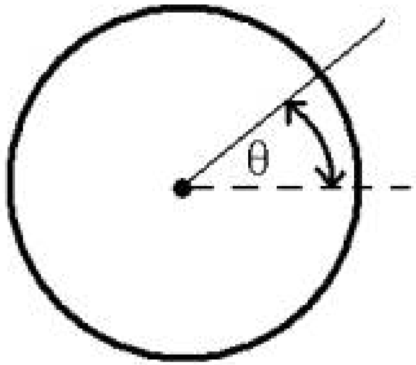
Lecture-6

Rotational Motion

Angular displacement

Angular displacement of a body is the angle in (radians, degrees, revolutions) through which a point or line has been rotated in a specified sense about a specified axis. If a body rotating about the rotation axis changes the angular position of the reference line from θ_1 to θ_2, the body undergoes an angular displacement $\Delta\theta$ given by,

$$\Delta\theta = \theta_2 - \theta_1$$



Angular displacement can be either positive or negative, depending on whether the body is rotating in the direction of increasing θ or decreasing θ.

Angular Velocity

- It is defined as the rate of change of angular displacement. Let a rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2. Then the average angular velocity of the body may be defined as

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

- Where $\Delta\,\theta$ is the angular displacement that occurs during the time interval Δt. When Δt approach to zero, average angular velocity is called instantaneous angular velocity and is given by,

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular acceleration

- It is the rate of change of angular velocity with time. If the angular velocity of rotating body is not constant, then the body has an angular acceleration. Let ω_1 and ω_2 be the angular velocities at time t_1 and t_2 respectively. Then the average angular acceleration of the body is defined as

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

- where $\Delta\omega$ is the angular velocity that occurs during the time interval Δt. When Δt approach to zero, average angular acceleration is called instantaneous angular acceleration and is given by,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Torque or moment of a force

Torque is the turning or twisting action on a body about a rotation axis due to a force F. If a force F acts on a single particle at a point P whose position with respect to the origin O of the inertial reference frame is given by the displacement vector r, the torque τ acting on the particle with respect to the origin O is defined as,

$$\tau = r \times F$$

Torque is a vector quantity. Its magnitude is given by

$$\tau = r\,F \sin\theta$$

where θ is the angle between r and F. Its direction is normal to the plane formed by r and F. The SI unit of torque is Newton-meter. A torque is positive if it tends to rotate a body at rest counterclockwise and negative if it tends to rotate the body in the clockwise direction.

Moment of inertia

Moment of inertia is a property of rotating bodies that defines its resistance to a change in angular velocity about an axis of rotation. It is the inertia of a rotating body with respect to its rotation. . Moment of inertia applies to an extended body in which the mass is constrained to rotate around an axis.

The moment of inertia of a body about an axis can be defined as the sum of the product of the mass of each particle and the square of its distance from the axis of rotation. It is given by

$$I = \sum mr^2$$

Moment of inertia

- ***Example:*** Let A be a rigid body which is rotating around a fixed axis YY/ with a uniform angular velocity as shown in fig. Let the body be composed of particles of masses m1, m2, m3, ……….mn and the particles are respectively at distances r_1, r_2, r_3,………….r_n.. Now, according to the definition,

The moment of inertia of the 1^{st} particle $= m_1 r_1^2$

The moment of inertia of the 2^{nd} particle $= m_2 r_2^2$

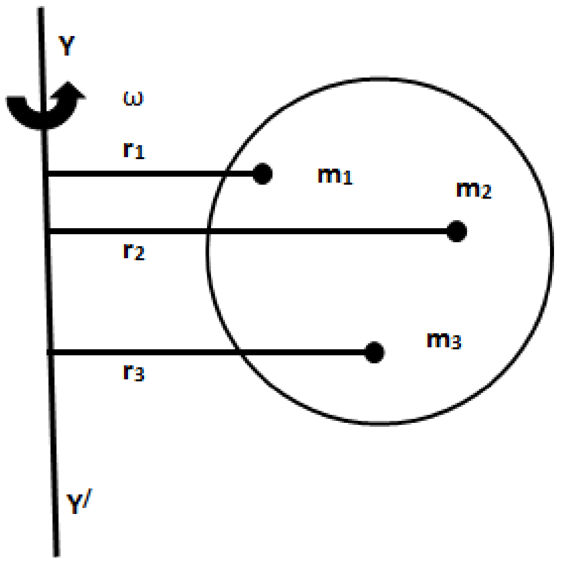
The moment of inertia of the 3^{rd} particle $= m_3 r_3^2$

The moment of inertia of the n^{th} particle $= m_n r_n^2$

So, the moment of inertia for the whole body is,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots\ldots\ldots + m_n r_n^2$$

$$I = \sum_{i=1}^{n} mr^2$$



In case of a body having a continuous and homogeneous distribution of matter,

$$I = \int r^2 dm$$

Where dm is the mass of infinitesimally small element of the body at distance r from the axis.

Radius of gyration

It is a distance of a point from an axis of rotation inside the body such that if the whole mass of a body is concentrated in a particle at that point,

its moment of inertia of the particle is equal to the moment of inertia of the original body about the same axis.

Explanation: In Fig-13, M is the mass of a rigid body and I is the moment of inertia of the body about the axis.

Now, by definition,

$$I = \sum mr^2 \qquad \ldots\ldots\ldots\ldots(1)$$

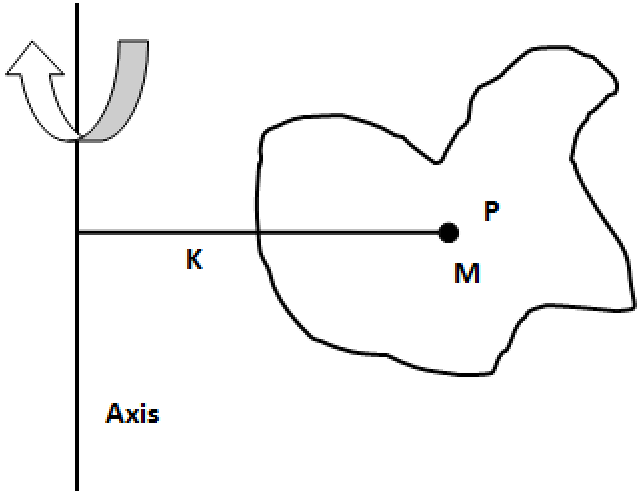
Let P be a point and M be the point mass placed at P.

The distance of the point mass from the axis of rotation = K.

So, moment of inertia of the point mass = MK^2 .....(2)

Radius of gyration

Equating eqns. (1) and (2), we get, $MK^2 = \sum mr^2 = I \therefore K = \sqrt{\frac{I}{M}}$, where K is called the radius of gyration.



Relation between angular momentum and angular velocity

Relation between angular momentum and angular velocity: Let an object rotates about an axis with angular velocity ω. If the object is composed of many small particles, then we can write, $L = L_1 + L_2 + L_3 + \ldots\ldots\ldots + L_n$, where $L_1, L_2, L_3, \ldots\ldots\ldots L_n$, are the angular momentum of individual particle.

Now, $L = r_1 p_1 + r_2 p_2 + r_3 p_3 + \ldots\ldots\ldots + r_n p_n$

$$= r_1 m_1 v_1 + r_2 m_2 v_2 + r_3 m_3 v_3 + \ldots\ldots\ldots + r_n m_n v_n.$$

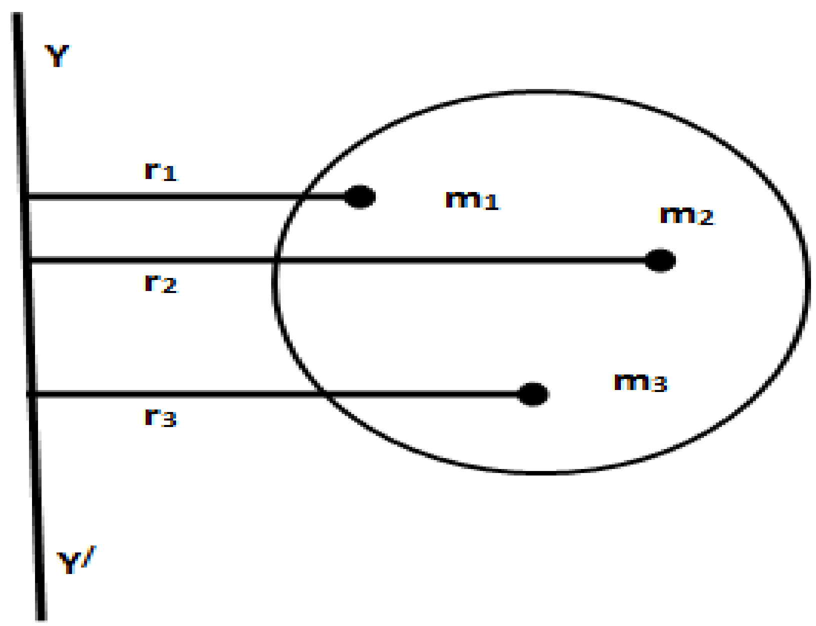
$$= r_1 m_1 (r_1\omega) + r_2 m_2 (r_2\omega) + r_3 m_3 (r_3\omega) + \ldots\ldots + r_n m_n (r_n\omega)$$

$$= \omega\,(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots\ldots\ldots + m_n r_n^2)$$

$$= \omega \sum m r^2 \quad = I\omega \qquad \therefore L = I\omega$$

Thus, the angular momentum of an object is the product of its momentum of inertia about the axis of rotation and its angular velocity.

Relation between angular momentum and angular velocity



Problems

<u>Example-7:</u> The mass of metal sphere is 6 g. It is rotated 4 times per second by fastening it at the end of a thread of length 3 m. What is its angular momentum?

<u>Solution:</u> We know, $L = I\omega = mr^2\omega = mr^2(2\pi/T)$ $[\because I = mr^2 \text{ and } \omega = 2\pi/T]$

$\therefore L = 0.006 \times (3)^2 \times 2 \times 3.14 / 0.25 = 1.356 \text{ kgm}^2/\text{s}.$ $[\because T = 1/n = ¼ = 0.25 \text{ s.}]$

Problems

Example-8: A heavy fly-wheel of radius 20 cm is set into motion by exerting a driving tension of 40 N in a belt passing round its circumference. If the fly-wheel reaches its operating speed of 140 rev/min in 10 sec. from rest, calculate its moment of inertia.

Here, $F = 40N$, $d = 20/100 = 0.2$ m

$\omega_0 = 0$,

$\omega = (150 \times 2\pi)/ 60 = 5\pi$ ra/sec

$t = 10$ s .

Solution: We have, $\tau = F \times d = 40 \times 0.2 = 8$ J.

Again, $\omega = \omega_0 + \alpha t \Rightarrow 5\pi = 0 + 10\alpha \Rightarrow \alpha = 5\pi/10 = 0.5\pi$

Again, $\tau = I \times \alpha \quad \Rightarrow 8 = I \times 0.5 \Rightarrow I = 8 / 0.5\pi$

$\Rightarrow I = 5.09$ kg m^2.

Kinetic energy of a rotating body

Let a rigid body rotate about a fixed axis at uniform angular velocity ω . In order to make the body rotational at this velocity from rest, work is to be done on the body which remains stored in the body as kinetic energy. This kinetic energy is the rotational kinetic energy.

Since the body is moving at uniform angular velocity ω, so every particle of the body rotates with the same angular velocity ω. But since the distance of each particle from the axis of rotation is different, so linear velocities will be different as well.

Kinetic energy of a rotating body

Let the body be composed of particles if masses $m_1, m_2, m_3, \ldots\ldots m_n$ and the distances from the axis of rotation are respectively $r_1, r_2, r_3, \ldots\ldots\ldots r_n$., the linear velocity of the particle m_1 is $v_1 = r_1\omega$.

So, kinetic energy $= \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$. Similarly, we can write,

Kinetic energy for particle of mass $m_2 = \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_2r_2^2\omega^2$

Kinetic energy for particle of mass $m_3 = \frac{1}{2}m_3v_3^2 = \frac{1}{2}m_3r_3^2\omega^2$

Kinetic energy for particle of mass $m_n = \frac{1}{2}m_nv_n^2 = \frac{1}{2}m_nr_n^2\omega^2$

Kinetic energy of a rotating body

So, kinetic energy for the whole body is,

$$K.E. = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots\ldots\ldots\ldots + \frac{1}{2} m_n r_n^2 \omega^2$$

$$= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots\ldots\ldots\ldots\ldots + m_n r_n^2)\, \omega^2$$

$$= \frac{1}{2} \sum mr^2\, \omega^2 \quad = \quad \therefore K.\,E. = \frac{1}{2} I\omega^2 .$$

When $\omega = 1$ unit, $K.\,E. = I/2 \Rightarrow I = 2 \times K.\,E.$

That is moment of inertia of a body rotating about a fixed axis at uniform velocity is numerically equal to twice the kinetic energy of the body. Alternately, kinetic energy of a body rotating about a fixed axis at uniform angular velocity equal to half of its moment of inertia.

Determination of moment of inertia and radius of gyration for some special cases:

1) ***Moment of inertia of a thin uniform rod about an axis through its centre and perpendicular to its length:***

Let AB be a thin uniform rod of length l and mass M free to rotate about the axis CD which is passing through the centre and perpendicular to the length of the rod as shown in Fig-14.

Since the rod is uniform, the mass per unit length = M/l. So, at a distance x from the axis CD let dx be a small length whose mass, $dM = (M/l)\ dx$.

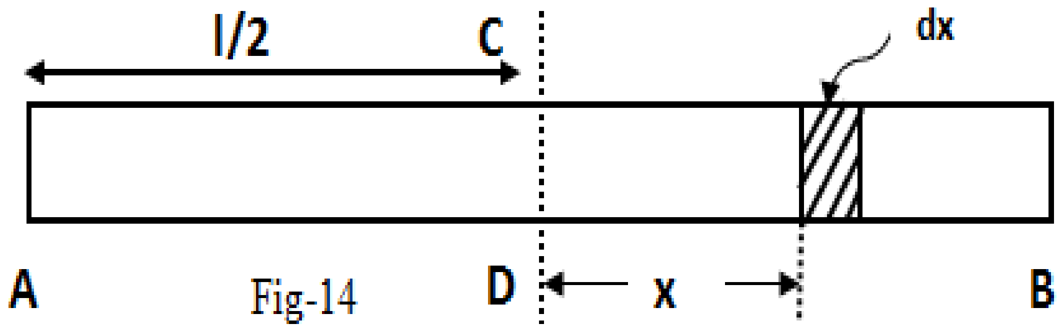


Fig-14

As dx is very small, we can consider all the particles in dx are at same distance from CD. So, moment of inertia of dx about the axis CD = $dM \times x^2 = (M/l)\ dx \times x^2$.

As dx is very small, we can consider all the particles in dx are at same distance from CD. So, moment of inertia of dx about the axis CD = dM × x^2 = (M/l) dx × x^2.

Now integrating the above equation within limits x = l/2 and x = - l/2, we get the moment of inertia for the entire rod. So, moment of inertia of the rod about the axis CD is,

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \left(\frac{M}{l}\right) x^2\, dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l}\left[\frac{l^3}{24} - \frac{l^3}{24}\right] = \frac{M}{l} \times \frac{2l^3}{24} = \frac{M}{12} l^2$$

$$\therefore I = = \frac{M}{12} l^2.$$

Let K be the radius of gyration. $\therefore MK^2 = I = \frac{Ml^2}{12} \Rightarrow K = \frac{l}{\sqrt{12}} = \frac{l}{2\sqrt{3}}$

2) ***Moment of inertia of a thin uniform rod about an axis at one end and perpendicular to its length:***

If the moment of inertia of the rod about one end, A, is required, then a similar calculation measuring x from A gives,

$$I = \int_0^l \left(\frac{M}{l}\right) x^2\, dx = \frac{M}{l}\int_0^l x^2\, dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{M}{l}\left[\frac{l^3}{3} - 0\right] = \frac{Ml^3}{3l} = \frac{Ml^2}{3}$$

$$\therefore I = \frac{Ml^2}{3}$$

If K is the radius of gyration, then $\therefore MK^2 = I = \frac{Ml^2}{3} \Rightarrow K = \frac{l}{\sqrt{3}}$

3) ***Moment of inertia of a uniform circular disc:*** We consider a uniform circular disc of mass M and radius R rotating about an axis passing through its centre and perpendicular to its plane.

Mass of the disc = M, Area of the disc = πR^2, Mass per unit area = $M/\pi R^2$.

Now, we consider a thin element of disc of radial thickness dx at a distance x from the centre.

The area of the element = $2\pi x \times dx$

So, the mass of the element = $\left(\frac{M}{\pi R^2}\right) 2\pi x\, dx = \left(\frac{2M}{R^2}\right) x\, dx$

Now, the moment of inertia of the element about the axis of rotation = $\left[\left(\frac{2M}{R^2}\right) x\, dx\right] x^2 = \left(\frac{2M}{R^2}\right) x^3\, dx.$

So, the moment of inertia of the whole disc about the axis of rotation,

$$I = \int_0^R \left(\frac{2M}{R^2}\right) x3\, dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4}\right] = \frac{MR^2}{2}$$

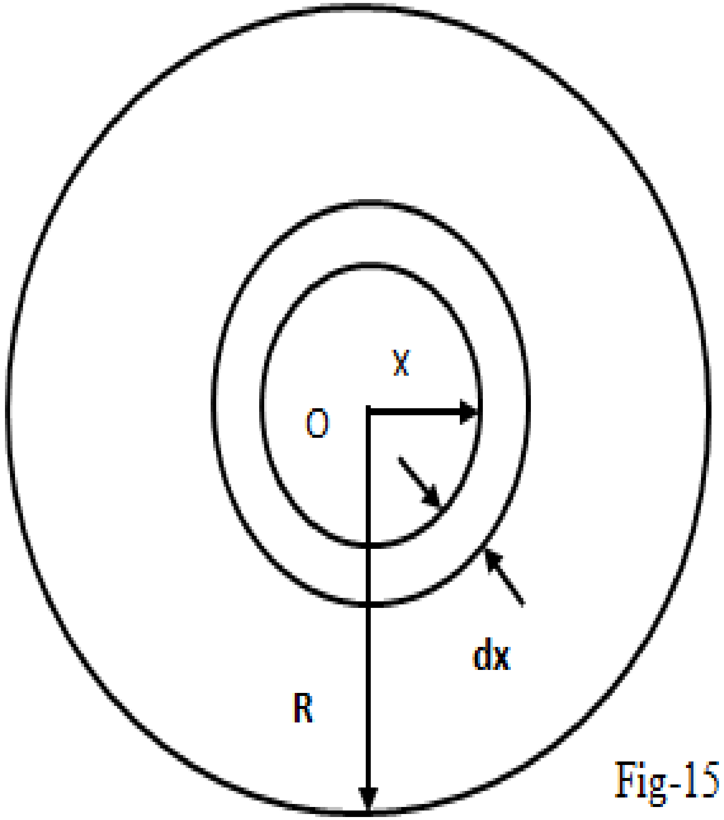


Fig-15

If K is the radius of gyration, then $\therefore MK^2 = I = \frac{MR^2}{2} \Rightarrow K = \frac{R}{\sqrt{2}}$

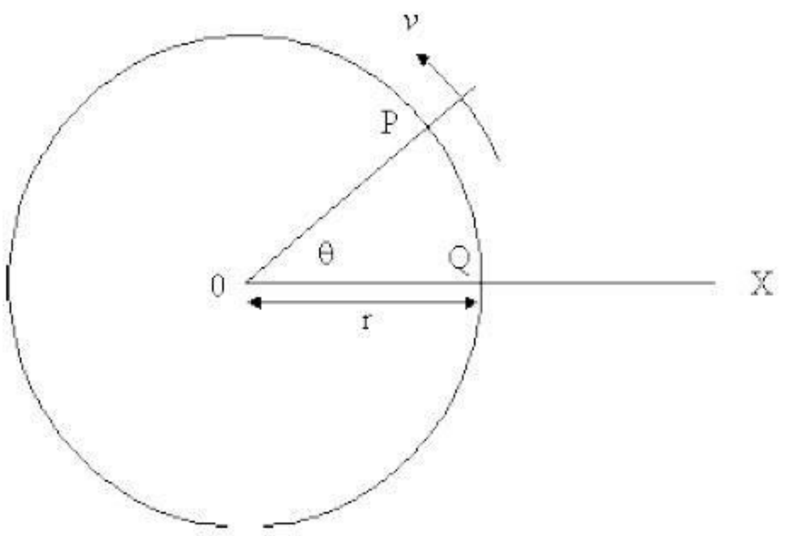
Relation between linear velocity and angular velocity:

If a point P move round a circle of radius r with constant linear velocity v , then the angular velocity will be

$\omega = \frac{\theta}{t}$……………..(1)

Where t is the time to move from Q to P along the arc QP of the curve. However, arc length QP is $r\,\theta$ when q is measured in radians. Hence linear speed v is

$$v = \frac{\text{Length of arc QP}}{t} = \frac{r\,\theta}{t} \;\ldots\ldots\ldots\ldots\ldots (2)$$

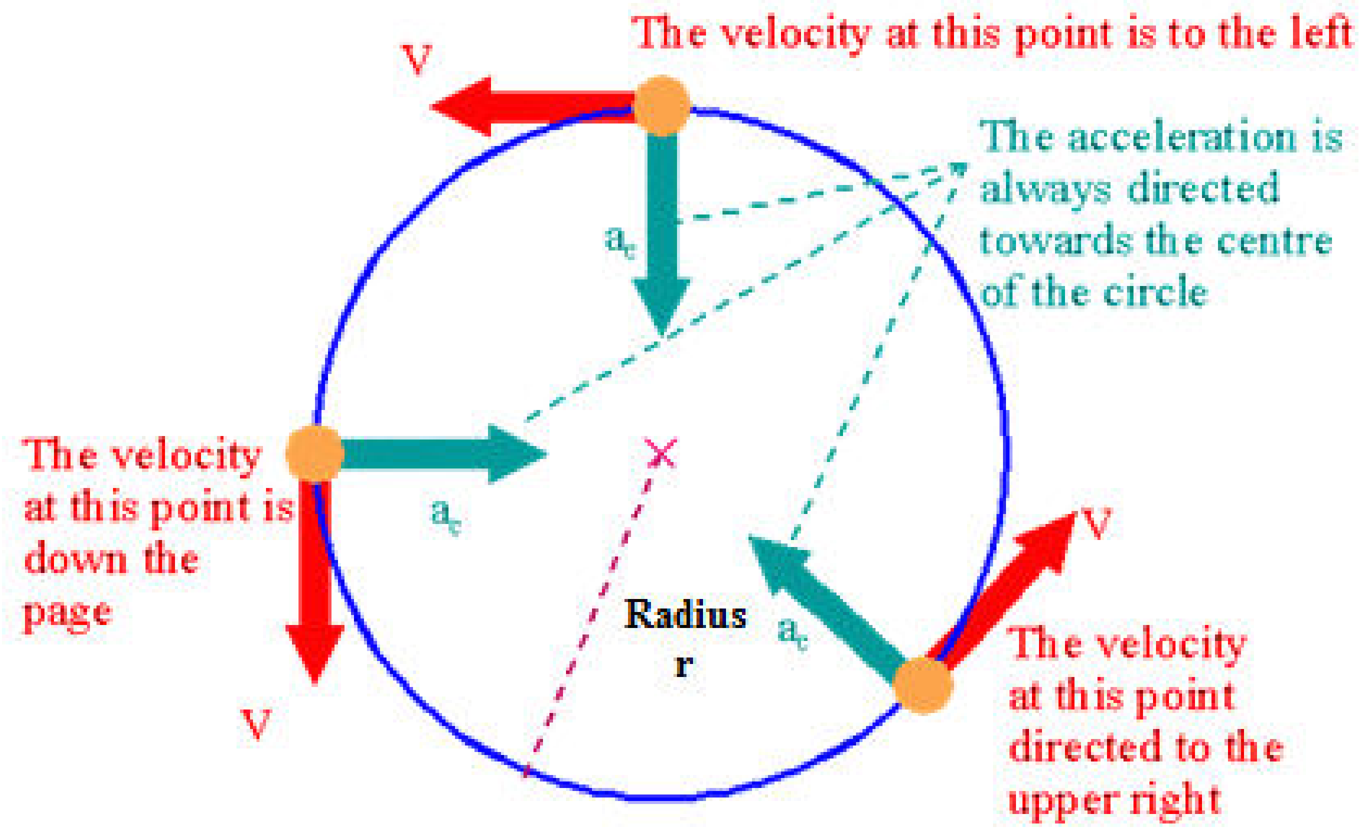


Substituting Equation (1) into Equation (2) leads to
 the relationship for circular motion

$V = \omega r$

Linear velocity = radius × Angular velocity

Centripetal Acceleration



Centripetal Acceleration

Centripetal acceleration is the rate of change of tangential velocity.

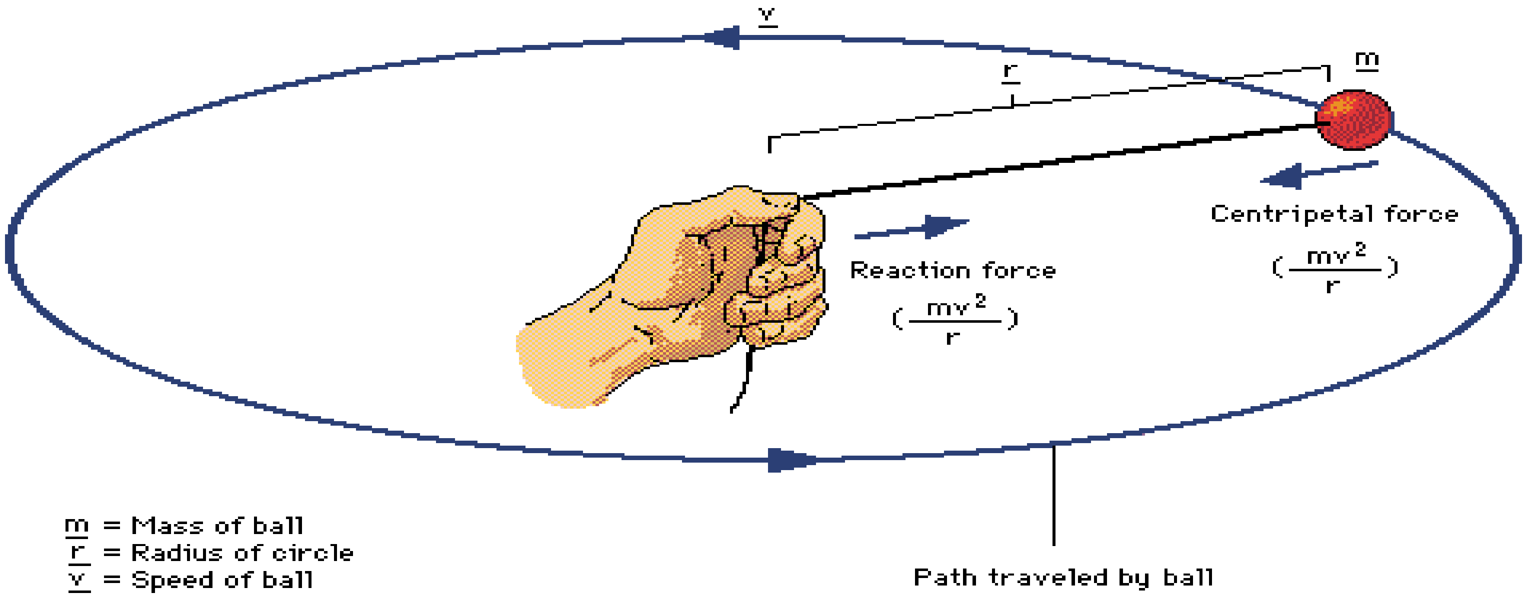
The direction of the centripetal acceleration is always inwards along the radius vector of the circular motion.

It can be denoted by **ac** and mathematically

$$\mathbf{a}_{\mathbf{centripetal}} = \frac{\mathbf{v}^2_{\mathbf{tangential}}}{\mathbf{r}} = \omega^2 \mathbf{r}$$

An object moving in a circular path of radius r with a constant speed v has an acceleration called centripetal acceleration. The acceleration directed towards the center of the circle.

Centripetal force is defined as the force which acts towards the center along the radius of a circular path on which the body is moving with a uniform velocity.

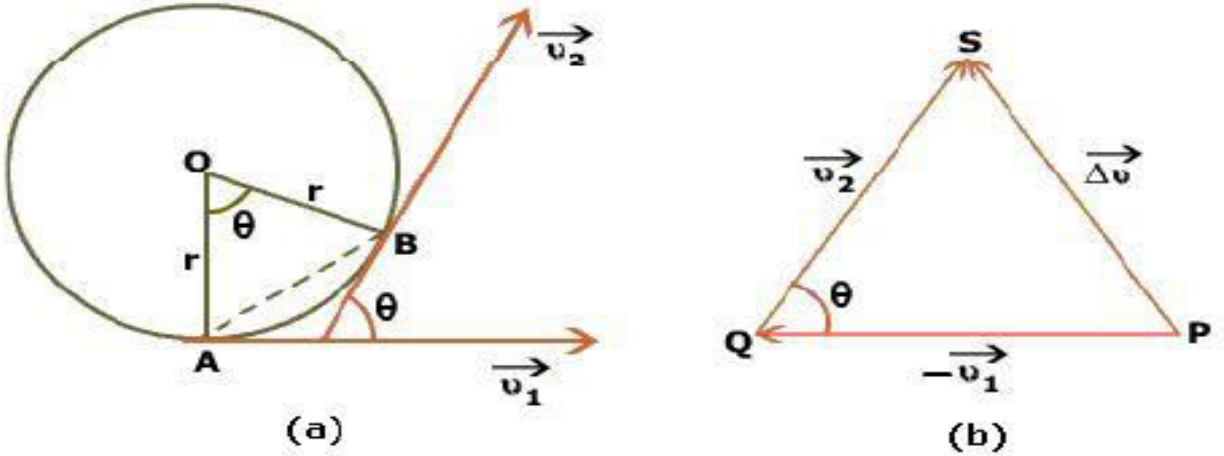


Relation between Centripetal Force and Acceleration

Consider, an object revolving a circle with constant speed v having radius r making an angle θ in the center as shown in figure (a).

Let, at point A the velocity is v_1 and after short time Δt at point B the velocity is v_2. So, the change in velocity is Δv.

From the figure (b), Velocity change $= \overrightarrow{v_2} - \overrightarrow{v_1} = \overrightarrow{v_2} + (-\overrightarrow{v_1}) = PS$



Expression for centripetal force

Relation between Centripetal Force and Acceleration

The magnitude of the acceleration is

$$a = \frac{\text{velocity change}}{\text{time}}$$

$$a = \frac{v \,.\, \Delta\theta}{\Delta t} \quad (\text{Since, } AB = v\,\Delta\theta\,)$$

$$a = v \,.\, \omega \quad (\because \text{ Angular velocity, } \omega = \frac{\Delta\theta}{\Delta t})$$

$$a = v \,.\, \frac{v}{r} \quad (\because v = \omega r)$$

$$a = \frac{v^2}{r} \ldots\ldots\ldots.(1)$$

or

$$a = r\,\omega^2$$

From Newton's 2^{nd} Law

Centripetal force = mass × acceleration

$$F = m \times \frac{v^2}{r} \qquad (\text{Using equation number 1})$$

$$F = \frac{m\,v^2}{r}$$

Problems

Example-9: A solid cylinder of mass 25 kg rotates about its axis with an angular speed 150 rad/s. The radius of the cylinder is 0.25 m. a) Calculate the moment of inertia, b) rotational kinetic energy and c) angular momentum of the cylinder?

Solution: a) We know, $I = MR^2/2 = 25 \times (0.25)^2 / 2 = 0.78\ kgm^2$.

b) We know, K. E. $= \frac{1}{2} I\omega^2 = 0.78 \times (150)^2 / 2 = 8775$ J

c) We know, $L = I\omega = 0.78 \times 150 = 117\ kgm^2/s$.

Problems

Example-10: The radius of a circular sheet is 0.3 m and its mass per unit square meter is 0.1 kg. Calculate moment of inertia and radius of gyration about the axis passing through the center and perpendicular to the plane of the sheet.

Solution: a) We know, $I = MR^2/2 = (\pi R^2 \times m \times R^2)/2 = \pi m \times R^4 = 3.14 \times 0.1 \times (0.3)^4 = 1.27 \times 10^{-3}\ kgm^2$.

b) We know, $K = \frac{R}{\sqrt{2}} = = \frac{0.3}{\sqrt{2}} = 0.212$ m.

Self Assesment-8: *A circular disc of mass 100 gm and radius 10 cm is making 120 revolution per minute about an axis passing through its center and perpendicular to its plane. Calculate it K.E. [Ans: 0.039 J]*

End of Lecture