Forward Kinematics

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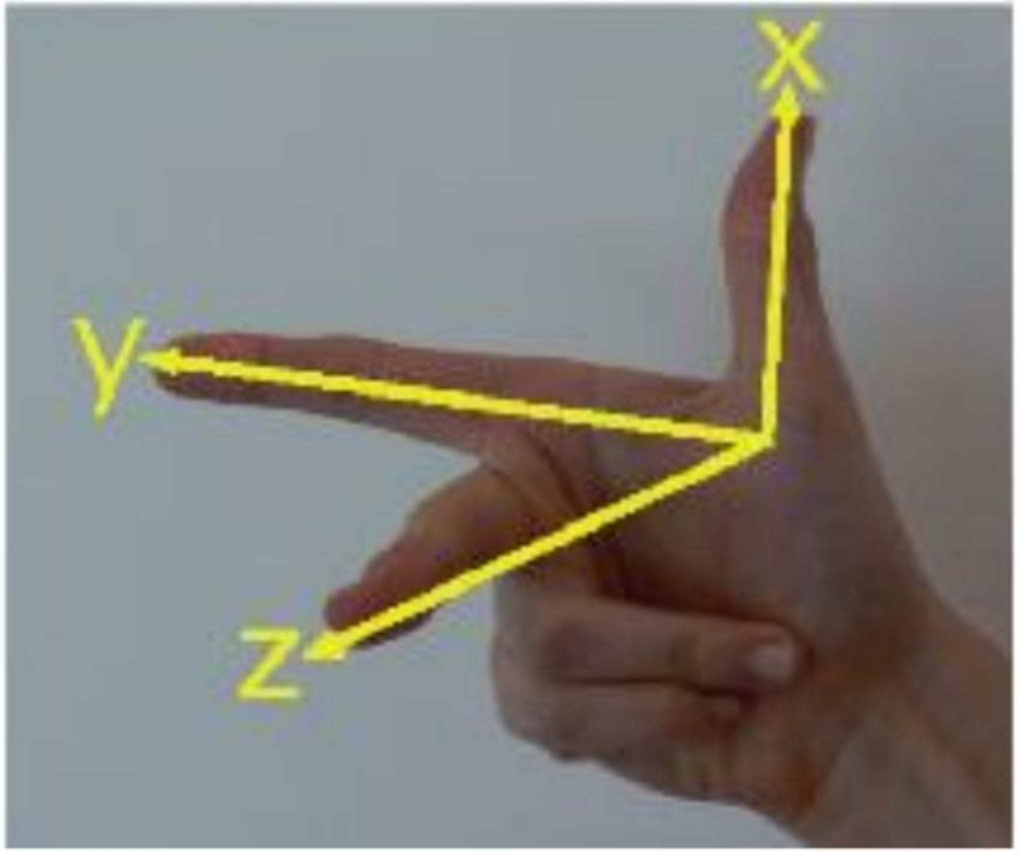
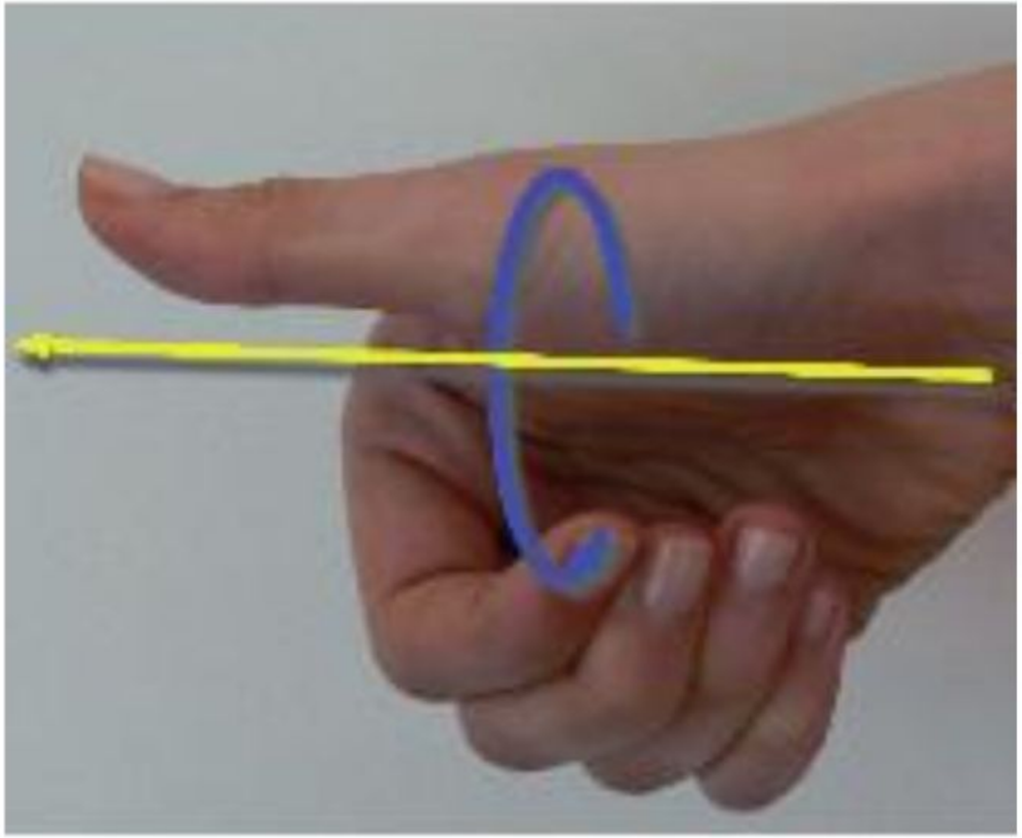
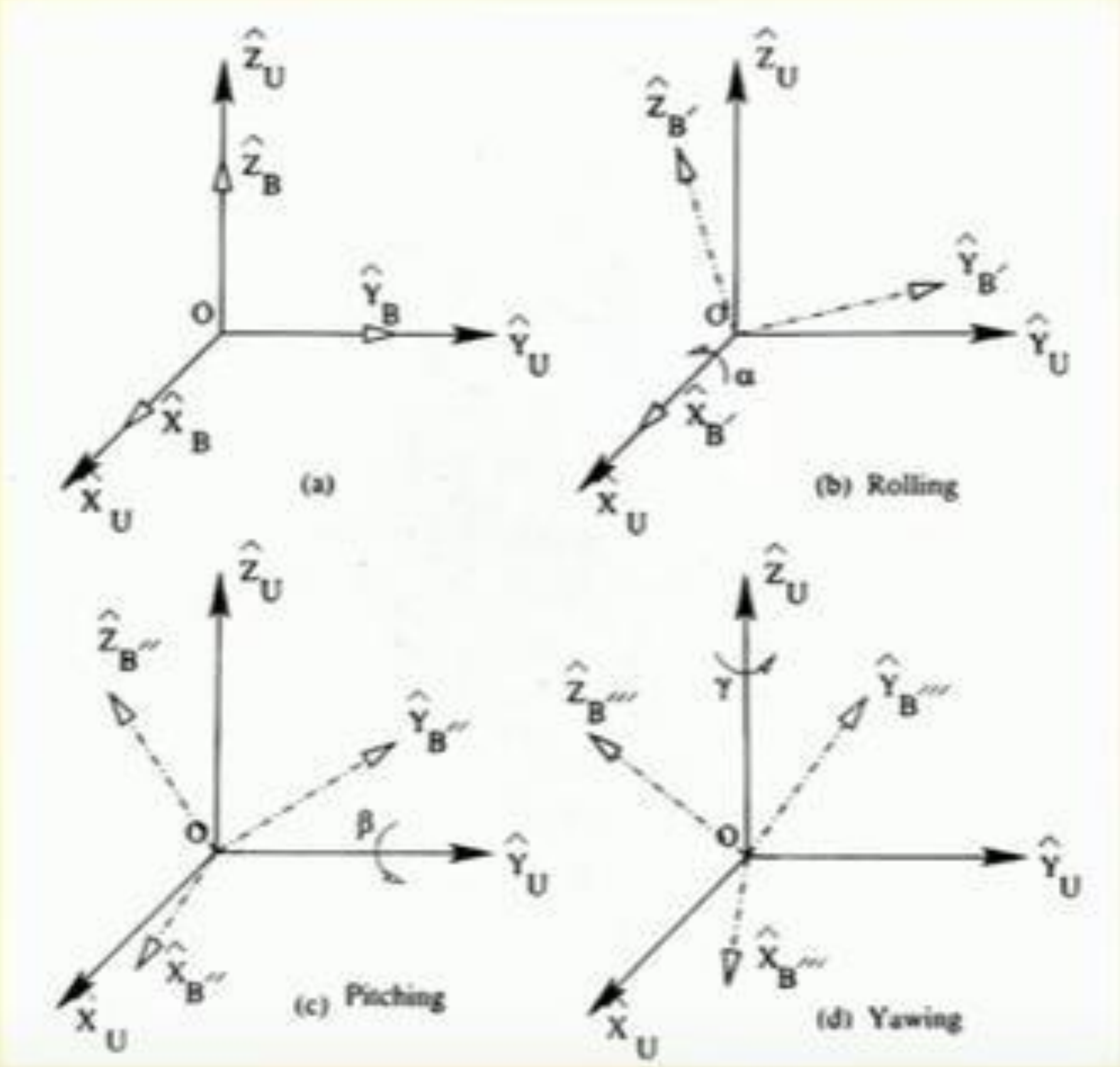
Right hand Rule

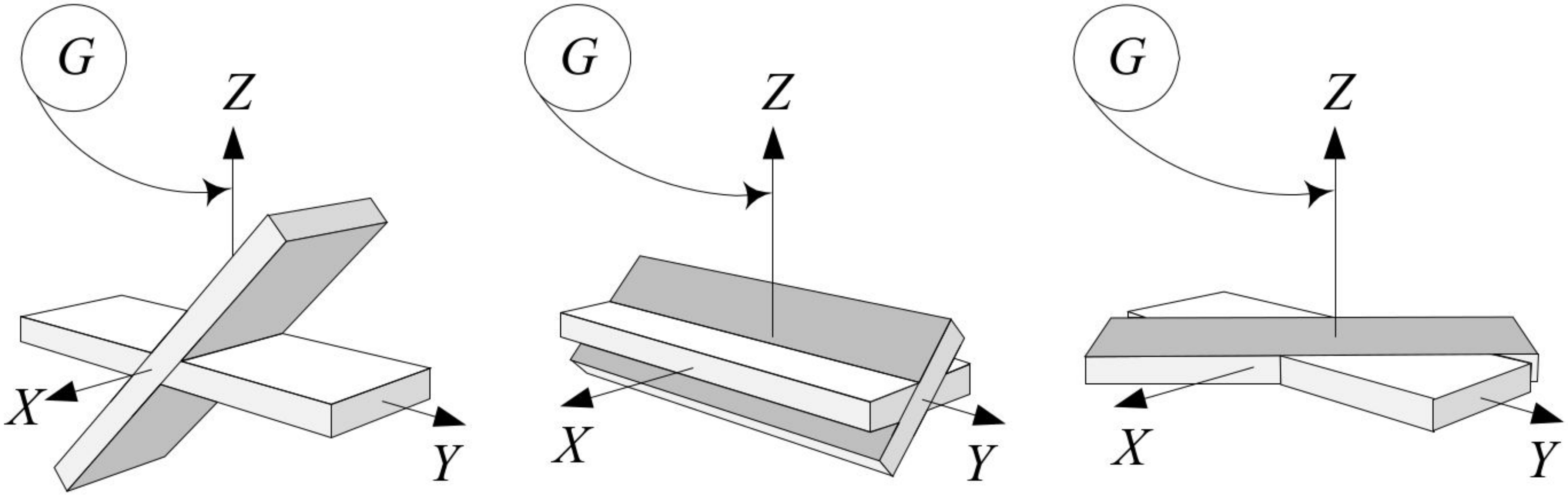


Rotate about vector



Roll, Pitch and Yaw Angles





Roll *Pitch* *Yaw*

$$ {}^{U}_{B}R_{composite:\, rpy} = ROT(\hat{Z}_U, \gamma) ROT(\hat{Y}_U, \beta) ROT(\hat{X}_U, \alpha) $$

$$ = \begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + s\alpha s\beta c\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha c\beta \end{bmatrix} $$

We compare with

$$ {}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

We get

$$ \alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right) $$

$$ \beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right) $$

$$ \gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) $$

A Numerical Example

The concept of roll, pitch and yaw angles has been used to represent the rotation of a frame {B} with respect to the reference frame {U}, that is ${}^U_B R$. Let us suppose that the above rotation can also be expressed by a 3X3 rotation matrix as given below.

$$\alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

$${}^U_B R = \begin{bmatrix} -0.250 & 0.433 & -0.866 \\ 0.433 & -0.750 & -0.500 \\ -0.866 & -0.500 & 0.000 \end{bmatrix}$$

Determine the angles of rolling, pitching and yawing.

Solution:

Angle of rolling $\alpha = \tan^{-1}\frac{r_{32}}{r_{33}} = \tan^{-1}\frac{-0.500}{0.000} = 90^\circ$

Angle of pitching $\beta = \tan^{-1}\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}$

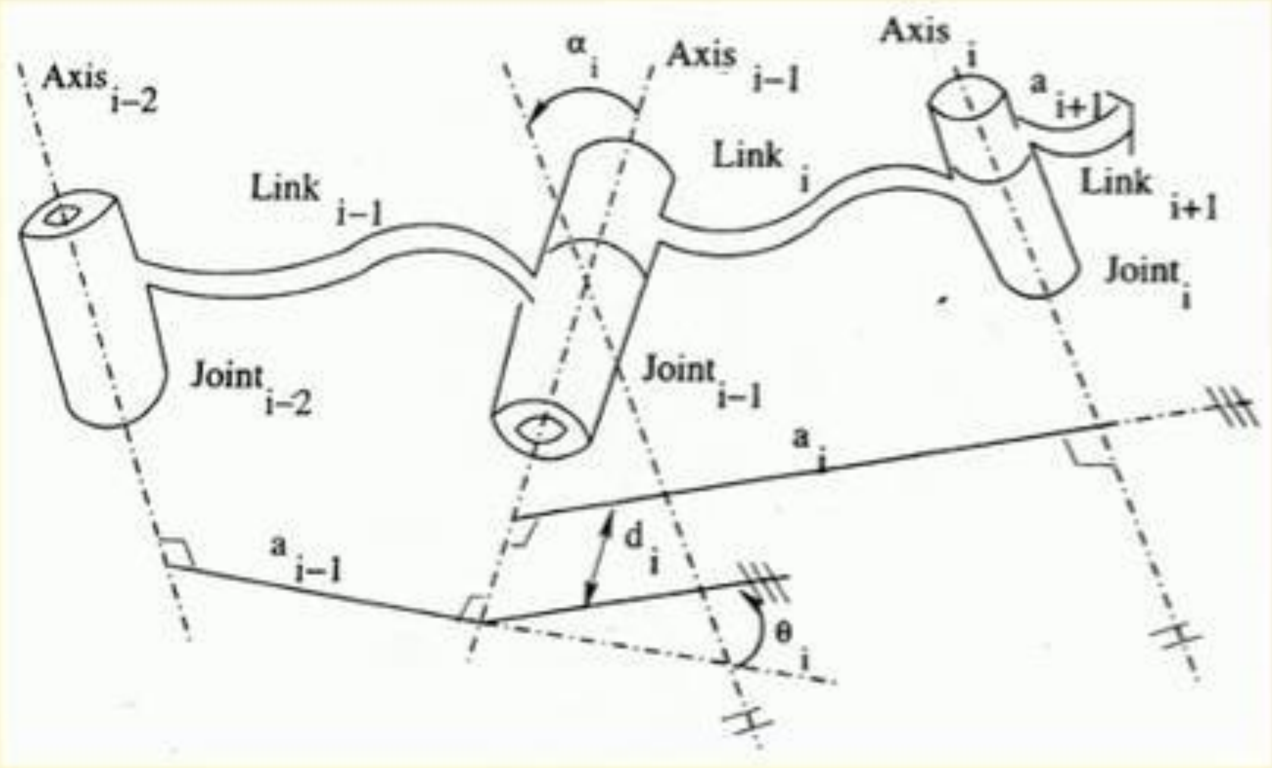
$$= \tan^{-1}\frac{0.866}{\sqrt{(-0.250)^2 + (0.433)^2}}$$

Angle of yawing $\gamma = \tan^{-1}\frac{-r_{21}}{r_{11}} = \tan^{-1}\frac{0.433}{-0.250}$

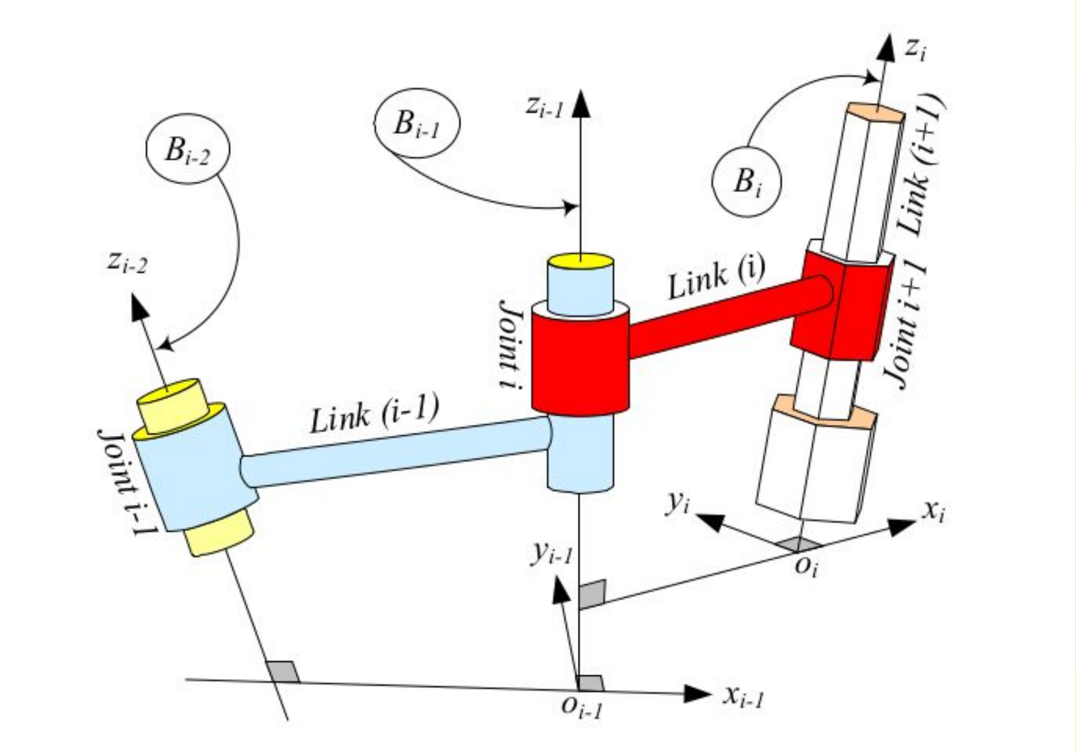
$$= -59.99 \approx -60^\circ$$

Denavit-Hartenberg Notations

Link and Joint Parameters



- **Length of link$_i$ (a_i):** It is the mutual perpendicular distance between Axis$_{i-1}$ and Axis$_i$
- **Angle of twist of link$_i$ (a_i):** It is defined as the angle between Axis$_{i-1}$ and Axis$_i$



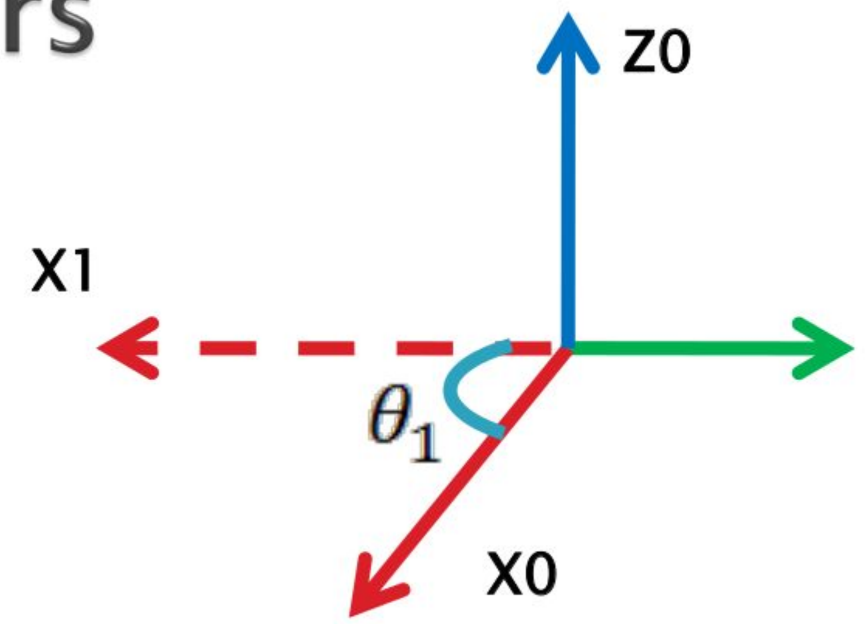
- **Offset of link$_i$ (d_i):** It is the distance measured from a point where a_{i-1} intersects the Axis$_{i-1}$ to the point where a_i intersects the Axis$_{i-1}$ measured along the said axis

- **Joint Angle (θ_i):** It is defined as the angle between the extension of a_{i-1} and a_i measured about the Axis$_{i-1}$

Notes:

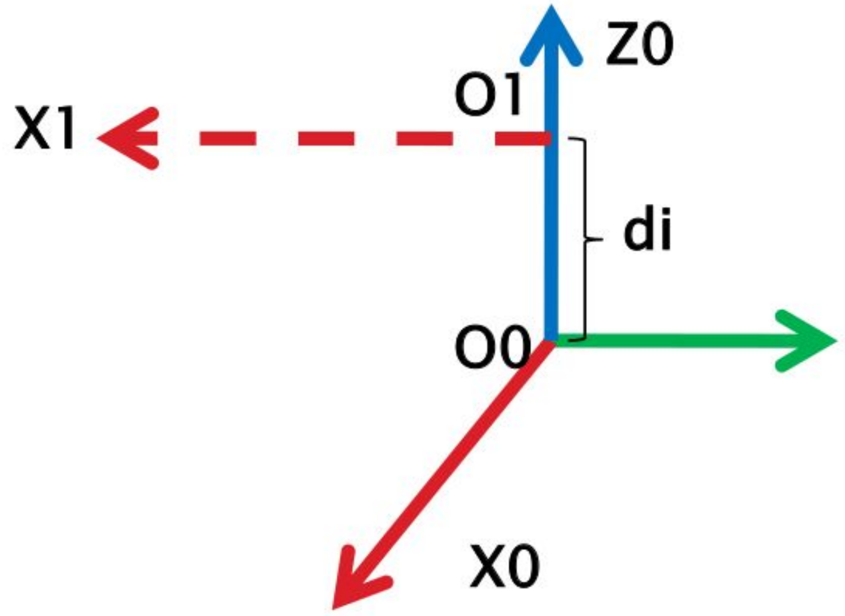
- **Revolute joint:** θ_i is variable
- **Prismatic joint:** d_i is variable

DH prameters



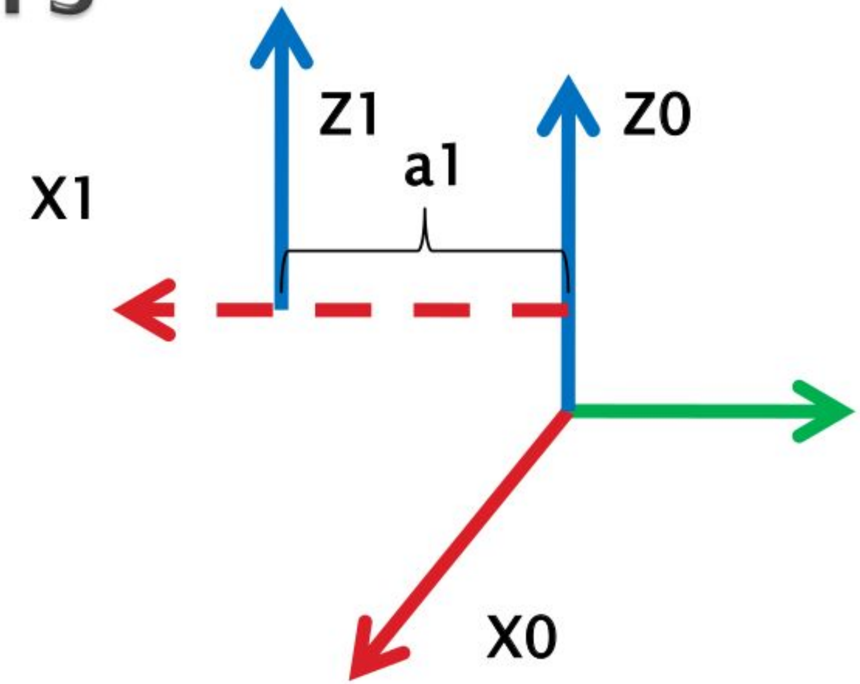
- (joint angle) θ_1 is angle from x0 to x1 measured about Z0

DH prameters



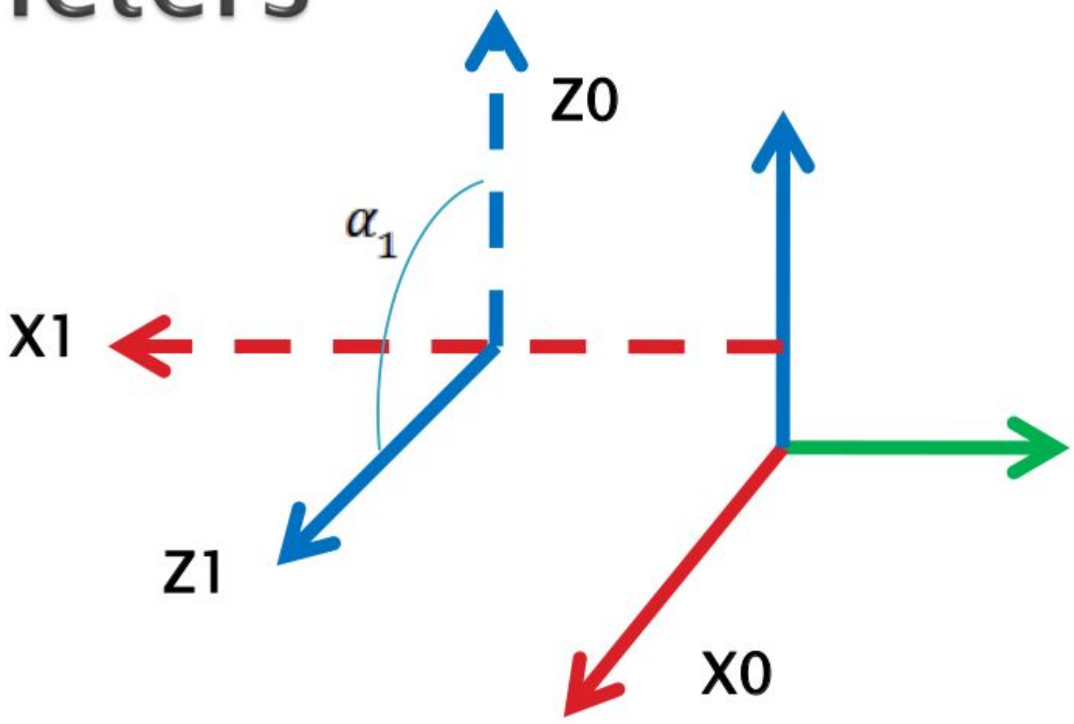
- (Link Offset) d1 distance from O0 to O1 measured along z0

DH prameters



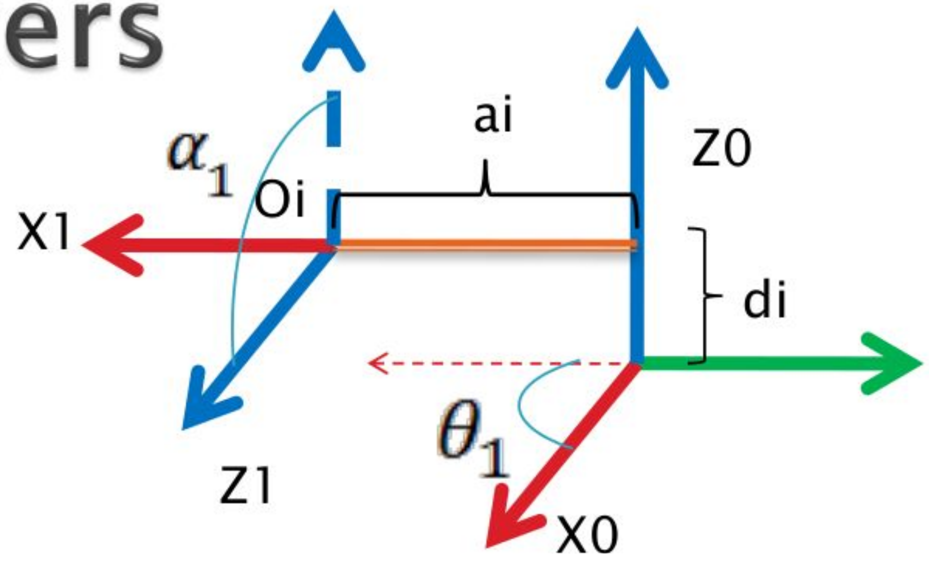
- (Link Length) a1 distance from z0 to z1 measured along x1

DH prameters



- (Link twist) α_1 is angle from z0 to z1 measured about x1

DH prameters



1. *Link length* a_i is the distance between z_{i-1} and z_i axes along the x_i-axis. a_i is the *kinematic length* of link (i).

2. *Link twist* α_i is the required rotation of the z_{i-1}-axis about the x_i-axis to become parallel to the z_i-axis.

3. *Joint distance* d_i is the distance between x_{i-1} and x_i axes along the z_{i-1}-axis. Joint distance is also called *link offset.*

4. *Joint angle* θ_i is the required rotation of x_{i-1}-axis about the z_{i-1}-axis to become parallel to the x_i-axis.

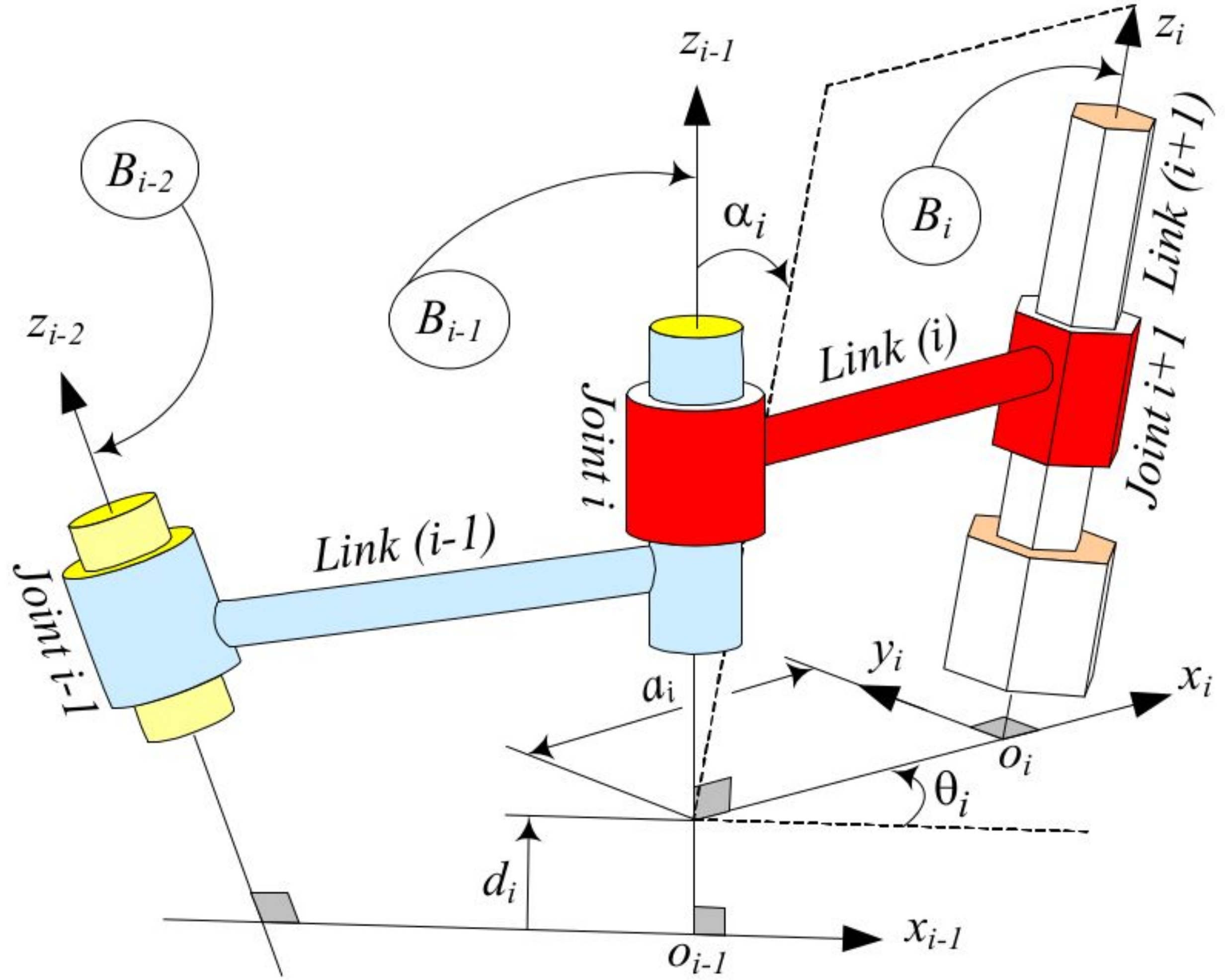


FIGURE 5.3. DH parameters $a_i, \alpha_i, d_i, \theta_i$ defined for joint i and link (i).

DH Techniques

- Matrix A_i representing the four movements is found by: four movements

1. Rotation of θ about current Z axis
2. Translation of d along current Z axis
3. Translation of a along current X axis
4. Rotation of α about current X axis

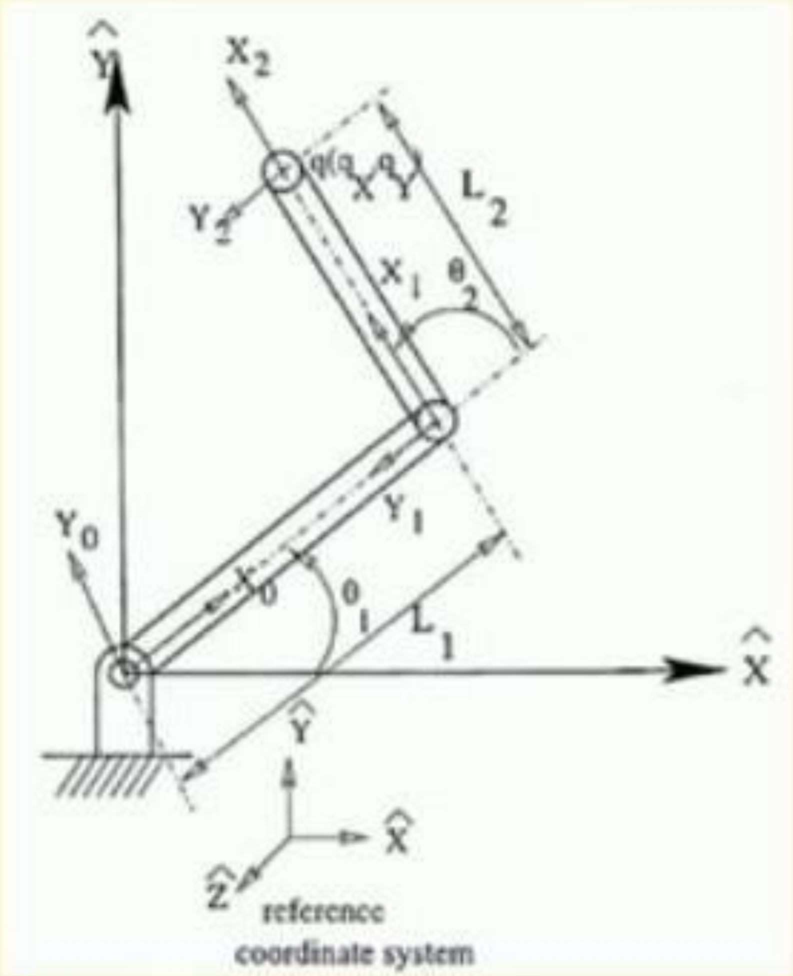
$$A_i = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$A_i = R_{z,\theta_i} \mathrm{Trans}_{z,d_i} \mathrm{Trans}_{x,a_i} R_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1

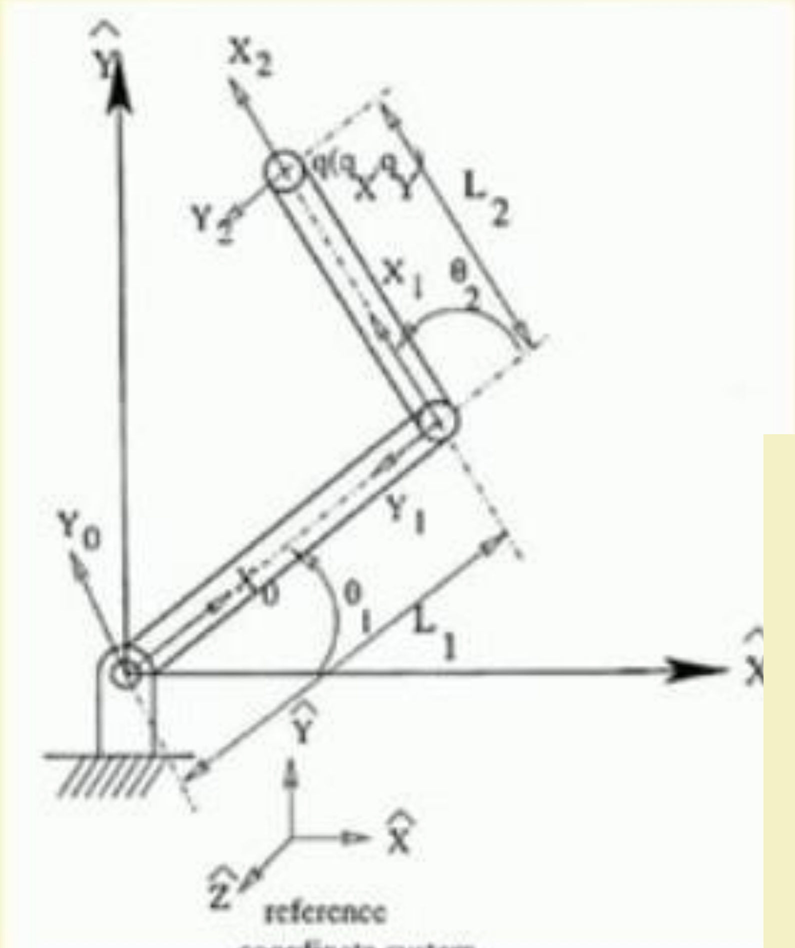


Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$^{Base}_{2}T = {}^{Base}_{1}T\,{}^{1}_{2}T$$

$$\begin{aligned} ^{Base}_{1}T &= ROT(\hat{Z},\theta_1)TRANS(\hat{X},L_1) \\ &= \begin{bmatrix} c_1 & -s_1 & 0 & L_1c_1 \\ s_1 & c_1 & 0 & L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

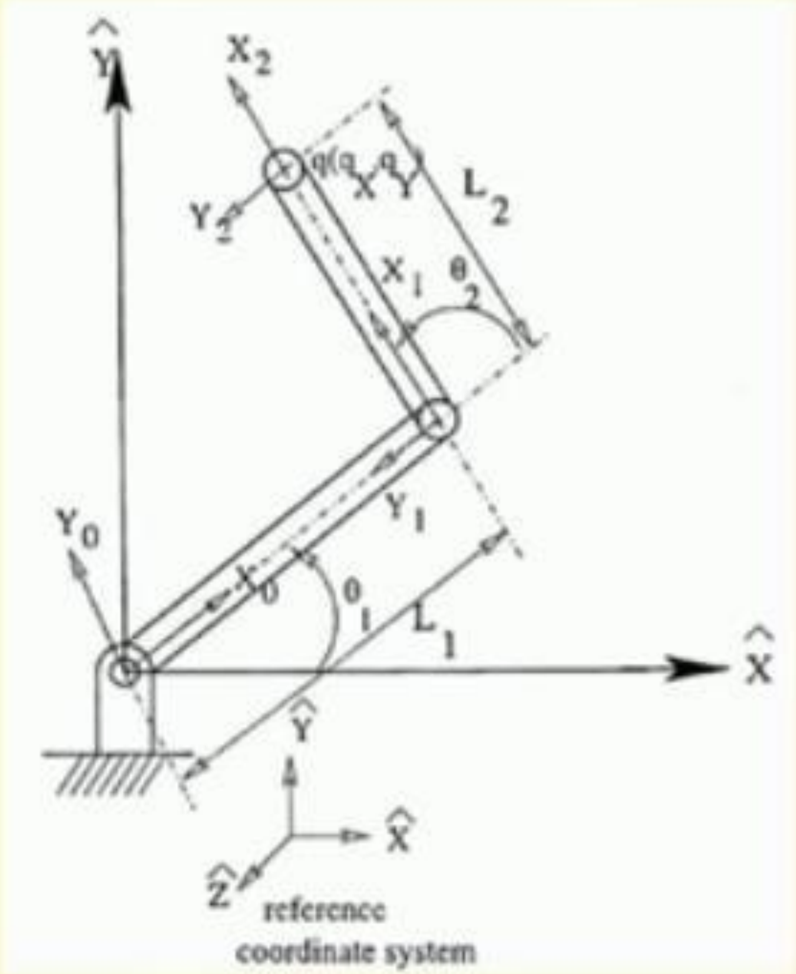
Example 1



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$
\begin{aligned}
{}^{1}_{2}T &= ROT(\hat{Z},\theta_2)TRANS(\hat{X},L_2) \\
&= \begin{bmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

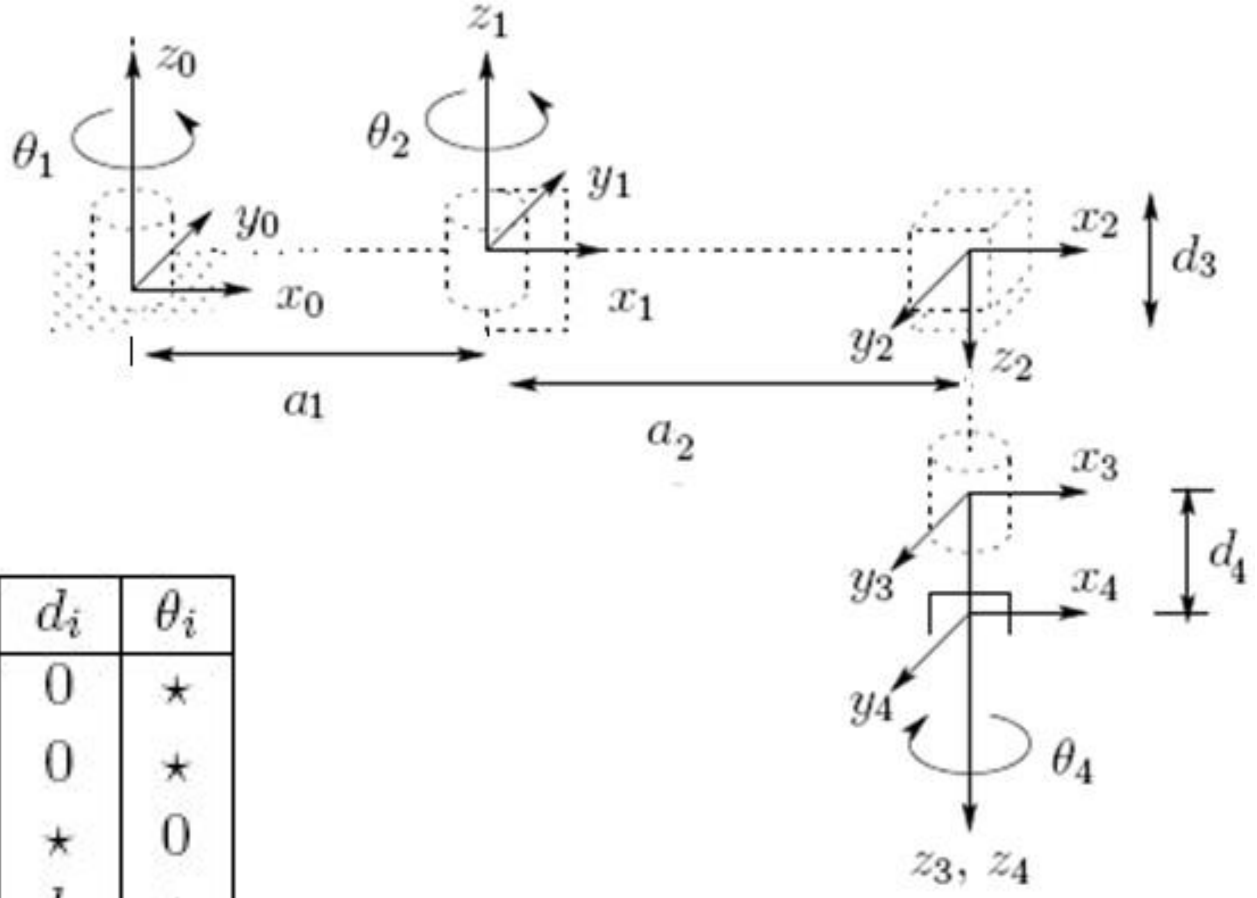
Example 1



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

$$
\begin{aligned}
{}^{Base}_{2}T &= {}^{Base}_{1}T\,{}^{1}_{2}T \\
&= \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & L_1c_1 + L_2c_{12} \\ s_{12} & c_{12} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example 2



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

Example 2

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

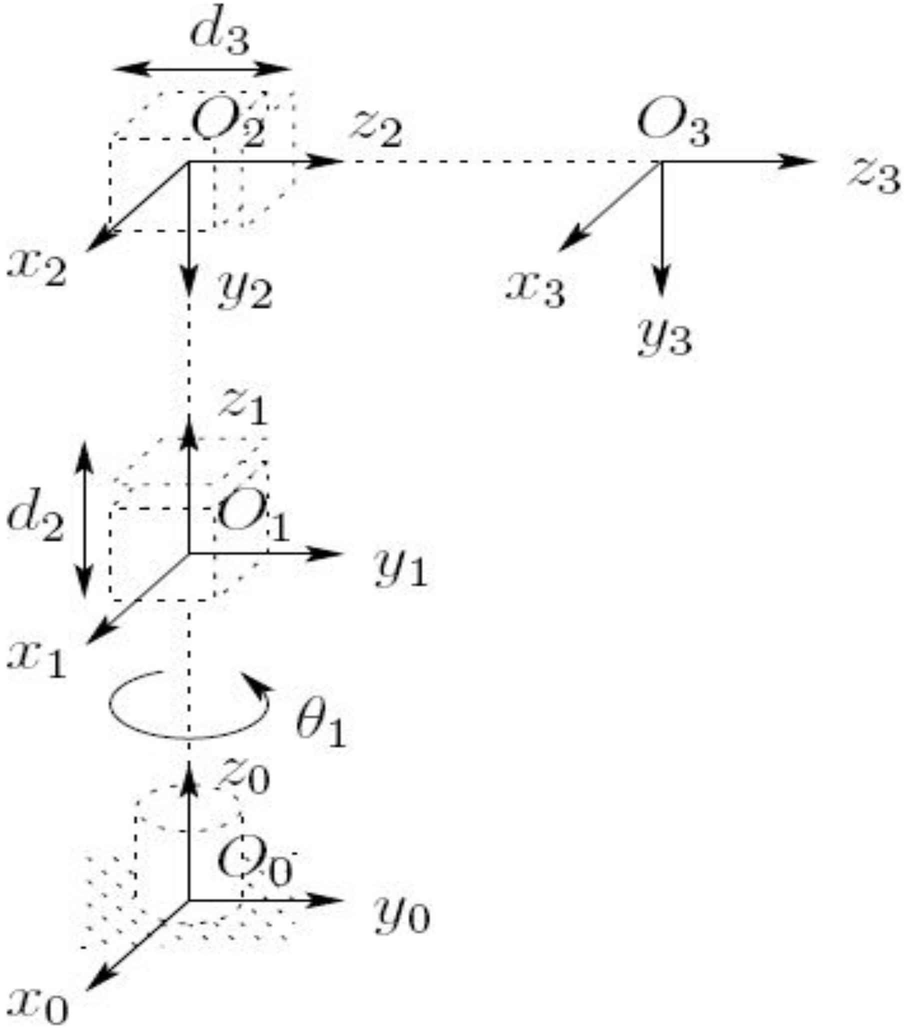
$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3
The three links cylindrical

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable



Example 3
The three links cylindrical

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

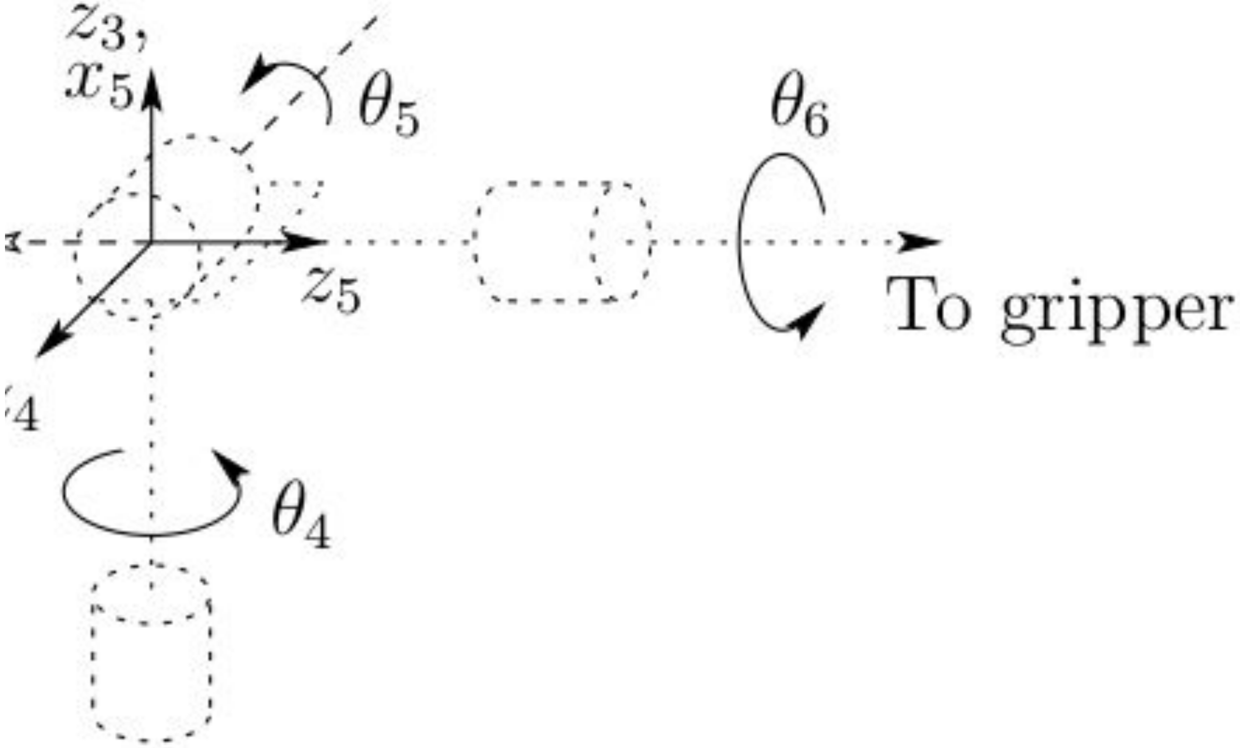
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4
Spherical wrist

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable



Example 4
Spherical wrist

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

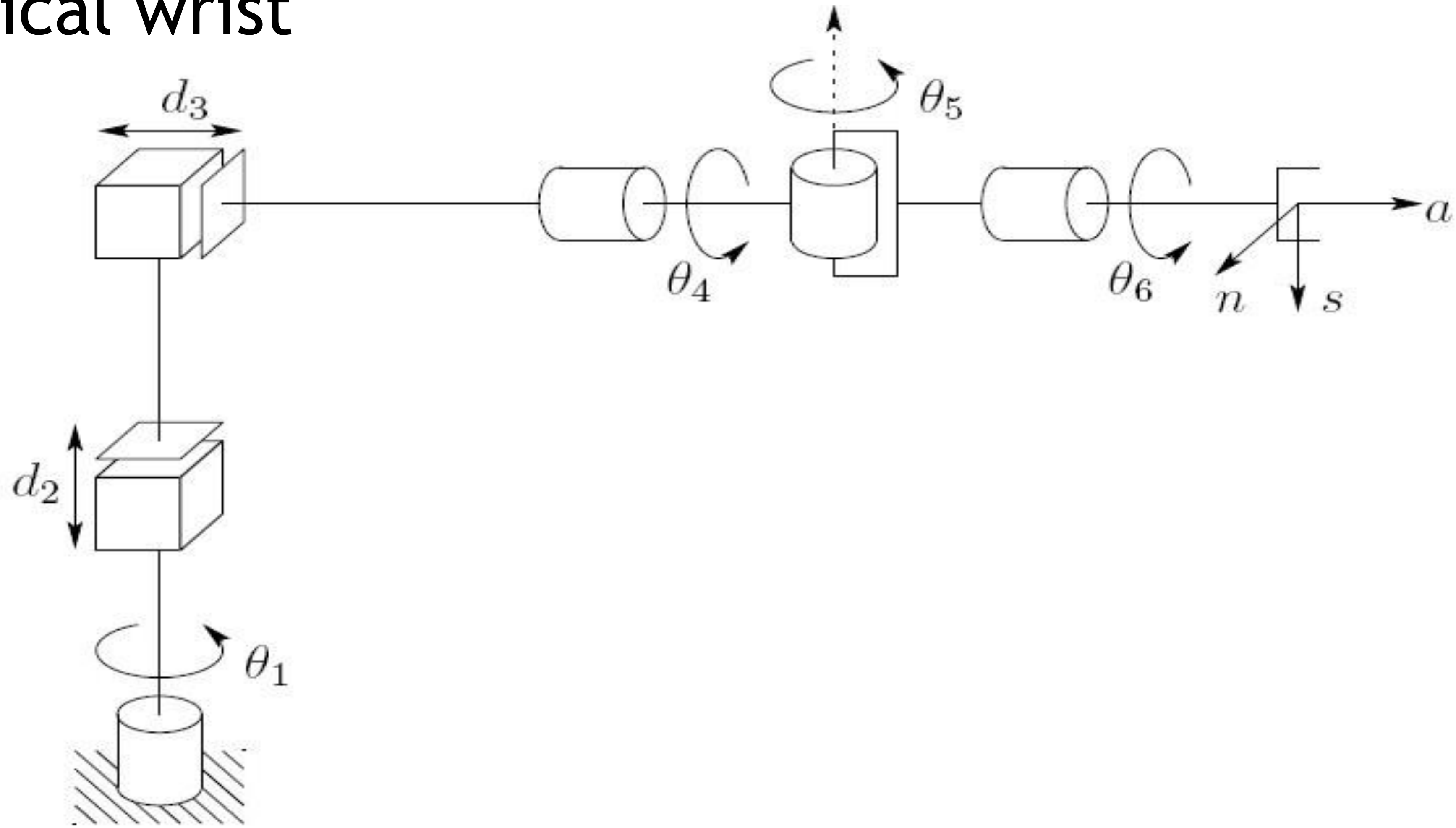
$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} R_6^3 & O_6^3 \\ 0 & 1 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 5

The three links cylindrical with

Spherical wrist



Example 5

The three links cylindrical with

Spherical wrist

$$T_6^0 = T_3^0 T_6^3$$

- given by example 3 given by example 4.

T_3^0 ${}_3T^{6}$

Example 5

The three links cylindrical with

Spherical wrist

$$T_6^0 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_1 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2.
\end{aligned}$$

Frame No.	a_i	α_i	d_i	θ_i
1	l_1	0	0	θ_1
2	l_2	0	0	θ_2
3	l_3	0	0	θ_3

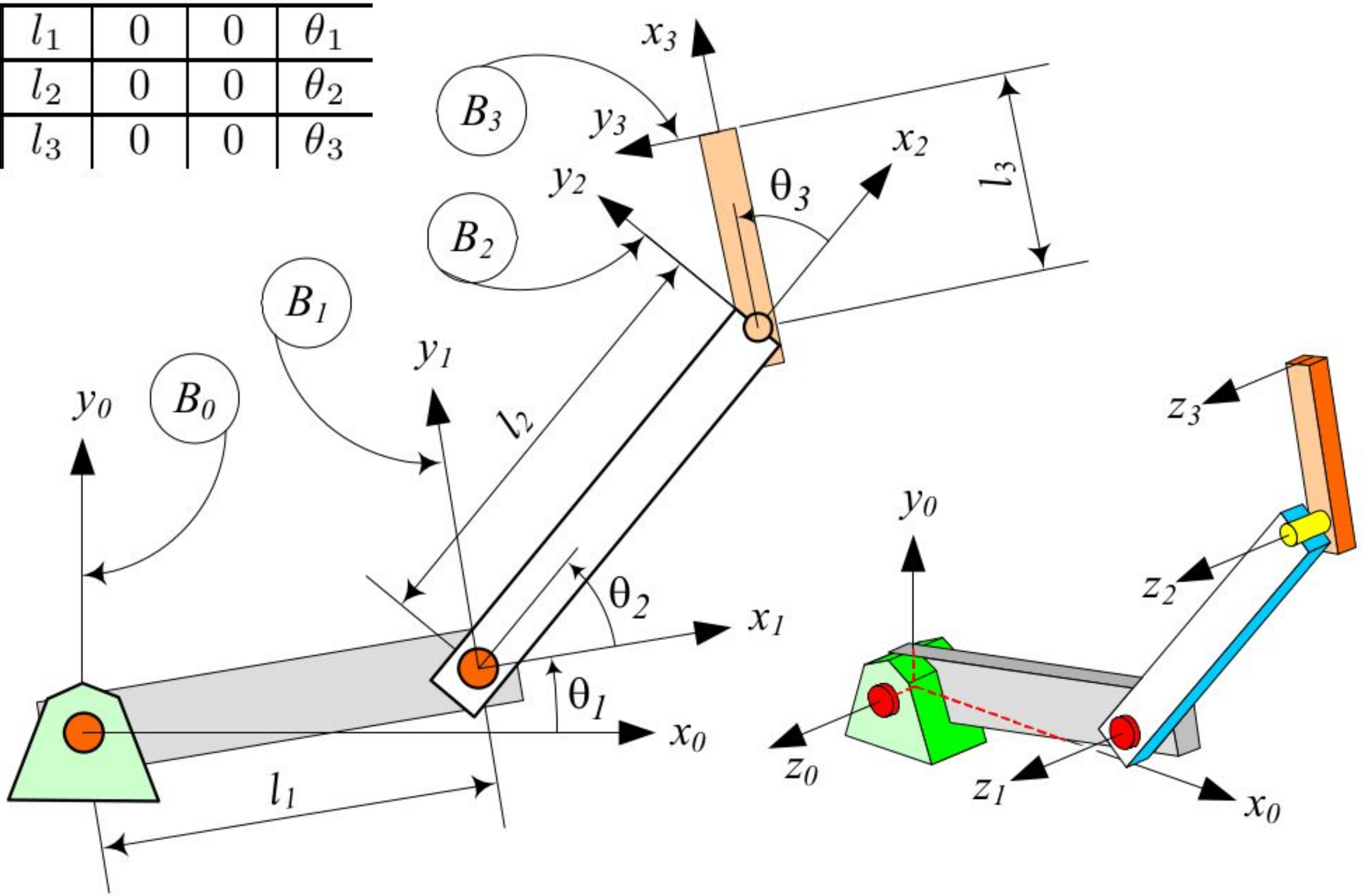
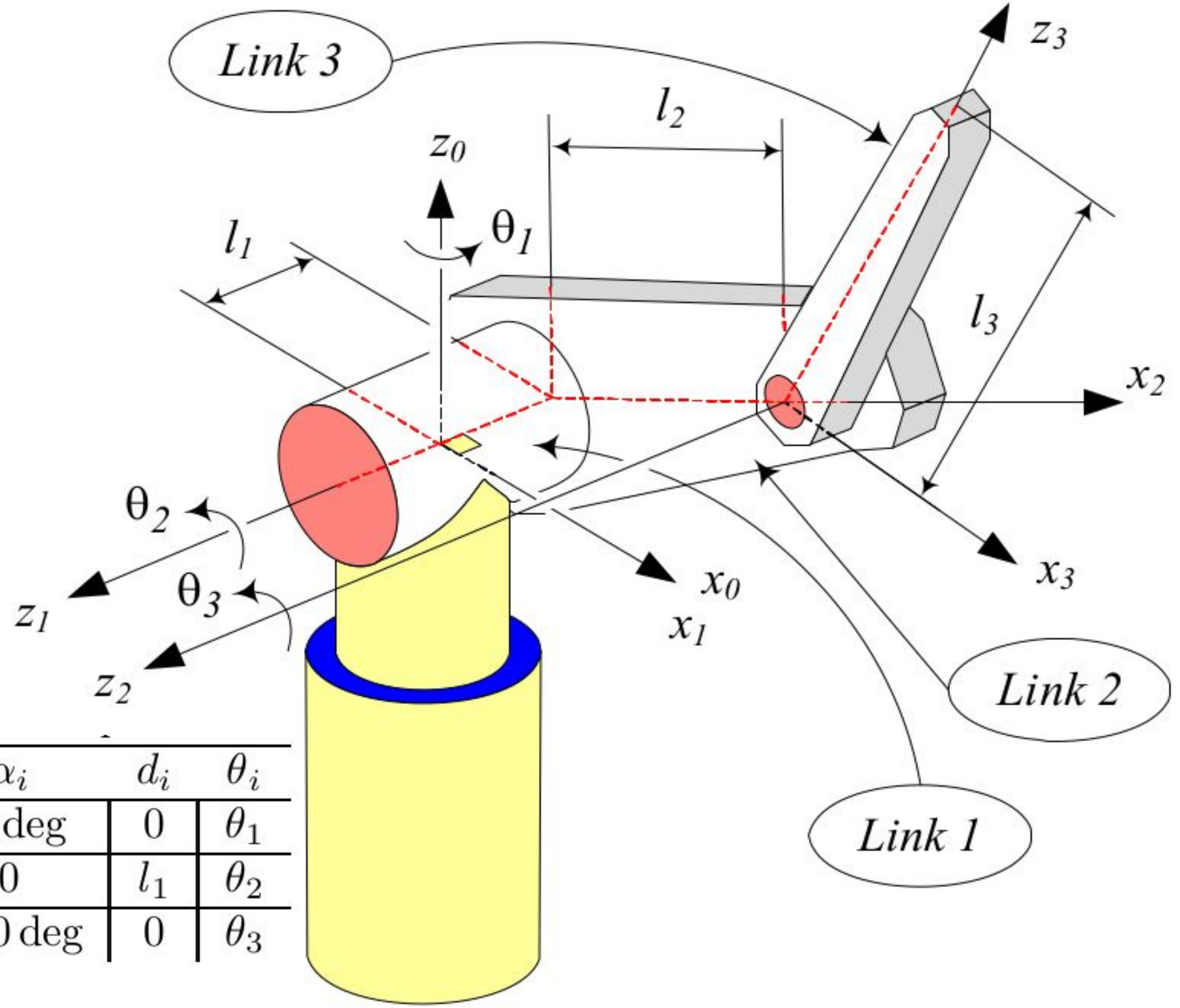


FIGURE 5.4. Illustration of a 3R planar manipulator robot and DH frames of each link.



Frame No.	a_i	α_i	d_i	θ_i
1	0	90 deg	0	θ_1
2	l_2	0	l_1	θ_2
3	0	-90 deg	0	θ_3

FIGURE 5.5. 3R $PUMA$ manipulator and links coordinate frame.

References

- Lecture on Kinematics-Fall2019 by Honorable Prof. Dr. Syed Akhter Hossain Sir

- Lectures by honourable Prof D K Pratihar of NPTEL
- https://youtu.be/6Wb0rmIvIlI
- https://youtu.be/AbRhzpReb2Q
- https://youtu.be/h4_2xAPj3y0