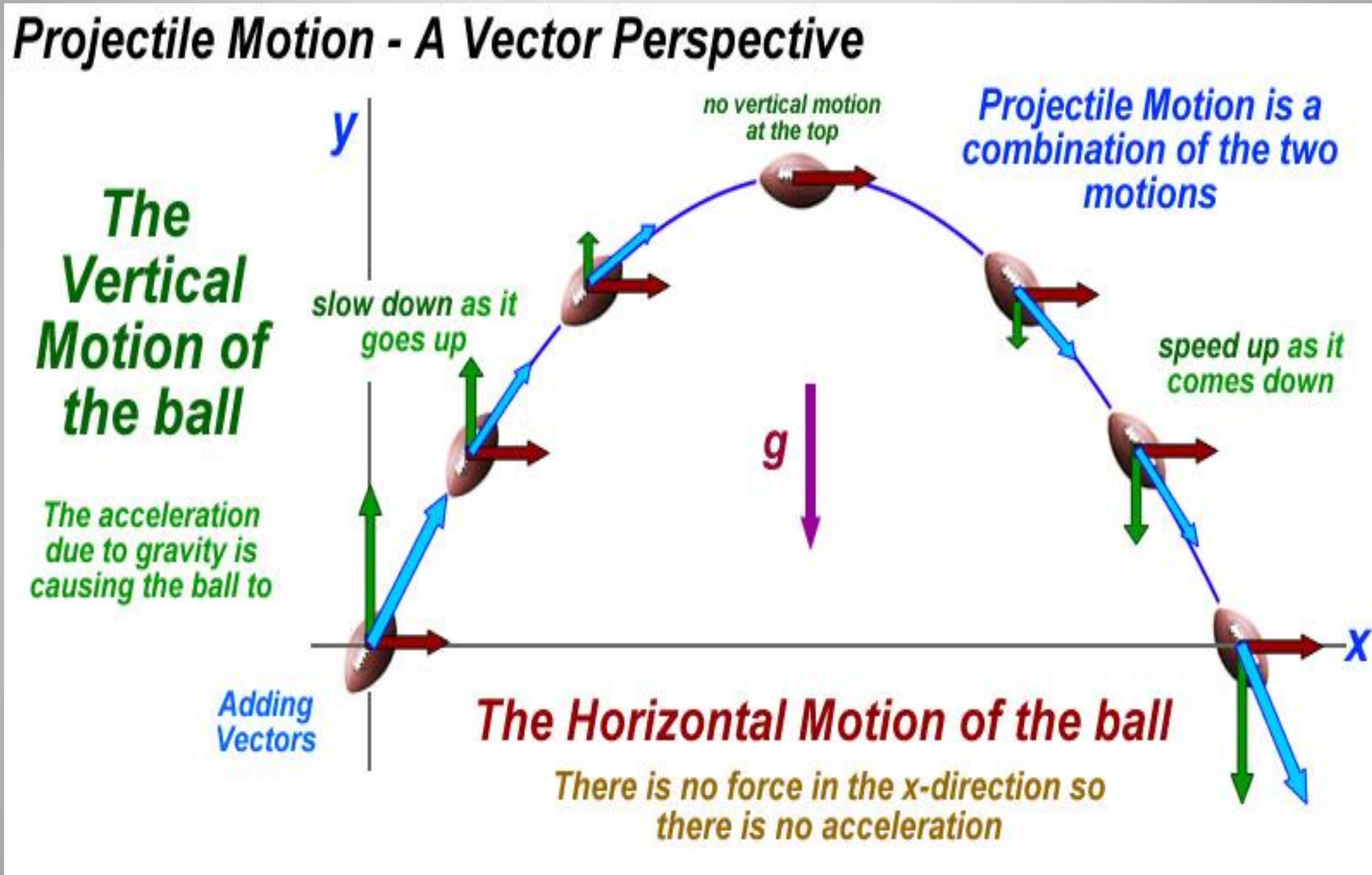
Lecture-3
Projectile Motion

- # Two dimensional motion

Projectile motion:

When an object is thrown obliquely into space, it is called projectile and its motion is called projectile motion. A projectile moves through the space under the influence of gravitational force. Two co-ordinate must be used to describe the projectile motion, since it moves horizontally as well as vertically. The motion of a football, cricket ball, missile etc. is examples of projectile motion.


Projectile Motion - A Vector Perspective
y
no vertical motion
at the top
Projectile Motion is a
combination of the two
motions
The
Vertical
Motion of
the ball
slow down as it
goes up
speed up as it
comes down
g
The acceleration
due to gravity is
causing the ball to
x
Adding
Vectors
The Horizontal Motion of the ball
There is no force in the x-direction so
there is no acceleration

- **Some definitions relating to projectile motion:**

- **Velocity of projection:** The initial velocity at which an object is thrown upward is called velocity of projection.

- **Angle of projection:** The angle between the velocity of projection and the horizontal plane is called angle of projection.

- **Time of flight:** The time taken from the point of projection and return to the ground is called time of flight.

- **Range:** The distance the point of projection and the point at which it falls on a plane is called the range.

- **Derivation of equation of motion of a projectile:**

- Let a projectile begins its flight from a point O with initial velocity v_0 and making an angle α with the horizontal direction as shown in fig-1 . Taking O as origin let the horizontal and vertical directions be considered along X and Y-axes. So, at $t=0$, the horizontal component of initial velocity,

Equation of motion of a projectile

$v_{x0} = v_0 \cos\alpha$ and vertical component, $v_{y0} = v_0 \sin\alpha$

Now from the equation of motion,

$v_x = v_{x0} + a_x t$, we get $v_x = v_{x0} = v_0 \cos\alpha \quad [\because a_x = 0]$

Equation of motion of a projectile

Let at t=t, the projectile reaches the point P, whose co-ordinate is (x,y) and where its velocity is $\vec{v}$. So, the

displacement of the projectile parallel to the ground i.e. along X-axis is, x= ON = $v_0 \cos\alpha \times t$

Or, $t = \dfrac{x}{v_0 \cos\alpha}$ ……………………(1)

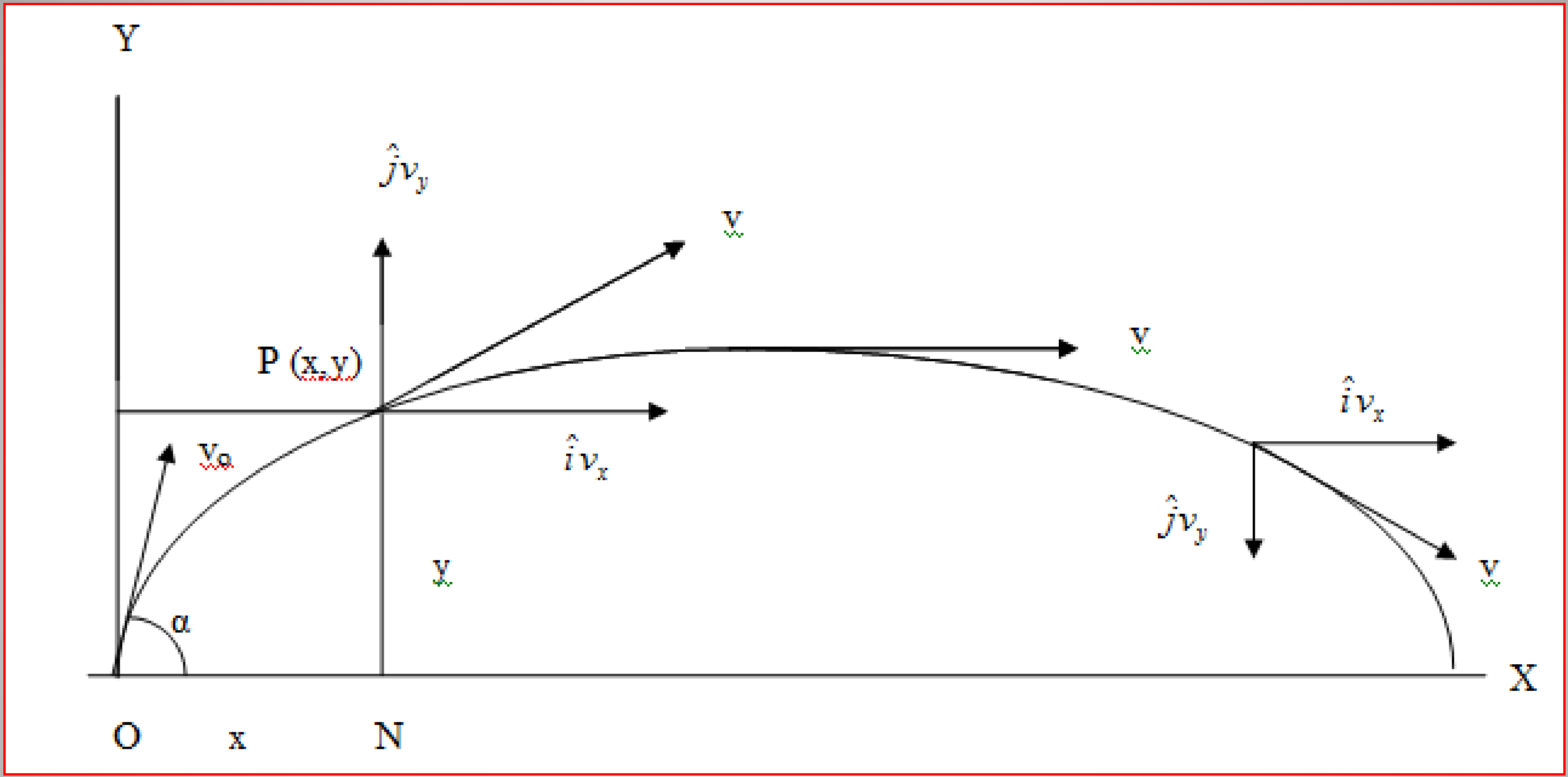


Figure: Projectile Motion

Equation of motion of a projectile

Now, the vertical component of the velocity is, $v_y = v_{y0} + a_y t$ or, $v_y = v_{y0} = v_0 \sin\alpha - gt$ $[\because a_y = -g]$

So, vertical displacement at t=t, $y = PN = y_0 + v_{y0} t + \frac{1}{2} a_y t^2 \Rightarrow y = 0 + v_0 \sin\alpha t - \frac{1}{2} g t^2$

$\Rightarrow y = v_0 \sin\alpha t - \frac{1}{2} g t^2$ ....(2)

Putting the value of t from equation (1) into equation (2), we have

$$y = v_0 \sin\alpha \times \frac{x}{v_0 \cos\alpha} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = x \tan\alpha - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha} = x \tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In equation (3) α, g and v_0 are constants. So, taking $\tan\alpha = b$ and $\frac{g}{2v_0^2 \cos^2\alpha} = c$ as constants eq. (3) can be written as, $y = bx - cx$ ……….(4). This is an equation of a parabola. Hence the path of motion (called trajectory) of a projectile is parabolic.

Maximum Height

Now, we will derive some important expressions relating projectile motion.

(i) Maximum height of the path of a projectile:

We know from equation of motion, $v^2 = v_0^2 + 2as$

Since the vertical component of the projectile is along Y-axis, so above equation reduces to,

$$v_y^2 = v_{y0}^2 + 2a_y y \text{ ....} (1)$$

Maximum Height

Let the maximum height reached by the projectile be H when final velocity v= 0 i.e., $v_y = 0$. So, we get from

equation (1), $0 = v_{y0}^2 - 2gH = v_0^2 \sin^2 \alpha - 2gH \Rightarrow H = \dfrac{v_0^2 \sin^2 \alpha}{2g}$ ---(2) $[\because a_y = -g \text{ and } v_{y0} = v_0 \sin \alpha]$

When $\alpha = 90^0$, the height will be maximum. So, $H = \dfrac{v_0^2}{2g}$ ....(3).

By knowing v_0 and g, H can be determined.

Time to reach maximum height

(ii) Time to reach maximum height: We know from equation of motion, $v = v_0 + at$.

Since the vertical component of the projectile is along Y-axis, so above equation reduces to,

$v_y = v_{y0} + a_y t$ ....(4)

Now, the vertical component of initial velocity $v_{y0} = v_0 \sin\alpha$ and at maximum height final velocity v = 0. So,

from eq. (4) we get, $0 = v_0 \sin\alpha - gt \Rightarrow t = \dfrac{v_0 \sin\alpha}{g}$ --------(5) $[\because a_y = -g]$

By knowing v_0, g and α, t can be determined.

Time of Flight

(iii) Time of flight: Let the time of flight be T. Now, we know the time of ascend to the maximum height = time of descent to the ground. So, $T = t + t = 2t = 2 \times \frac{v_0 \sin \alpha}{g} = \frac{2 v_0 \sin \alpha}{g}$ ....(6) [By using eq. (5)]

By knowing v_0, g and α, T can be determined.

Horizontal Range

(iv) Horizontal range: The linear distance from the oint of projection to the end of the flight is called the horizontal range. This is represented by R.

R= horizontal component of the initial velocity X time of flight

$$R = v_0 \cos\alpha \times T \Rightarrow R = v_0 \cos\alpha \times \frac{2v_0 \sin\alpha}{g} \Rightarrow R = \frac{v_0^2 \times 2\sin\alpha\cos\alpha}{g} \Rightarrow R = \frac{v_0^2 \sin 2\alpha}{g} \quad ....(7)$$

By knowing v_0, g and α, R can be determined.

Maximum Horizontal Range

(v) Maximum horizontal range: From eq. (7), it is evident that R will be maximum when $\sin 2\alpha = 1$

$\Rightarrow \sin 2\alpha = \sin 90^0 \Rightarrow 2\alpha = 90^0 \Rightarrow \alpha = 45^0$. In that case, $R_{max} = \dfrac{v_0^2}{g}$ ....(8)

That is, if an object is thrown at an angle 45^0 with the horizontal direction, the horizontal range will be maximum.

Problems Relating Projectile Motion

Problems relating projectile

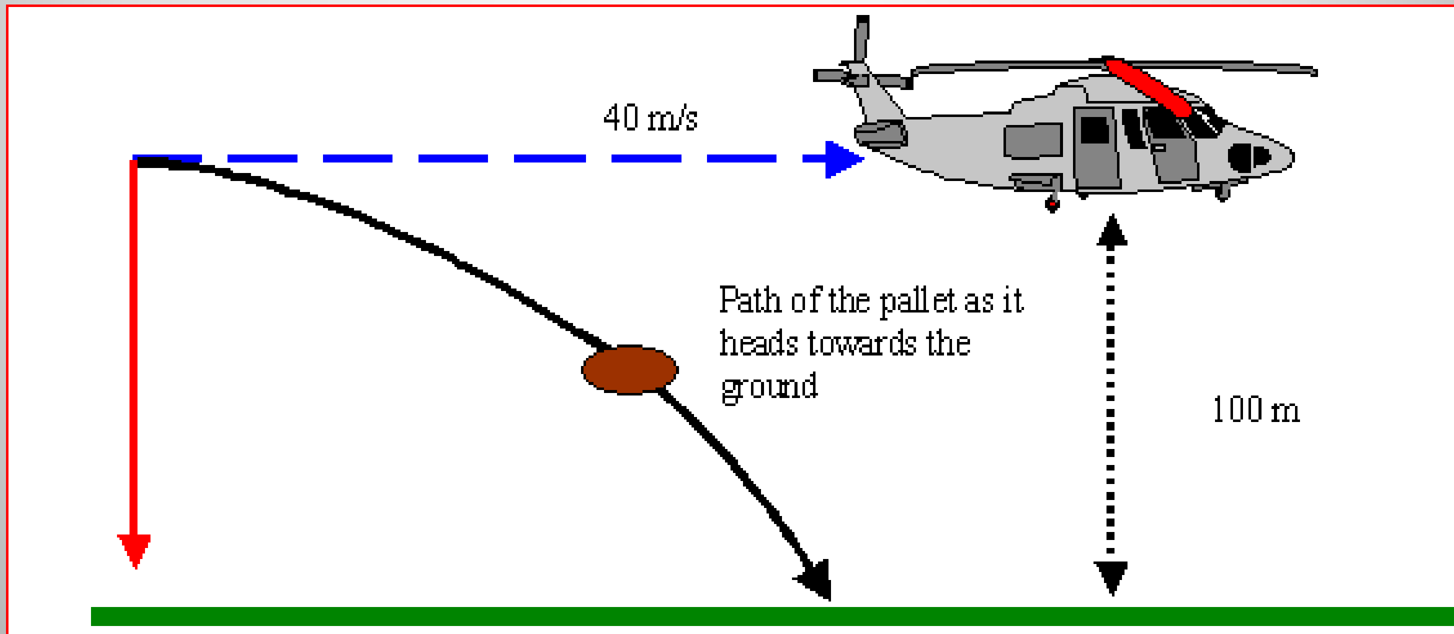
Prob-1: A bouncing ball leaves the ground with a velocity of 4.36 m/s at an angle of 81 degrees above the horizontal. a) How long did it take the ball to land? b) How high did the ball bounce? c) What was the ball's range?

Solution: a) We know, $T = \dfrac{2v_0 \sin\alpha}{g} = \dfrac{2 \times 4.36 \times \sin 81}{9.81} = 0.878$ s.

b) $H = \dfrac{v_0^2 \sin^2\alpha}{2g} = \dfrac{(4.36)^2 (\sin 81)^2}{2 \times 9.81} = 0.95 m/s$ c) $R = \dfrac{v_0^2 \sin 2\alpha}{g} = \dfrac{(4.36)^2 \sin(2 \times 81)}{9.81} = 0.60 m$

Problems Relating Projectile Motion

Prob-2: Look at the diagram below. A pallet is dropped from a helicopter to the ground. We will ignore the air resistance.



Problems Relating Projectile Motion

a) What is the horizontal velocity? B) Can you show that the vertical velocity is 44.7 m/s towards the ground? Note that the horizontal velocity is ignored, c) What is the resultant velocity of the pallet just before it hits the ground?

Solution: a) We know, $v_x = v_0 = 40m/s$ b) We have, $y = \frac{1}{2}gt^2 \Rightarrow t^2 = \frac{2y}{g} \Rightarrow t = \sqrt{\frac{2\times 100}{9.8}} = 4.52s$

And again $v_y = v_0 + gt = 0 + 9.80\times 4.52 = 44.3m/s \approx 44.7m/s(shown)$

c) $\therefore v = \sqrt{v_x^2 + v_y^2} = \sqrt{40^2 + 44.3^2} = 59.69m/s.$

Problems Relating Projectile Motion

Prob-3: The horizontal range of a projectile is 96m and its initial velocity is 66 m/s. What is the angle of projection?

Solution: We know,

$$R = \frac{v_0^2 \sin 2\alpha}{g} \Rightarrow \sin 2\alpha = \frac{Rg}{v_0^2} \Rightarrow 2\alpha = \sin^{-1}(\frac{Rg}{v_0^2}) \Rightarrow 2\alpha = \sin^{-1}(\frac{96 \times 9.8)}{(66)^2} \Rightarrow 2\alpha = 12.473$$

$$\Rightarrow \alpha = \frac{12.473}{2} \Rightarrow \alpha = 6.24^0$$

Self assessment: An object is thrown at velocity 40 m/s making an angle 600 with the horizontal plane. Find the maximum height and the horizontal range.