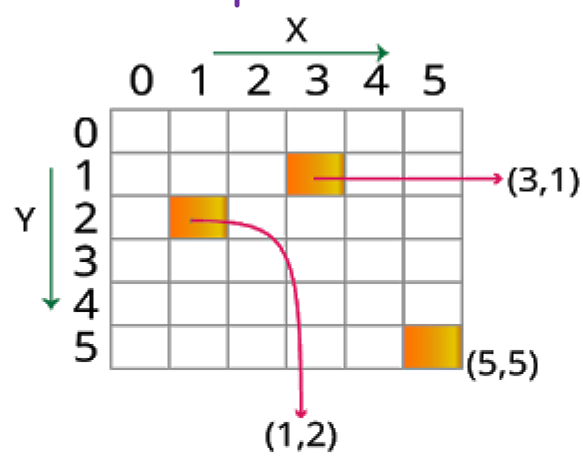
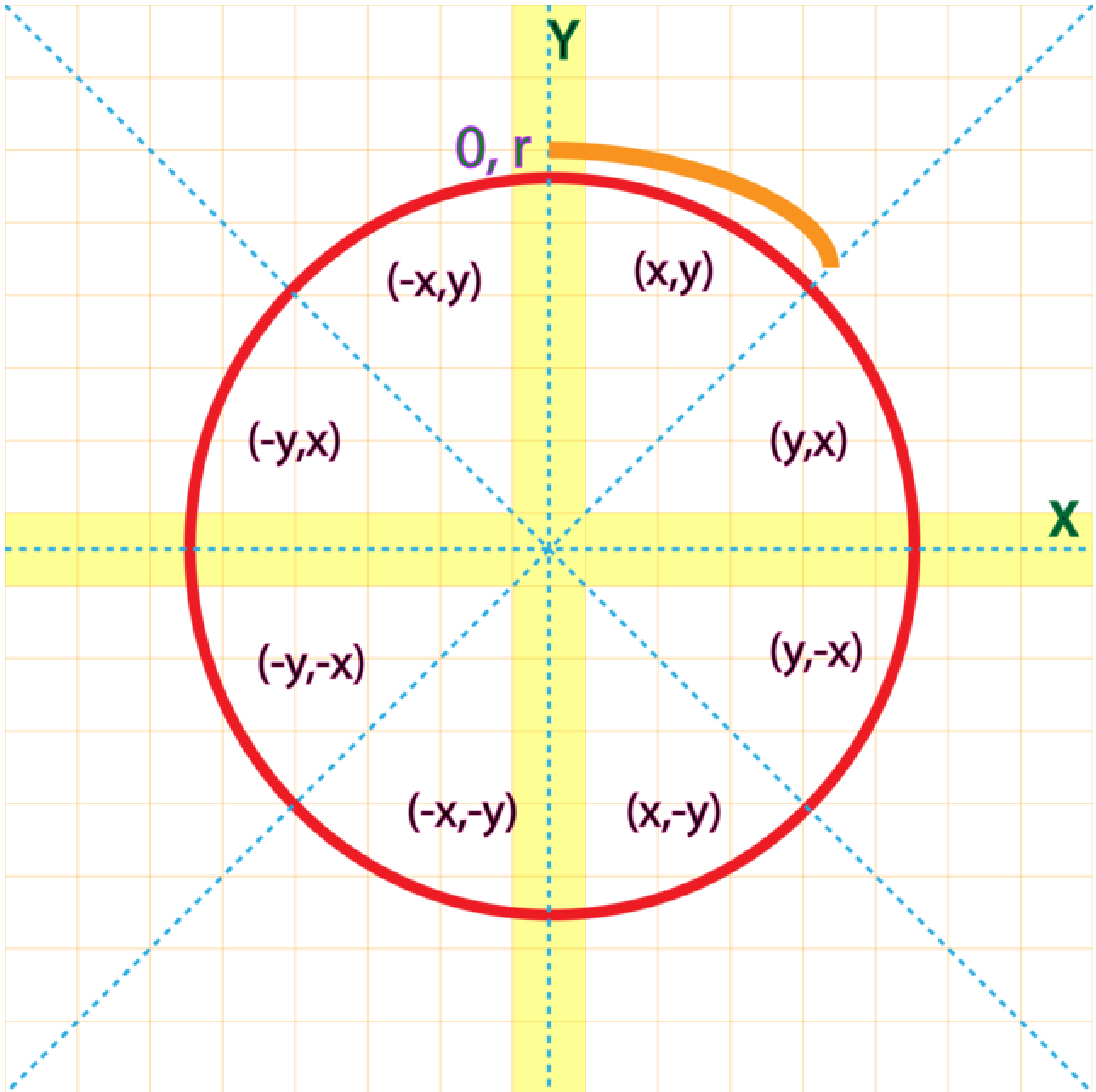
Mid Point Circle Drawing Derivation (Algorithm)

The mid point circle algorithm is used to determine the pixels needed for rasterizing a circle while drawing a circle on a pixel screen.
In this technique algorithm determines the mid point between the next 2 possible consecutive pixels and then checks whether the mid point in inside or outside the circle and illuminates the pixel accordingly.

This is how a pixel screen is represented:

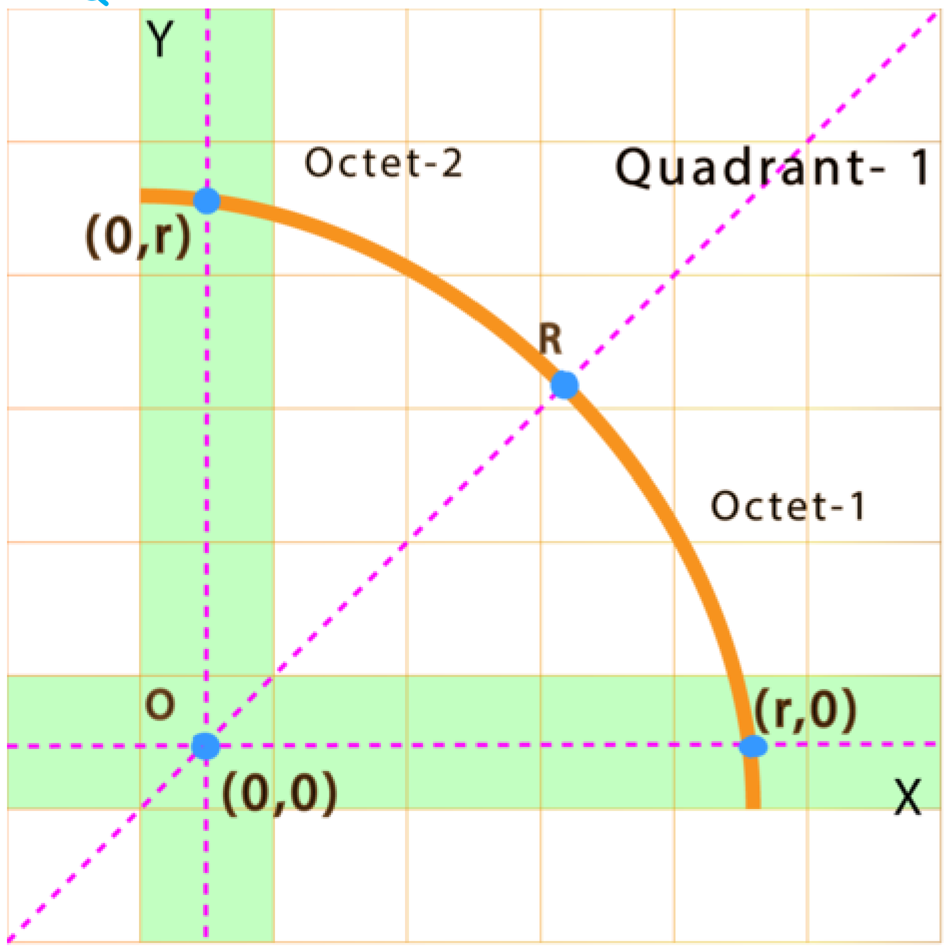


A circle is highly symmetrical and can be divided into 8 Octets on graph. Lets take center of circle at Origin i.e (0,0) :



We need only to conclude the pixels of any one of the octet rest we can conclude because of symmetrical properties of circle.

Let us Take Quadrant 1:



Radius = OR = r

Radius = x intercept = y intercept

At point R

coordinate of x = coordinate of y   or we can say x=y

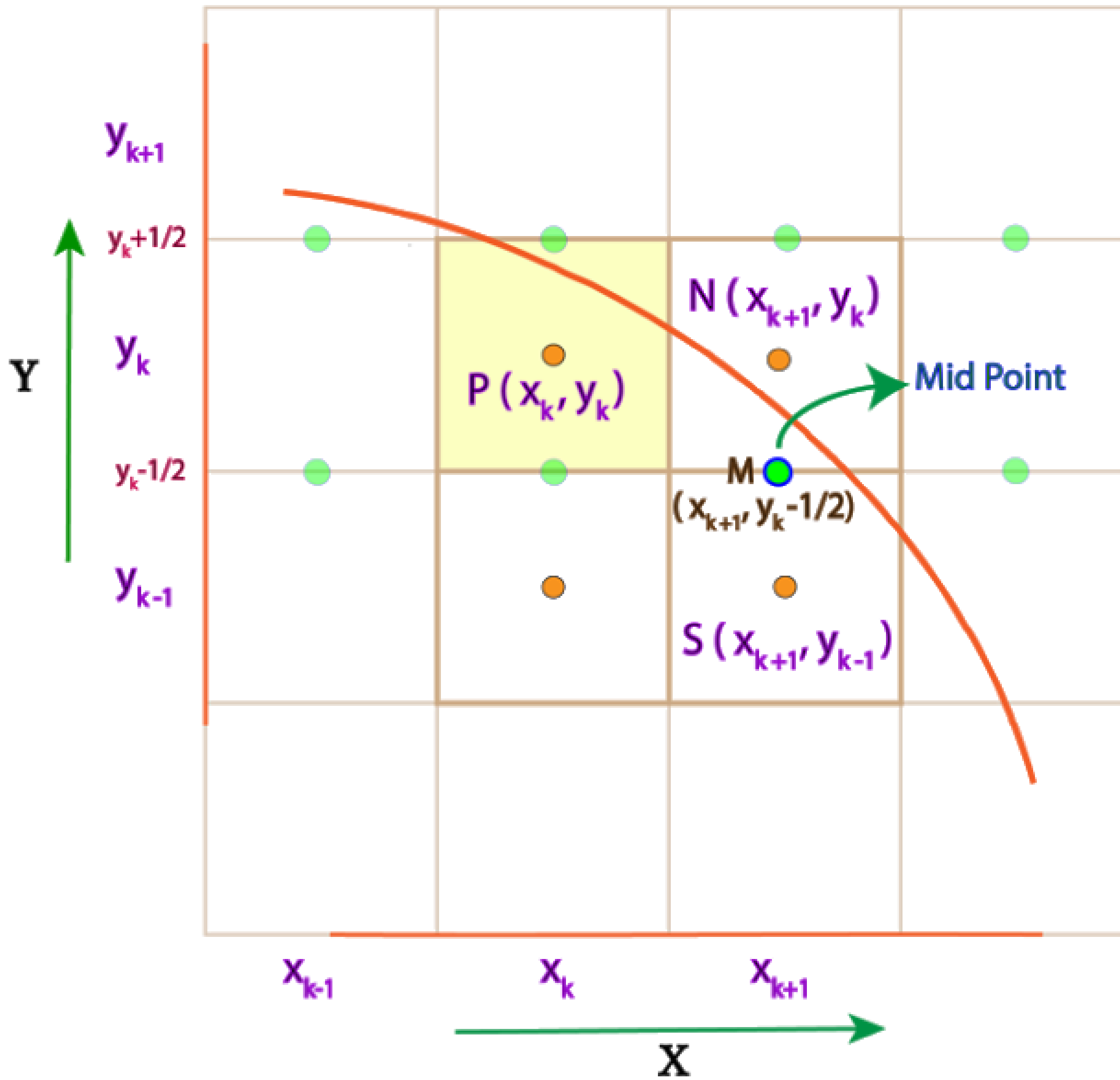
let us take Octet 2 of quadrant 1

here first pixel would be (0,y)

here value of y intercept = radius (r)

as circle's center is at origin

DERIVATION



let us assume we have plotted Pixel P whose coordinates are (x_k , y_k)
Now we need to determine the next pixel.

We have chosen octet 2 where circle is moving forward and downwards so y can never be increased, either it can be same or decremented. Similarly x will always be increasing as circle is moving forward too.

So y is needed to be decided.
Now we need to decide whether we should go with point N or S.

For that decision Mid Point circle drawing technique will us decide our next pixel whether it will be N or S.

As x_{k+1} is the next most pixel of x_k therefore we can write,

$$x_{k+1} = x_k + 1$$

And similarly $y_{k-1} = y_k - 1$ in this case.

Let M is the midpoint between $N(x_{k+1}, y_k)$ and $S(x_{k+1}, y_{k-1})$.
And coordinates of point (M) are

$$M = \frac{N + S}{2}$$

$$= \frac{x_k + 1 + x_k + 1}{2}, \frac{y_k + y_k - 1}{2}$$

$$= \left(x_k + 1, y_k - \frac{1}{2}\right)$$

Equation of Circle with Radius r

$$(x - h)^2 + (y - k)^2 = r^2$$

When coordinates of centre are at Origin i.e., (h=0, k=0)

$$x^2 + y^2 = r^2$$ (Pythagoras theorem)

Function of Circle Equation

$F(C) = x^2 + y^2 - r^2$

Function of Midpoint M $(x_{k+1}, y_k - 1/2)$ in circle equation

$$F(M) = x_{k+1}^2 + (y_k - 1/2)^2 - r^2$$
$$= (x_k+1)^2 + (y_k - 1/2)^2 - r^2$$

The above equation is too our decision parameter p_k

$$P_k = (x_k+1)^2 + (y_k - 1/2)^2 - r^2 \qquad ....(i)$$

To find out the next decition parameter we need to get P_{k+1}

$$P_{k+1} = (x_{k+1}+1)^2 + (y_{k+1} - 1/2)^2 - r^2$$

Now,

$$P_{k+1} - P_k = (x_{k+1}+1)^2 + (y_{k+1} - 1/2)^2 - r^2$$
$$-[(x_k+1)^2 + (y_k - 1/2)^2 - r^2]$$

$$= ((x_k+1)+1)^2 + (y_{k+1} - 1/2)^2$$
$$- (x_k+1)^2 - (y_k - 1/2)^2$$

$$= (x_k+1)^2 + 1 + 2(x_k+1) + y_{k+1}^2 + (1/4) - y_{k+1}$$
$$- (x_k+1)^2 - y_k^2 - (1/4) + y_{k+1}$$

$$= 2(x_k+1) + y_{k+1}^2 - y_k^2 - y_{k+1} + y_k + 1$$

$$= 2(x_k+1) + (y_{k+1}^2 - y_k^2) - (y_{k+1} - y_k) + 1$$

$$P_{k+1} = P_k + 2(x_k+1) + (y_{k+1}^2 - y_k^2) - (y_{k+1} - y_k) + 1 \qquad .....(ii)$$

Now let us conclude the initial decision parameter

For that we have to choose coordinates of starting point i.e. (0,r)

Put this in (i) i.e. P_k

$P_k = (x_k+1)^2 + (y_k-1/2)^2 - r^2$

$P_0 = (0+1)^2 + (r-1/2)^2 - r^2$

$= 1 + r^2 + ¼ - r - r^2$

$= 1 + ¼ - r$

.....(initial decision parameter)

Now If $P_k \geq 0$ that means midpoint is outside the circle and S is closest pixel so we will choose S (x_{k+1}, y_{k-1})

That means $y_{k+1} = y_{k-1}$

Putting coordinates of S in (ii) then,

$P_{k+1} = P_k + 2(x_k+1) + (y_{k-1}^2 - y_k^2) - (y_{k-1} - y_k) + 1$

$= P_k + 2(x_k+1) + (y_k-1)^2 - y_k^2) - ((y_k-1) - y_k) + 1$

$= P_k + 2(x_k+1) + y_k^2 + 1 - 2y_k - y_k^2 - y_k + 1 + y_k + 1$

$= P_k + 2(x_k+1) - 2y_k + 2 + 1$

$= P_k + 2(x_k+1) - 2(y_k - 1) + 1$

As we know $(x_k+1 = x_{k+1})$ and $(y_k-1 = y_{k-1})$

Therefore,

$P_{k+1} = P_k + 2x_{k+1} - 2y_{k-1} + 1$

And if $P_k < 0$ that means midpoint is inside the circle and N is closest pixel so we will choose N (x_{k+1}, y_k)
i.e. $y_{k+1} = y_k$

Now put coordinates of N in (ii)

$$P_{k+1} = P_k + 2(x_k+1) + (y_k^2 - y_k^2) - (y_k - y_k) + 1$$
$$= P_k + 2(x_k+1) + (y_k^2 - y_k^2) - (y_k - y_k) + 1$$
$$= P_k + 2(x_k+1) + 1$$

as $x_k+1 = x_{k+1}$, therefore,

$$P_{k+1} = P_k + 2x_{k+1} + 1$$

Hence we have derived the mid point circle drawing algorithm.