CSE444: Introduction to ROBOTICS

Forward Kinematics

SPRING 2021

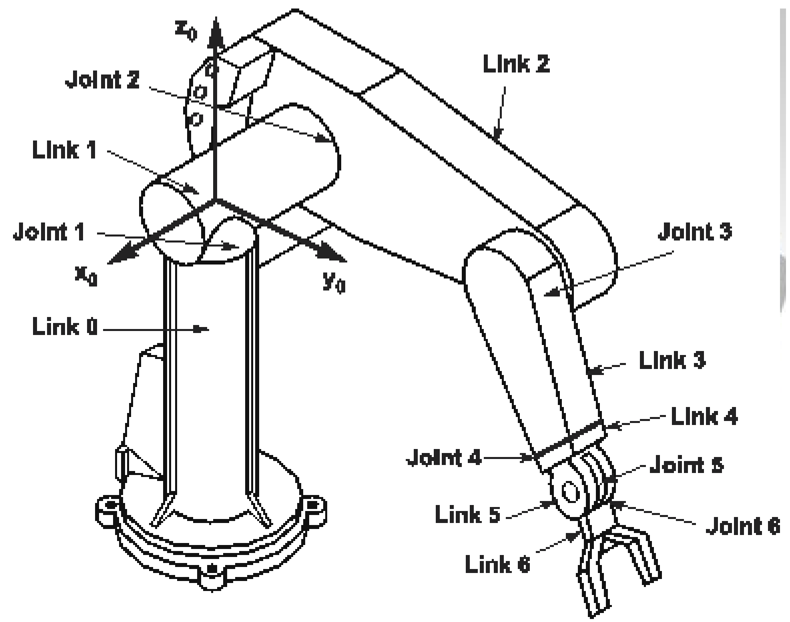
Kinematic

- Forward (direct) Kinematics
- Given: The values of the joint variables.
- Required: The position and the orientation of the end effector.

- Inverse Kinematics
- Given : The position and the orientation of the end effector.
- Required : The values of the joint variables.

Why DH notation

- Find the homogeneous transformation ***H*** relating the tool frame to the fixed base frame



Why DH notation

- A very simple way of modeling robot links and joints that can be used for any kind of robot configuration.

- This technique has became the standard way of representing robots and modeling their motions.

DH Techniques

1. Assign a reference frame to each joint (x-axis and z-axis). The D-H representation does not use the y-axis at all.
2. Each homogeneous transformation A_i is represented as a product of four basic transformations

DH Techniques

- Matrix A_i representing the four movements is found by: four movements

1. Rotation of θ about current Z axis
2. Translation of d along current Z axis
3. Translation of a along current X axis
4. Rotation of α about current X axis

$$A_i = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$R_{x,\theta} = Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \quad R_{z,\theta} = Rot(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} \mathrm{c}\theta_\mathrm{i} & -\mathrm{c}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{s}\alpha_\mathrm{i}\mathrm{s}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{c}\theta_\mathrm{i} \\ \mathrm{s}\theta_\mathrm{i} & \mathrm{c}\theta_\mathrm{i}\mathrm{c}\alpha_\mathrm{i} & -\mathrm{s}\alpha_\mathrm{i}\mathrm{c}\theta_\mathrm{i} & \mathrm{a}_\mathrm{i}\mathrm{s}\theta_\mathrm{i} \\ 0 & \mathrm{s}\alpha_\mathrm{i} & \mathrm{c}\alpha_\mathrm{i} & \mathrm{d}_\mathrm{i} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

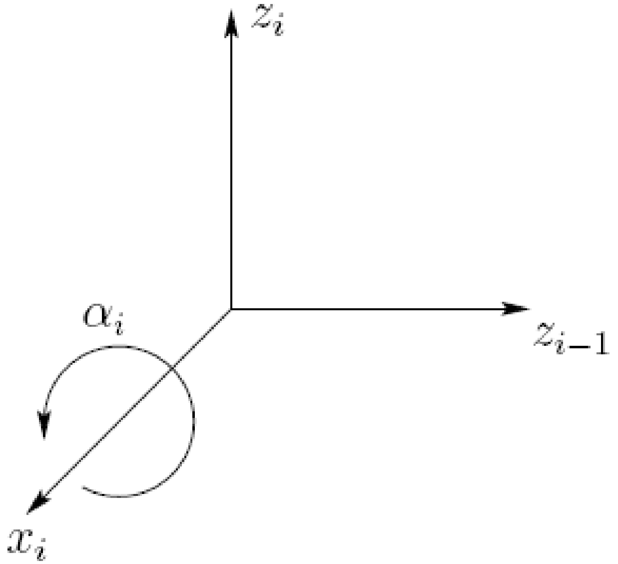
DH Techniques

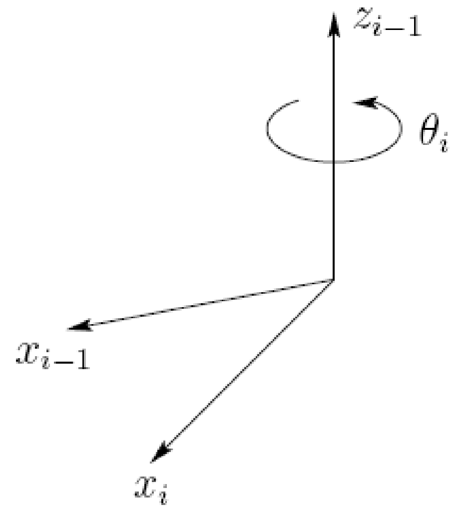
- The link and joint parameters :

- ***Link length*** **a_i** *: the offset distance between the Z_{i-1} and Z_i axes along the X_i axis.*

- ***Link offset d***$_i$ *the distance from the origin of frame i−1 to the X_i axis* along the *Z_{i-1} axis.*

DH Techniques





•***Link twist*** α_i *:the angle from the Z_{i-1} axis to the Z_i axis about the X_i axis.* The positive sense for α is determined from z_{i-1} and z_i by the right-hand rule.

•***Joint angle*** θ_i *the angle between the X_{i-1} and X_i axes about the Z_{i-1} axis.*

DH Techniques

- The four parameters:

a_i: link length, α_i: Link twist , d_i : Link offset and θ_i : joint angle.

- The matrix A_i is a function of only a single variable q_i , it turns out that three of the above four quantities are constant for a given link, while the fourth parameter is the joint variable.

DH Techniques

- With the i[th] joint, a joint variable is q_i associated where

$$q_i = \begin{cases} \theta_i & : \quad \text{joint i revolute} \\ d_i & : \quad \text{joint i prismatic} \end{cases}$$

All joints are represented by the z-axis.

- If the joint is revolute, the z-axis is in the direction of rotation as followed by the right hand rule.
- If the joint is prismatic, the z-axis for the joint is along the direction of the liner movement.

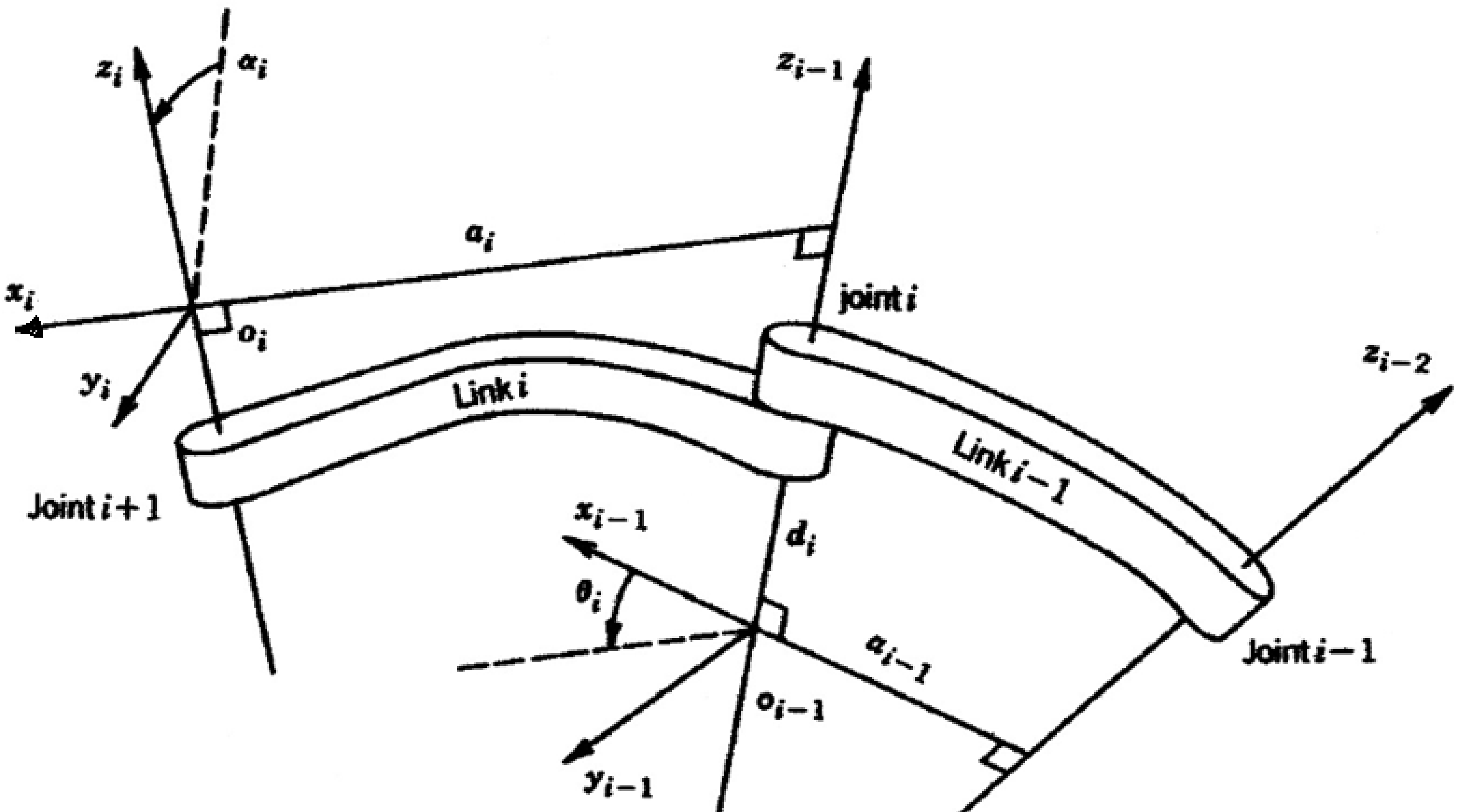
DH Techniques

3. Combine all transformations, from the first joint (base) to the next until we get to the last joint, to get the robot's *total transformation matrix*.

$$T_n^0 = A_1 . A_2 .... A_n$$

4. From T_n^0, the position and orientation of the tool frame are calculated.

DH Techniques



DH Techniques

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin o_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate o_i at this intersection. If z_i and z_{i-1} are parallel, locate o_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through o_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

DH Techniques

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin o_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i

DH Techniques

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} . θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

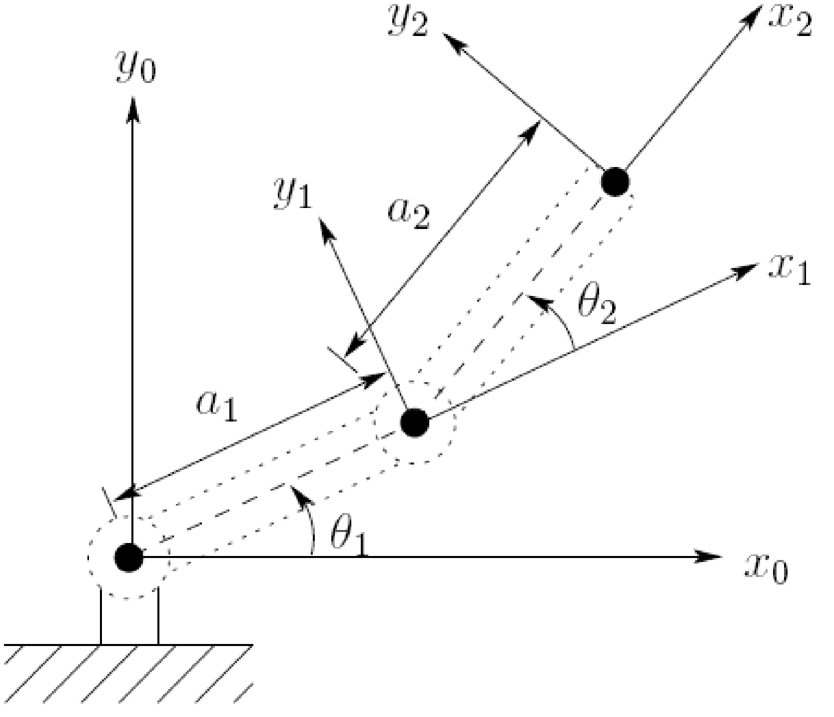
Example I
The two links arm

- Base frame O_0

•All Z 's are normal to the page

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable



Example I
The two links arm

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

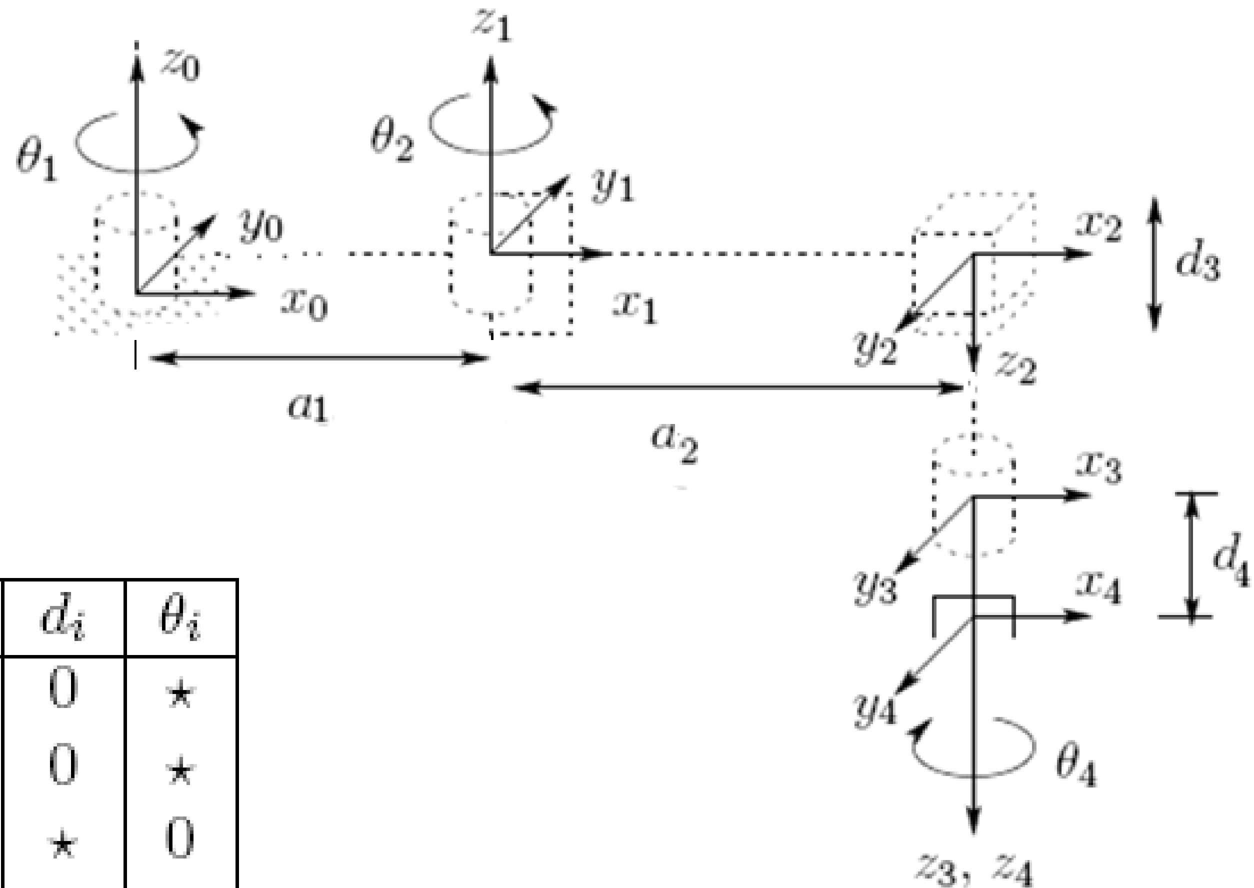
Where $(\theta_1 + \theta_2)$ denoted by θ_{12} and $\cos(\theta_1 + \theta_2)$ by c_{12}

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^0 = A_1.$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	⋆
2	a_2	180	0	⋆
3	0	0	⋆	0
4	0	0	d_4	⋆

* joint variable

Example 2

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

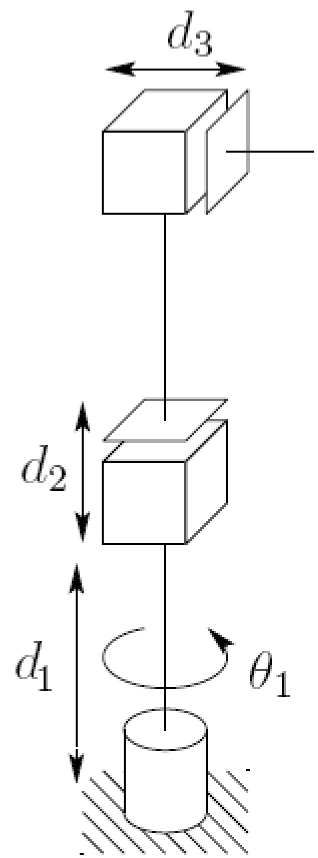
$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

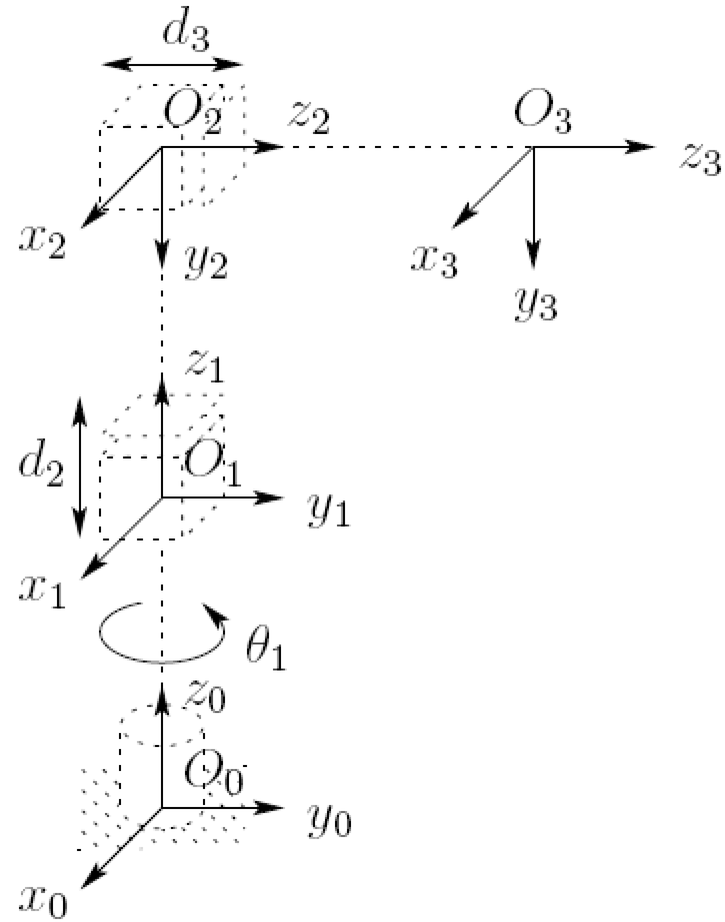
Example 3
The three links cylindrical



Example 3
The three links cylindrical

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable



Example 3
The three links cylindrical

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

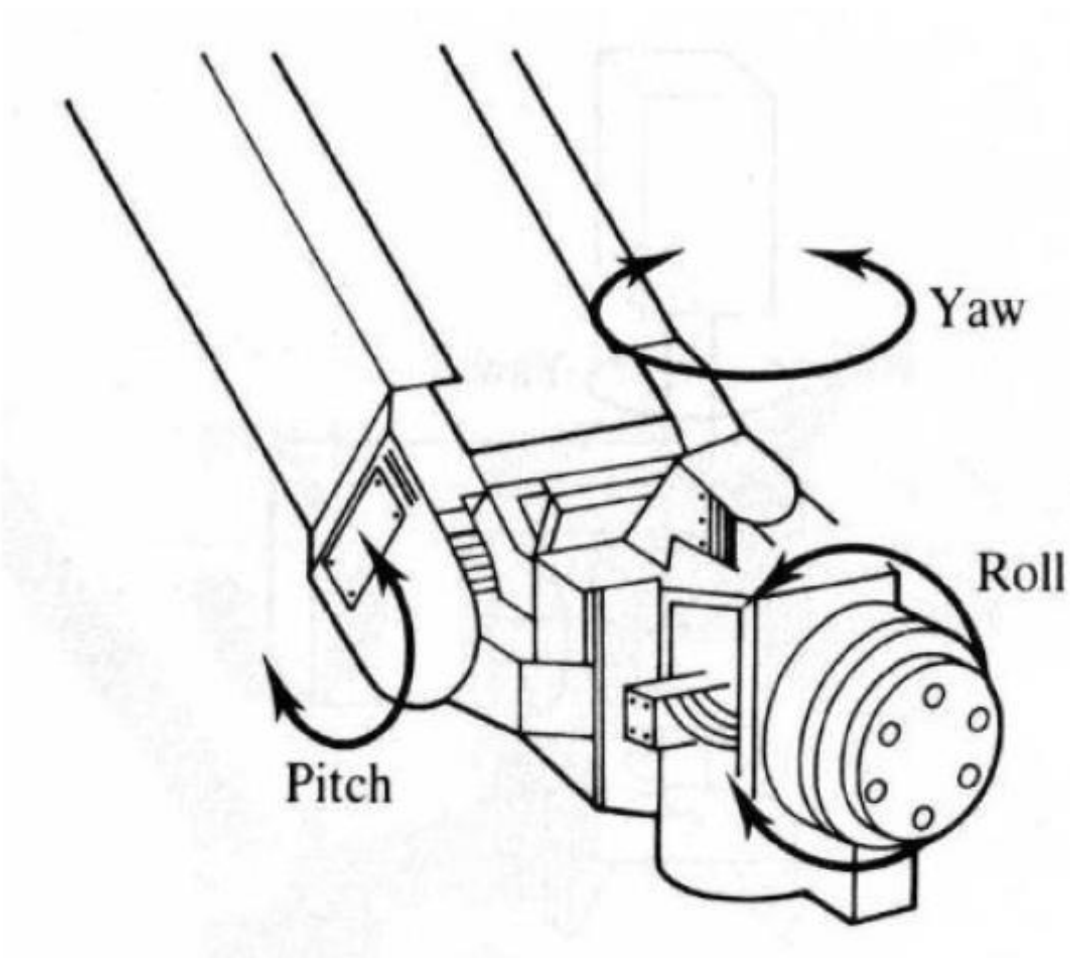
$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3
The three links cylindrical

$$T_3^0 = A_1 A_2 A_3 \quad = \quad \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4
Spherical wrist

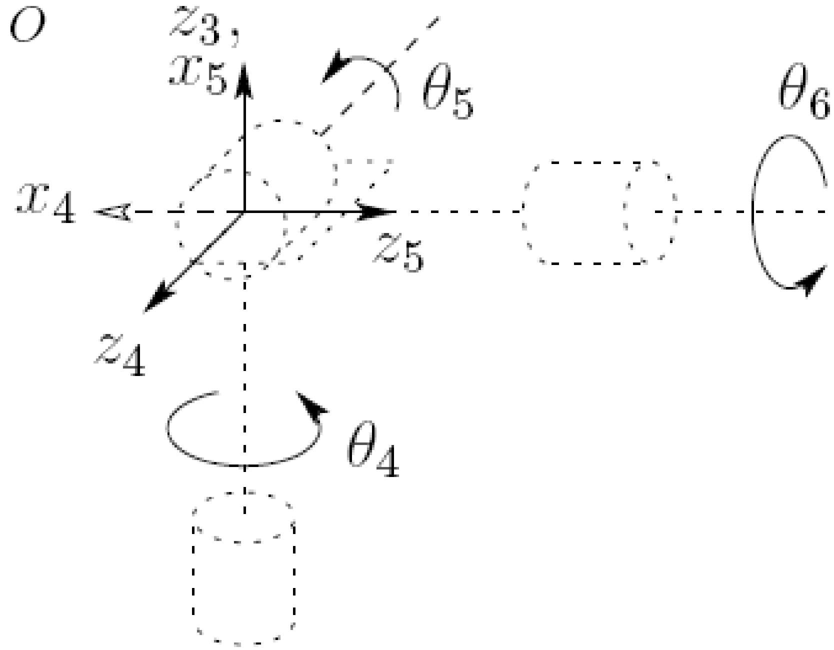


Example 4
Spherical wrist

joint axes z_3, z_4, z_5 intersect at O

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable



Example 4
Spherical wrist

$$
A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

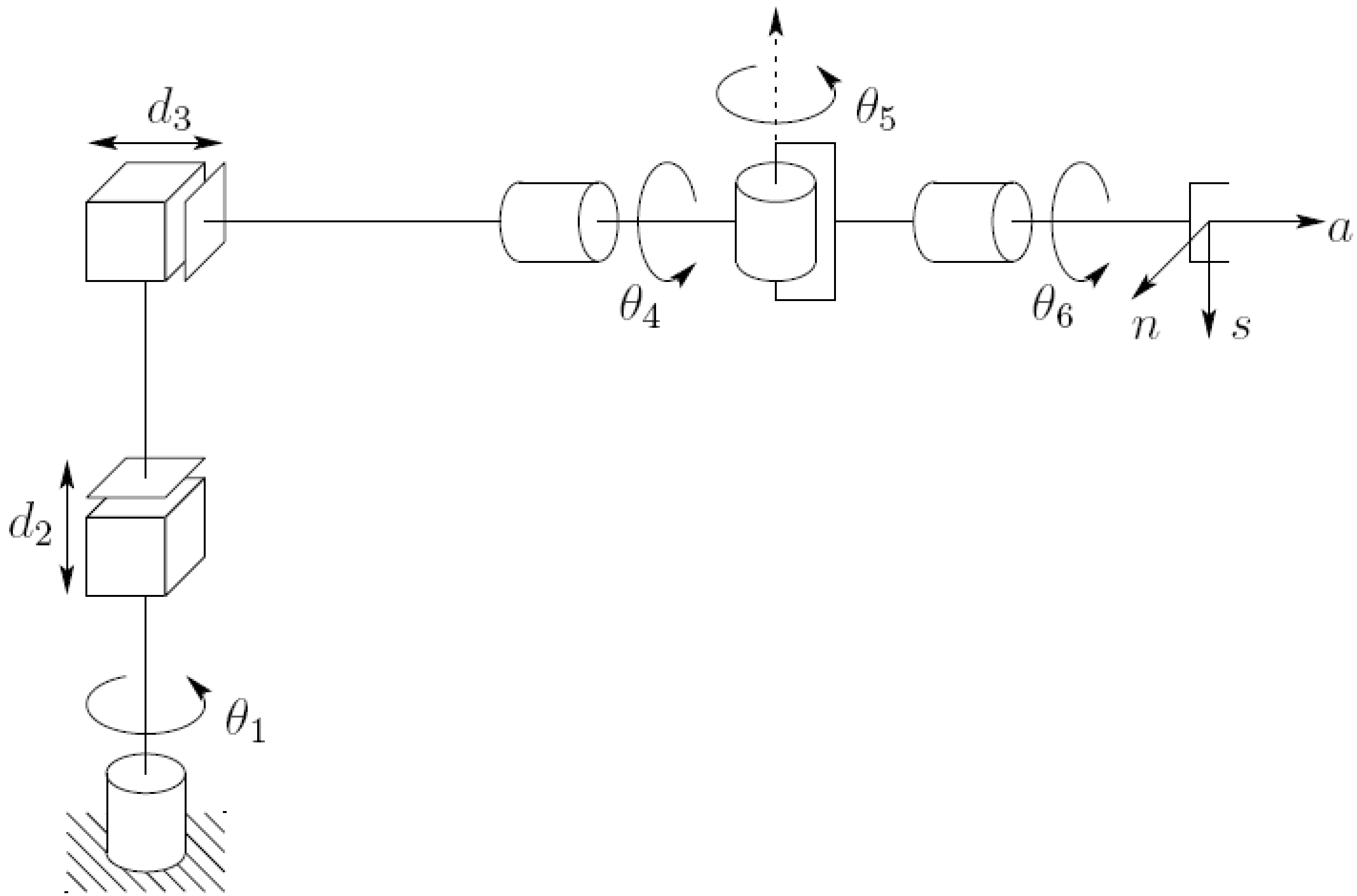
$$
A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Example 4
Spherical wrist

$$
\begin{aligned}
T_6^3 = A_4 A_5 A_6 &= \begin{bmatrix} R_6^3 & O_6^3 \\ 0 & 1 \end{bmatrix} \qquad (1) \\
&= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example 5

The three links cylindrical with Spherical wrist



Example 5

The three links cylindrical with Spherical wrist

$$T_6^0 = T_3^0 T_6^3$$

- T_3^0 given by example 2, and T_6^3 given by example 3.

Example 5

The three links cylindrical with Spherical wrist

$$T_6^0 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_1 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 c_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 5

The three links cylindrical with Spherical wrist

$$
\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2.
\end{aligned}
$$

Example 5

The three links cylindrical with Spherical wrist

- Forward kinematics:

1. The position of the end-effector: (d_x ,d_y ,d_z)

2. The orientation {Roll, Pitch, Yaw }

Rotation ψ about X axis{ROLL}

Rotation θ about fixed Y axis{PITCH}

Rotation φ about fixed Z axis{YAW}

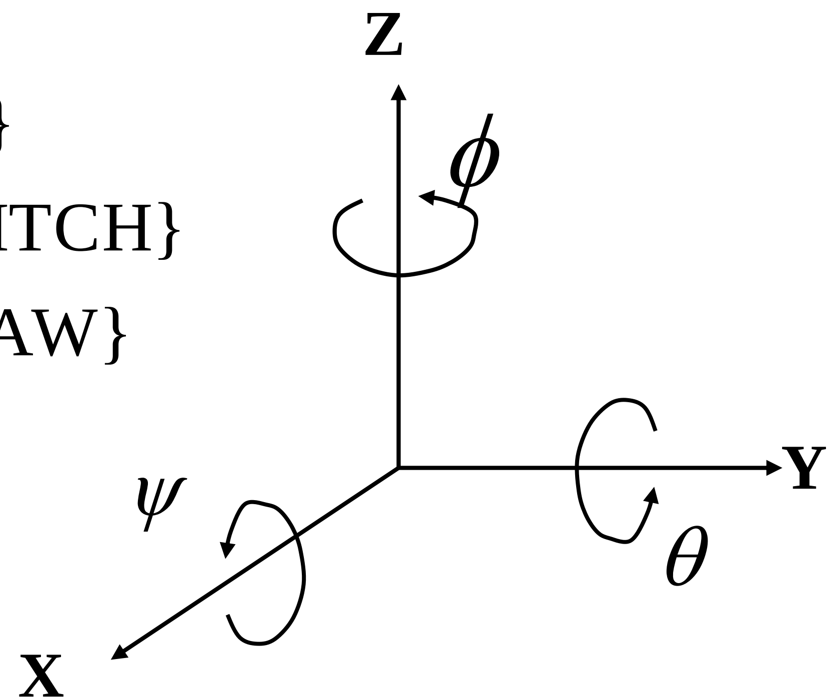
Roll Pitch Yaw

- The rotation matrix for the following operations:

$$\text{Rotation } \psi \text{ about X axis}\{\text{ROLL}\}$$

$$\text{Rotation } \theta \text{ about fixed Y axis}\{\text{PITCH}\}$$

$$\text{Rotation } \phi \text{ about fixed Z axis}\{\text{YAW}\}$$



$$R = Rot(z,\phi)Rot(y,\theta)Rot(x,\psi)$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{C}\theta & 0 & \mathrm{S}\theta \\ 0 & 1 & 0 \\ \text{-S}\theta & 0 & \mathrm{C}\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & -S\psi \\ 0 & S\psi & C\psi \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & -S\phi S\psi + C\phi S\theta S\psi & C\phi S\theta C\psi + S\phi S\psi \\ S\phi C\theta & C\phi S\theta S\psi + C\phi S\psi & -C\phi S\psi + S\phi S\theta C\psi \\ -S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix}$$

Example 4

The three links cylindrical with Spherical wrist

- How to calculate $\varphi, \psi, and \theta$

- Compare the matrix *R with*

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Of the matrix T_6^0

$$R = \begin{bmatrix} C\varphi C\theta & -S\varphi S\psi + C\varphi S\theta S\psi & C\varphi S\theta C\psi + S\varphi S\psi \\ S\varphi C\theta & C\varphi S\theta S\psi + C\varphi S\psi & -C\varphi S\psi + S\varphi S\theta C\psi \\ -S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix}$$

$$-S\theta = r_{31} \qquad C\theta S\psi = r_{32} \qquad S\varphi C\theta = r_{21}$$

$$\theta = Sin^{-1}(-r_{31}) \qquad \psi = Sin^{-1}(\frac{r_{32}}{C\theta}) \qquad \varphi = \sin^{-1}(\frac{r_{21}}{C\theta})$$

Module 1
RRR:RRR

Links	**α**	**a**	**θ**	**d**
1	90	0	*	10
2	0	10	*	0
3	-90	0	*	0
4	90	0	*	10
5	-90	0	*	0
6	0	0	*	0