of P in $\{B\}$ is transformed to position vector of P in $\{A\}$

- description of frame$\{B\}$ as seen from an observer in $\{A\}$



$$ {}^{A}\mathbf{r}_{P} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{r}_{P} + {}^{A}\mathbf{r}_{O'} $$

Rotation of $\{B\}$ with respect to $\{A\}$

Translation of the origin of $\{B\}$ with respect to origin of $\{A\}$

Homogeneous Representation

- Coordinate transformation from $\{B\}$ to $\{A\}$

$$^{A}r_{P} = {}^{A}R_{B}\,{}^{B}r_{p} + {}^{A}r_{o'}$$

Can be written as $$^{A}P = {}^{A}H_{B}\,{}^{B}P$$

Rotation matrix (3*3)

$$^{A}H_{B} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & p_{B}^{A} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]$$

Position vector (3*1)

Homogeneous Representation

$$^{A}P = {}^{A}H_{B}\,{}^{B}P$$

$$^{A}P = \begin{bmatrix} {}^{A}x_{p} \\ {}^{A}y_{p} \\ {}^{A}z_{p} \\ 1 \end{bmatrix} \qquad {}^{B}P = \begin{bmatrix} {}^{B}x_{p} \\ {}^{B}y_{p} \\ {}^{B}z_{p} \\ 1 \end{bmatrix}$$

Rotation matrix (3*3)

$$^{A}H_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & p_{B}^{A} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]$$

Position vector of the origin of frame B wrt frame A (3*1)

Homogeneous Transformation

- Special cases

1. Translation

$$^{A}H_{B} = \begin{bmatrix} I_{3\times3} & p_{B}^{A} \\ 0_{1\times3} & 1 \end{bmatrix}$$

2. Rotation

$$^{A}H_{B} = \begin{bmatrix} {}^{A}R_{B} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}$$

Homogeneous Transformation

- Composite Homogeneous Transformation Matrix
- Rules:
  - Transformation (rotation/translation) w.r.t fixed frame, using pre-multiplication
  - Transformation (rotation/translation) w.r.t current frame, using post-multiplication

Example 5

- Find the homogeneous transformation matrix (H) for the following operations:

$\text{Rotation } \alpha \text{ about OX axis}$

$\text{Translation of a along OX axis}$

$\text{Translation of d along OZ axis}$

$\text{Rotation of } \theta \text{ about OZ axis}$

$$H = Rot_{z,\theta} Trans_{z,d} Trans_{x,a} Rot_{x,\alpha}$$

$$Answer: = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remember those double-angle formulas…

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$

Review of matrix transpose

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \longrightarrow p^T = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

Important property: $\mathbf{A}^T\mathbf{B}^T = (\mathbf{BA})^T$

and matrix multiplication…

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Can represent dot product as a matrix multiply:

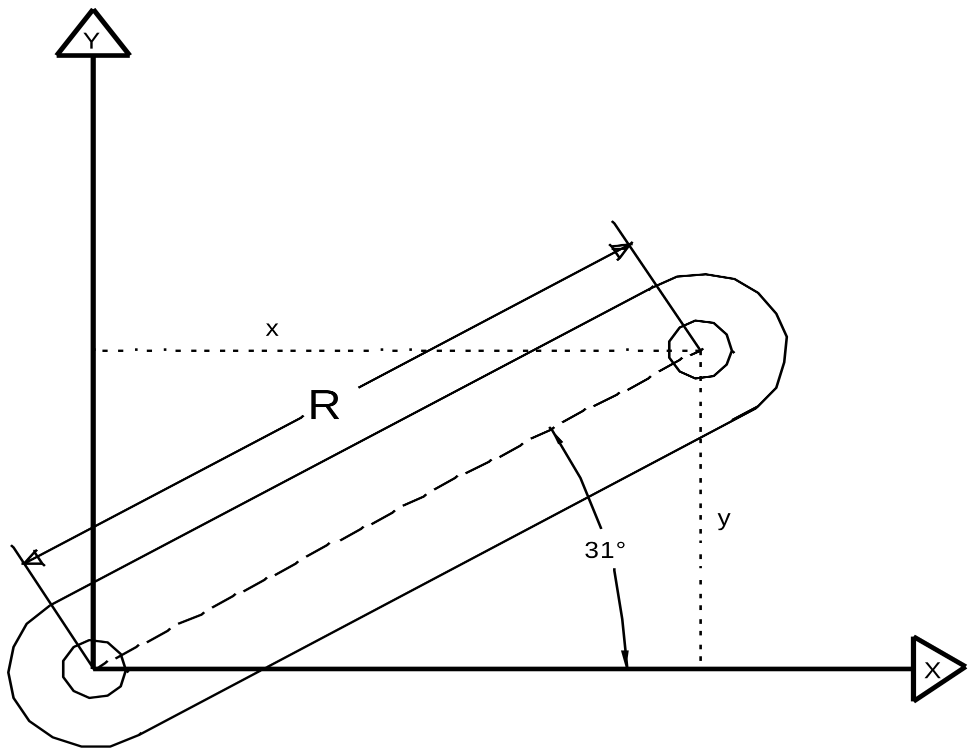
$$a \cdot b = a_x b_x + a_y b_y = \begin{bmatrix} a_x & a_y \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a^T b$$

Kinematics

- Kinematics is the science of motion without regard to forces.
- We study the position, velocity, acceleration, jerk etc of objects
- Concerned with the location of Objects
- We will define coordinate systems or frames to define there location

Forward Kinematics

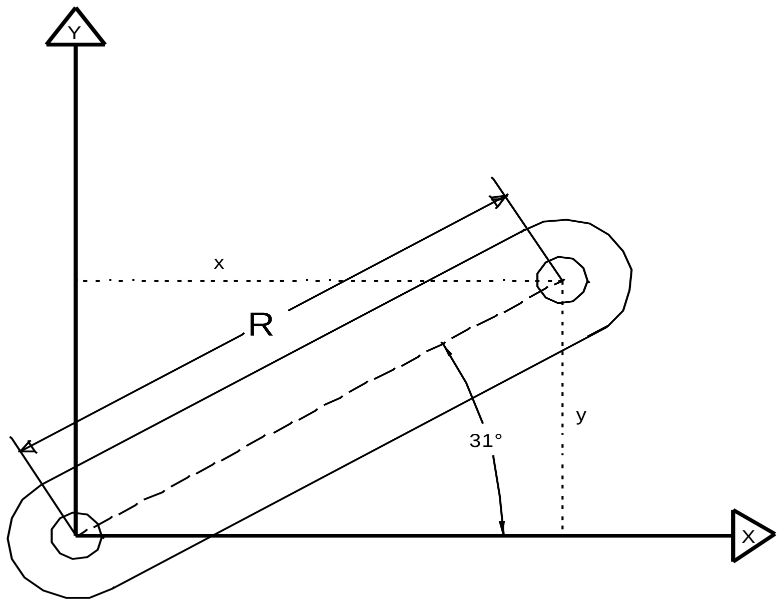
- Lets look at a simple link (1DOF)



Forward Kinematics

- Want to know the end point of link in terms of X and Y
- We have R and Theta
- From Geometry we can determine the position:

$$x = R \cdot \cos(\theta)$$
$$y = R \cdot \sin(\theta)$$

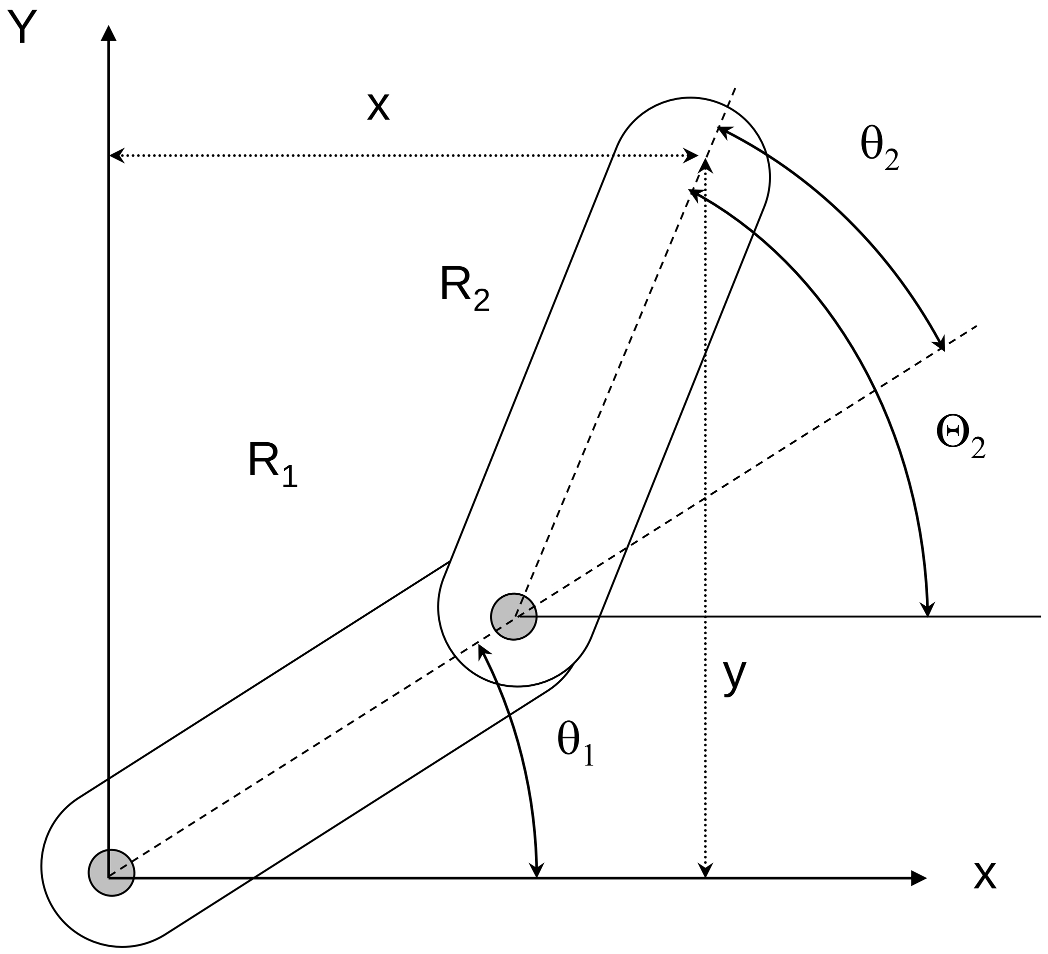


- and then the velocity

$$\dot{x} = -R \cdot \sin(\theta) \cdot \dot{\theta}$$

$$\dot{y} = R \cdot \cos(\theta) \cdot \dot{\theta}$$

Two Link Example



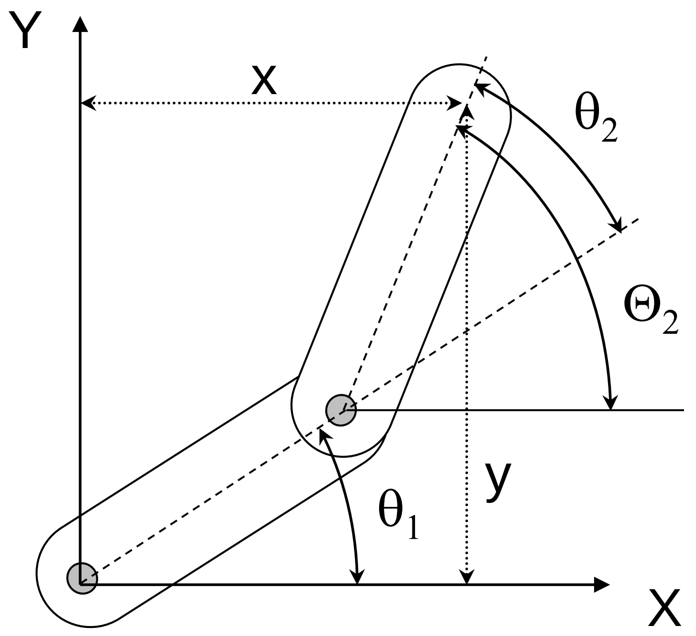
Two Link Example

$\Theta_1 = \theta_1 \qquad \Theta_2 = \theta_1 + \theta_2$

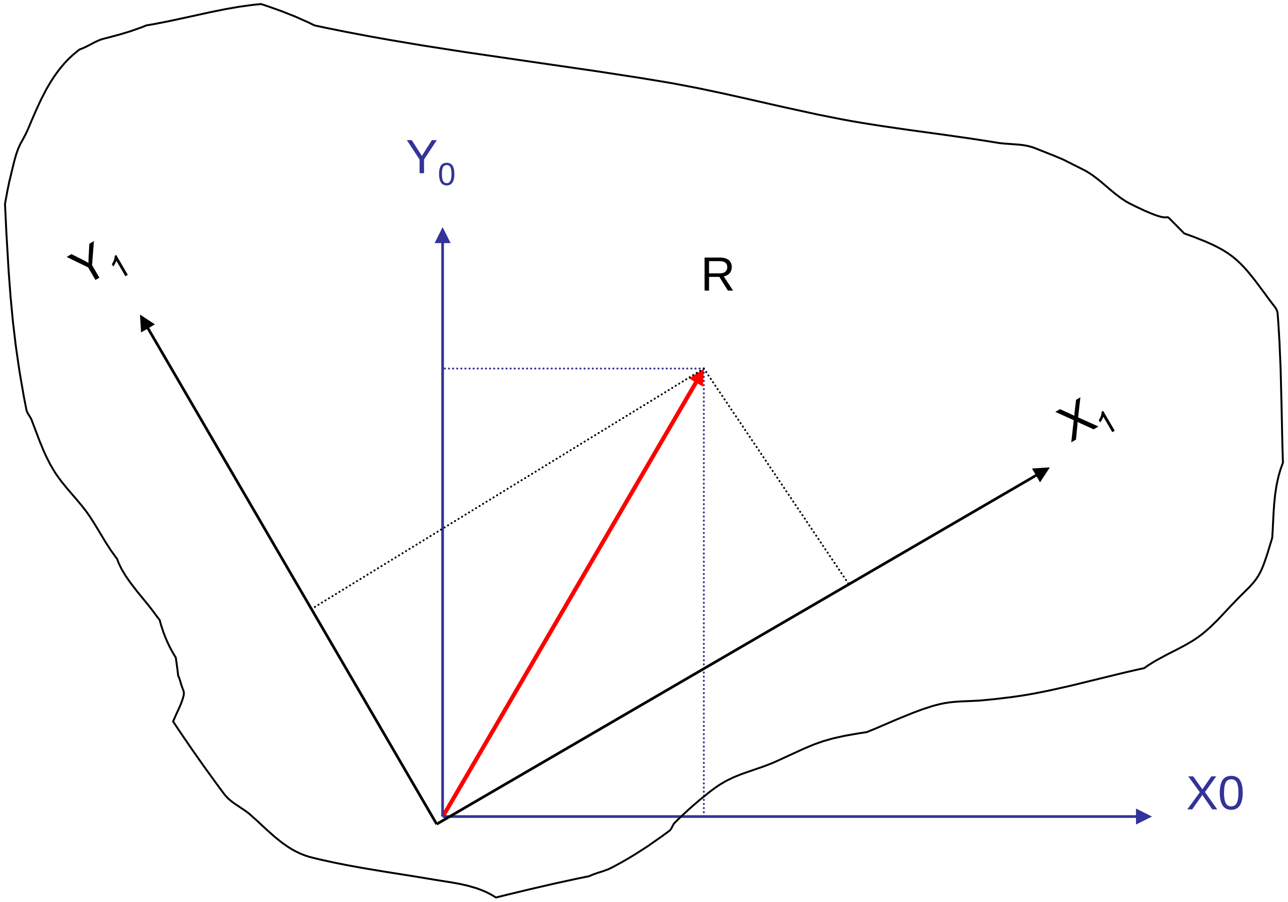
$$x = R_1 \cdot \cos(\Theta_1) + R_2 \cdot \cos(\Theta_2)$$

$$y = R_1 \cdot \sin(\Theta_2) + R_2 \cdot \sin(\Theta_2)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\Theta_1) & \cos(\Theta_2) \\ \sin(\Theta_1) & \sin(\Theta_2) \end{bmatrix} \cdot \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$



Coordinate Frames



Coordinate Frames

- Now lets put it in matrix form

$$x_0 = x_1 \cdot \cos(\theta_1) - y_1 \cdot \sin(\theta_1)$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad y_0 = x_1 \cdot \sin(\theta_1) + y_1 \cdot \cos(\theta_1)$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = [T] \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

- So what if we want to map the other way?

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [T]^{-1} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

- What is the inverse of T? Why?

Coordinate Frames

- If we look at the columns and rows of T we see that they have a norm of one.
- Also if we take the dot product of the columns we find they are orthogonal to each other.
- So T is an ortho-normal Matrices. Thus its transpose is its inverse.
- $$[T] = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \qquad [T]^{-1} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) \\ -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$
  - This was a simple 2DOF example what about 3.
  - If we project a Z axes out the plane generated by the X and Y axes, then a rotation around the Z axes will not affect the Z position of the vector R.

Coordinate Frames

- The 3D transformation axes about the Z axes is:

$$[T]_z = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \qquad [T]_Y = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

- Similarly for rotations around the X or Y axes we get

Homogeneous Transformation Matrices

- 3x3 Rotation Matrix

$$T_1 = \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- 3x1 Displacement Vector

$$R_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

- For a displacement and a rotation

$$[R_0] = [T] \cdot [R_1] + [\Delta R]$$

4x4 Homogeneous Matrix

- If we want to perform a rotation and a translation with one operation

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = [T] \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \qquad T = \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We can create a homogeneous Transformation Matrix

$$T_H = \left[\begin{array}{ccc|c} C1 & -S1 & 0 & \Delta x_0 \\ S1 & C1 & 0 & \Delta y_0 \\ 0 & 0 & 1 & \Delta z_0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \end{bmatrix} = [T_H] \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{bmatrix}$$

Homogeneous Transformation Matrices

- What does a pure translation look like

$$T_T = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- What does a pure rotation look like

$$T_R = \begin{bmatrix} C1 & -S1 & 0 & 0 \\ S1 & C1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$