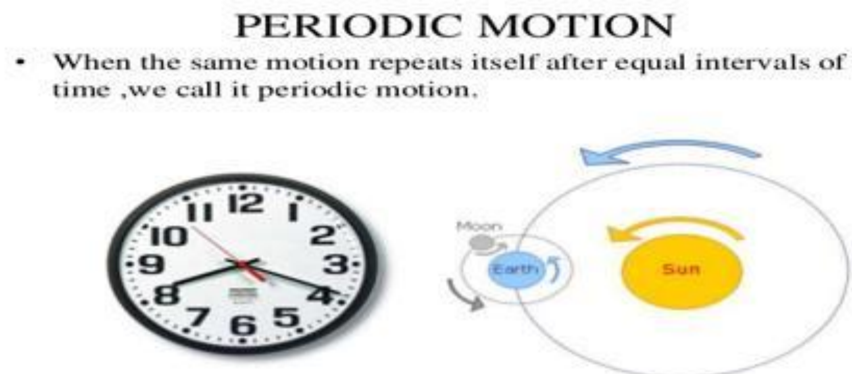
Periodic Motion:

If an object repeats its motion along a certain path, about a certain point, in a fixed interval of time, the motion of such an object is known as **periodic motion**. Examples of periodic motions are the motion of a pendulum, the motion of a spring, the vibration of a guitar string, the rotation of the Earth over its axis, the revolving of the Earth around the Sun, the revolving of the Sun around the center of the Galaxy,



Simple Harmonic Motion :

Whenever a force acting on a particle and hence the acceleration of the particle is proportional to its displacement from its equilibrium position or any other fixed point in its path, but is always directed in a direction opposite to the direction of the displacement and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute a simple harmonic motion.

Hooke's law:

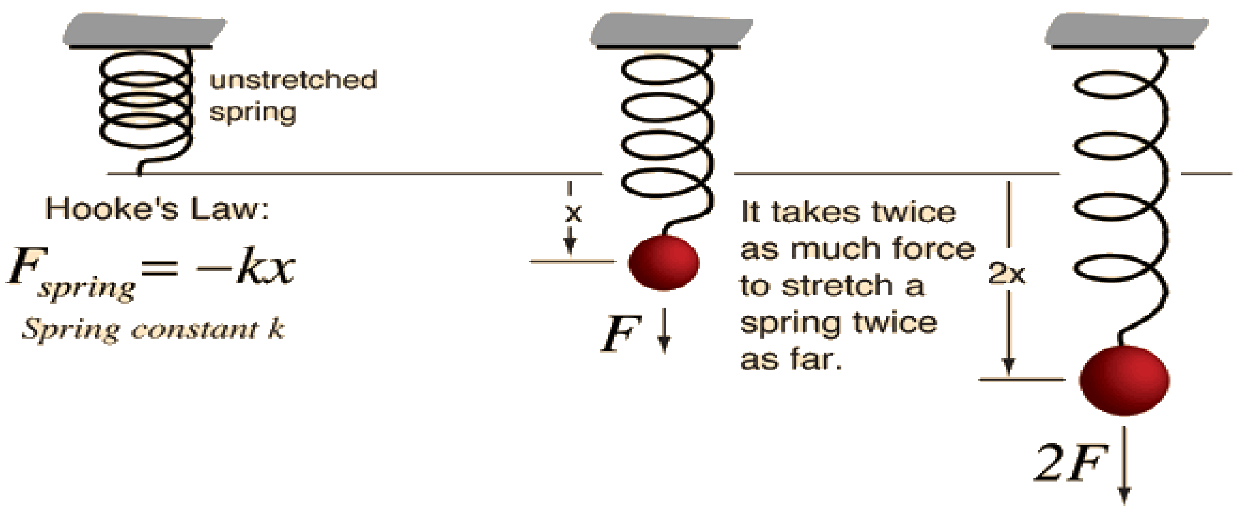
Hooke's law, law of elasticity discovered by the English scientist Robert Hooke in 1660.

This law states that, for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load. Under these conditions the object returns to its original shape and size upon removal of the load.

Mathematically, Hooke's law states that the restoring force of the spring (F) is directly proportional to the displacement or change in length x ,

That means, $F \propto -x$. or $F=-Kx$

The value of k depends not only on the kind of elastic material under consideration but also on its dimensions and shape.



Differential equation of simple harmonic motion:

If "F" be the force acting on a particle executing simple harmonic motion and "Y" it's displacement from its mean position, then

$$F \propto -Y$$

$$\therefore \; F = -KY \; \ldots\ldots\ldots\ldots(1)$$

According to newton's second law of motion, we have

$$F = ma \quad \ldots\ldots\ldots\ldots(2)$$

From equation (1) and (2)

$$ma = - ky$$

$$\Rightarrow m\frac{d^2y}{dt^2} = - ky$$

$$\Rightarrow \frac{d^2y}{dt^2} = - \frac{k}{m}y$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad \ldots\ldots\ldots\ldots (3)$$

When , $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. This equation is called the differential equation of motion of a body execating simple harmonic motion.

To obtain a general solution of the differential equation of simple harmonic motion , let us multiply both sides of equation (3) by $2\frac{dy}{dt}$,

we get

$$2\frac{dy}{dt} . \frac{d^2y}{dt^2} = \omega^2 y . 2\frac{dy}{dt}$$

$$\Rightarrow 2\frac{dy}{dt} . \frac{d^2y}{dt^2} = -2\omega^2 . y\frac{dy}{dt}$$

Integrating with respect to time, we have

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + c \quad \ldots\ldots\ldots\ldots(4)$$

Where, "C" is a constant of integration.

At maximum displacement, the velocity is zero.

$$\frac{dy}{dt} = 0 \text{ when } Y = a$$

From equation (4), we have, $0 = -\omega^2 y^2 + c$

$$\therefore c = \omega^2 a^2$$

Substituting the value of "c" in equation (4)

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + \omega^2 a^2$$

$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = \omega^2 (a^2 - y^2)$$

$$\Rightarrow \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{a^2 - y^2}} = \omega dt$$

$$\Rightarrow \int \frac{dy}{\sqrt{a^2 - y^2}} = \int \omega dt$$

$$\Rightarrow \sin^{-1}\frac{y}{a} = \omega t + \varphi$$

$$\Rightarrow \frac{y}{a} = \sin(\omega t + \varphi)$$

$$\therefore y = a \sin(\omega t + \varphi)$$

This is the general solution of the differential equation of simple harmonic motion.

Energy of a body executing simple harmonic motion :

The mechanical energy "E" of a particle executing simple harmonic motion is partly elastic and partly potential. If no non-conservative forces act on the particle, the sum of its kinetic energy and potential energy remains constant.

$$E = K + U = \text{constant}$$

Let the displacement of a particle executing simple harmonic motion at any instant be "Y". If the mass of the particle be "m" and its velocity at that instant be "V" then its kinetic energy is $\frac{1}{2}mv^2$.

The potential energy of the particle at the same is the amount of work that must be done in overcoming the force through a displacement "Y" and is given by the relation, $\int_0^y Fdy$ where "F" is the force required to maintain the displacement.

The displacement, $y = a\sin(\omega t + \varphi)$

∴ velocity, $\frac{dy}{dt} = a\omega\cos(\omega t + \varphi)$

∴ Acceleration, $\frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t + \varphi)$

$$= -\omega^2 y.$$

∴ Force, F = mass × acceleration $= m(-\omega^2 y) = -m\omega^2 y$

∴Potential energy of the particle , P.E $= \int_0^y Fdy$

$$= \int_0^y m\omega^2 y\,dy$$

$$= \frac{1}{2}m\omega^2 . y^2$$

$$= \frac{1}{2}m\omega^2 . a^2 \sin^2(\omega t + \varphi)$$

$$= \frac{1}{2}k\,a^2 \sin^2(\omega t + \varphi).$$

Kinetic energy of the particle is given by,K.E = $\frac{1}{2}mv^2$

$$=\frac{1}{2}m(\frac{dy}{dt})^2$$

$$=\frac{1}{2}m[\omega a\cos(\omega t + \varphi)]^2$$

$$=\frac{1}{2}\,\mathrm{m}\omega^2\,\mathrm{a}^2\,cos^2(\omega t + \varphi)$$

$$=\frac{1}{2}\,\mathrm{k}\,\mathrm{a}^2\,cos^2(\omega t + \varphi)$$

∴ Total energy, E = K+U

$$=\frac{1}{2}\,\mathrm{k}\,\mathrm{a}^2\,sin^2(\omega t + \varphi) + \quad =\frac{1}{2}\,\mathrm{k}\,\mathrm{a}^2\,cos^2(\omega t + \varphi)$$

$$=\frac{1}{2}\,\mathrm{ka}^2.$$

The total energy of the system is the same as the maximum value of any one of the two forms of energy. At the maximum displacement, the kinetic energy is zero but the potential energy has the value $\frac{1}{2}$ ka^2. At the position of equilibrium, the potential energy is zero but the kinetic energy has the value $\frac{1}{2}$ ka^2.

Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phase and amplitude.

Let two simple harmonic motion of the same frequency but of amplitude "a" and "b" and having their vibrations mutually perpendicular to one another.

If "φ" is the phase difference between the two vibrations, then

$x = \mathrm{a}\sin(\omega t + \varphi)$ ....(1)

$y = \mathrm{b}\sin\omega t$ .... (2)

From equation (1), $\frac{x}{a} = \sin(\omega t + \varphi) = \sin\omega t \cos\varphi + \cos\omega t \sin\varphi$

$$= \sin\omega t \cos\varphi + \sqrt{1 - sin^2\omega t} \sin\varphi \text{ .... (3)}$$

From equation (2), $\sin\omega t = \frac{y}{b}$

In equation (3) $\frac{x}{a} = \frac{y}{b}\cos\varphi + \sqrt{1 - \frac{y}{b^2}^2}\sin\varphi.$

$$\Rightarrow (\frac{x}{a} - \frac{y}{b}\cos\varphi)^2 = (1 - \frac{y}{b^2}^2)\, sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}\, cos^2\varphi - \frac{2xy}{ab}\cos\varphi = (1 - \frac{y}{b^2}^2)\, sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}\,(cos^2\varphi + sin^2\varphi) - \frac{2xy}{ab}\cos\varphi = sin^2\varphi$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\varphi = sin^2\varphi.$$

This is the general equation of the resultant vibration which gives rise to an ellipse.

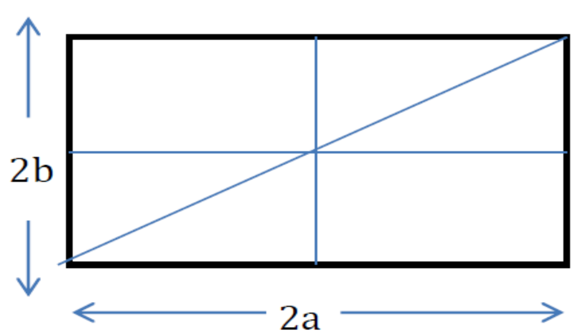


Fig 1: Φ= 0

Case I:
Φ= 0,2π,4π...........=2nπ where n= 0,1,2 ....
Since there is no phase difference between the two vibrations,

Φ= 0 and sin Φ=0 and cos Φ=1,
Putting these values in equation (4), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$$

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2=0$$

$$\frac{x}{a}-\frac{y}{b}=0$$

$$y=\frac{b}{a}x$$

This is the equation of a straight line passing through the origin and inclined to the direction of first motion that is the X axis.

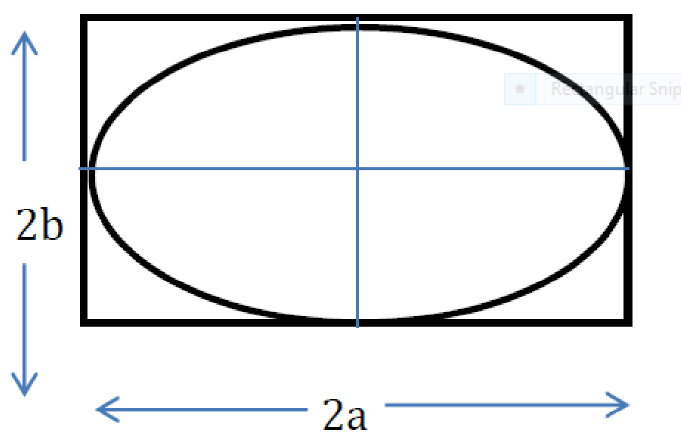


Fig 2: $\Phi=\frac{\pi}{2}$

Case II:
$\Phi=\frac{\pi}{2}$ radian, sinΦ=1 and cos Φ=0

Hence equation (4) becomes,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This equation represents a symmetrical ellipse whose center coincides with the origin.

Composition of two simple harmonic vibrations in a straight line:

Let two simple harmonic vibrations be represented by the equations.

$$y_1 = a_1 \sin(\omega t + \varphi_1)$$

$$y_2 = a_2 \sin(\omega t + \varphi_2)$$

Where, y_1 and y_2 are the displacement of the particle due to the individual vibrations of amplitudes a_1 and a_2 respectively.

$$\therefore y = y_1 + y_2$$

$$= a_1 \sin(\omega t + \varphi_1) + a_2 \sin(\omega t + \varphi_2)$$

$$= a_1(sin\omega t \cos\varphi_1 + cos\omega t\ sin\varphi_1) + a_2(sin\omega t \cos\varphi_2 + cos\omega t\ sin\varphi_2)$$

$$= (a_1 \cos\varphi_1 + a_2 \cos\varphi_2)\ sin\omega t + (a_1 sin\varphi_1 + a_2 sin\varphi_2)\ cos\omega t$$

The amplitudes a_1 and a_2 are constant .Hence putting,

$$a_1 \cos\varphi_1 + a_2 \cos\varphi_2 = A \cos\varphi$$

$$a_1 sin\varphi_1 + a_2 sin\varphi_2 = A \sin\varphi$$

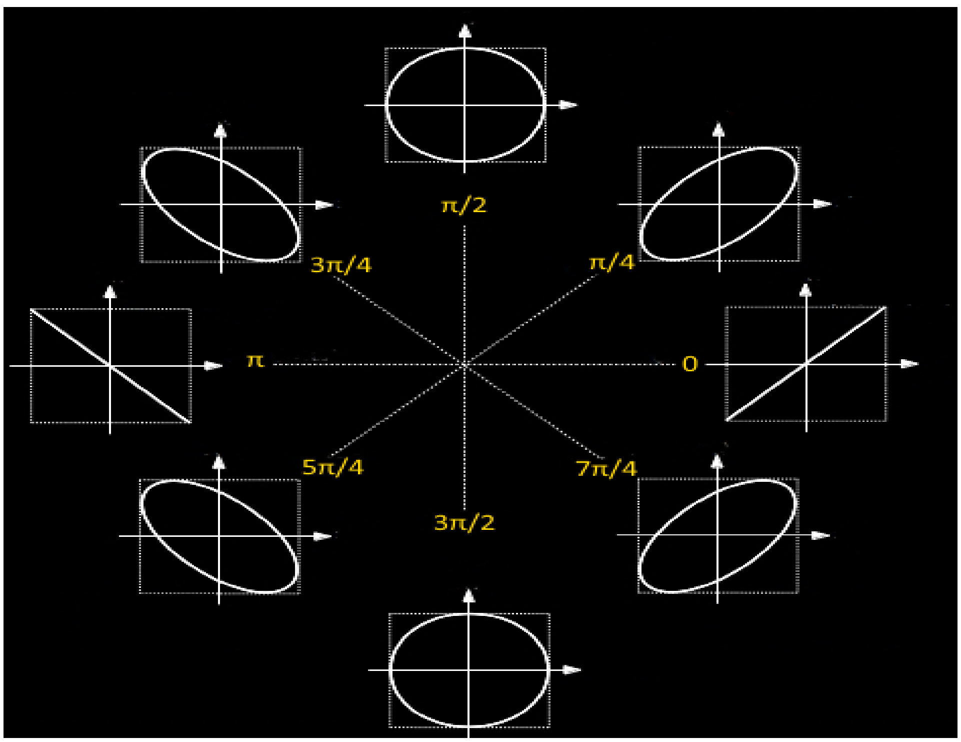
∴The resultant amplitude can be written as,

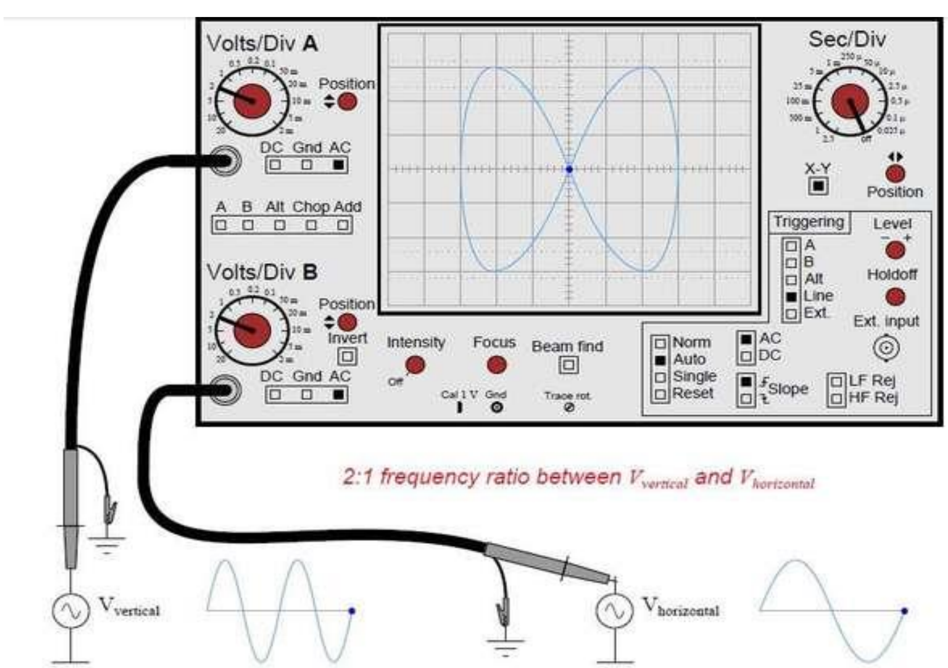
$$Y = A \cos\varphi \sin \omega t + A \sin\varphi \cos \omega t .$$

$$= \mathrm{A} \sin (\omega t + \varphi).$$

Lissajous Figure:

The composition of two simple harmonic vibrations in mutually perpendicular directions gives rise to an elliptical path. The actual shape of the curve will depend on time period, phase difference and on the amplitudes of the constituent vibrations. These figures or curves are known as Lissajous figures. Lissajous figures are helpful in determining the ratio of the time periods of two vibrations and to compare the frequencies of two tuning fork and can be used to detect the phase shift of two signals with a same frequency.





Some terms related to simple harmonic motion:

Time period: It is the time required for one complete oscillation.

$$T=\frac{2\pi}{\omega} \Rightarrow T = 2\pi/\omega \Rightarrow T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \quad \therefore T = 2\pi\sqrt{\frac{m}{k}}$$

Frequency: It is the number of complete oscillations per unit time. It is denoted by f and is the reciprocal of time period T.

$\therefore f = 1 / T = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ and angular frequency, $\omega = 2\pi f = \frac{2\pi}{2\pi}\sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m}}$

Velocity: We have

displacement, $x = A \sin(\omega t + \theta_0)$

$\Rightarrow \sin(\omega t + \theta_0) = x/A$

velocity, $v = dx/dt = \frac{d}{dt}\{A \sin(\omega t + \theta_0)\} = A\omega \cos(\omega t + \theta_0) = A\omega,$

$$V = \sqrt{1 - \sin^2(\omega t + \theta_0)} = A\omega\sqrt{1 - \frac{x^2}{A^2}}$$

$$= A\omega\sqrt{\frac{A^2 - x^2}{A^2}} = \frac{A\omega}{A}\sqrt{A^2 - x^2} = \omega\sqrt{A^2 - x^2} \quad \therefore v = \omega\sqrt{A^2 - x^2}$$

Acceleration:

$$a = dv/dt = \frac{d}{dt}\{A\omega \cos(\omega t + \theta 0)\} = -\omega^2 A \sin(\omega t + \theta 0) = -\omega^2 x \quad \therefore A = -\omega^2 x$$

From this we see that, acceleration $a \propto -x$ which is the condition of SHM.

Amplitude:

Amplitude is the maximum displacement on both sides of an object from its equilibrium position. The SI unit for amplitude is meter (m).

Phase:

Phase can be defined as an expression of relative displacement between two corresponding features (for example, peaks or zero crossings) of two waveforms having the same frequency.

Problems:

Prob-1: The amplitude and frequency of an object executing SHM are 0.01 m and 12 Hz, respectively. What is the velocity of the object at displacement 0.005 m? What is the maximum velocity of the object?

Solution:

a)

Weknow,

$$v = \omega\sqrt{A^2 - x^2} \Rightarrow v = 2\pi f\sqrt{A^2 - x^2} \Rightarrow v = 2\times 3.14\times 12\times\sqrt{(0.001)^2 - (0.005)^2}$$

=0.653m/s.

b) We know, when x = 0, then v = v_{max}.

so, $v_{max} = \omega A = 2\times 3.14\times 12\times 0.01 = 0.7536 m/s$

Prob-2: A spring hung vertically, is found to be stretched by 0.02 m from its equilibrium position when a force of 4 N acts on it. Then a 2 kg body is attached to the end of the spring and is pulled 0.04 m from its equilibrium

position along the vertical line. The body is then released and it executes simple harmonic motion. Find: 1) force constant 2) force executed by the spring on the 2 kg mass 3) period and frequency of oscillation 4) amplitude of motion and 5) maximum velocity.

Solution: 1) F =kx, k = F/x = 4/0.02 = 200 N/m.

2) F =kx = 200 × 0.04 = 8 N

3) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{200}}$ = π /5 = 0.628 s; f = 1/T = 1/ 0.628 = 1.59 Hz

$\therefore \omega = 2\pi f = 2 \times 3.14 \times 1.59 = 10$ Hz

4) Amplitude, A = 0.04 m

5) Maximum velocity occurs when the body passes the mean position where x = 0.

Hence $v = \pm\, \omega\sqrt{A^2 - x^2} = \omega A = 10 \times 0.04 = 0.4$ m/s

Prob-3: The displacement of an oscillating particle at an instant is given by x = a cosωt + b sinωt.

Show that it is executing a simple harmonic motion. If a = 5 cm, b = 12 cm and ω = 4 radian/second, calculate (i) the amplitude, (ii) the time period, (iii) the maximum velocity and (iv) the maximum acceleration of the particle.

Solution: We have, $x = a\cos\omega t + b\sin\omega t$

$$\frac{dx}{dt} = -a\omega\ sin\omega t + b\omega\ cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 a\ cos\omega t - \omega^2 b\ sin\omega t\ = -\omega^2(a\ cos\omega t + b\ sin\omega t)\ = -\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

Hence, the motion is simple harmonic. (Shown).

Now, (i) Let $a = A\sin\alpha$ ....(1) and $b = A\cos\alpha$.......(2)

Then, $x = A\sin\alpha\cos\omega t + A\cos\alpha\sin\omega t = A\sin(\omega t + \alpha)$, That is $x = A\sin(\omega t + \alpha)$

This represents a simple harmonic motion with amplitude A.

∴ Now squaring eqs. (1) and (2) and then adding, we obtain

$$A^2\sin^2\alpha + A^2\cos^2\alpha = a^2 + b^2 \quad => A^2 = a^2 + b^2$$

$$=> A = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = 13\text{ cm}$$

(ii) we have, $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ second

(iii) Again, $v = \omega A = 4 \times 13 = 52$ cm/s

(iv) $a_{max} = -\omega^2 A = -(4)^2 \times 13 = -208$ cm/s^2.

Damped Vibrations/Oscillations:

In actual practice, a simple harmonic oscillator always vibrates in a resisting medium. Consequently, when the oscillator vibrates in such a medium, energy is dissipated in each vibration goes on decreasing progressively with time. Such forces which are non-conservative in nature have a damping effect on the oscillation.

Damping Coefficient:

A body executing simple harmonic oscillations in a damping medium will be simultaneously subjected to the following two opposing forces:

The restoring force acting on the body which is proportional to the displacement of the body and acts in a direction opposite to the displacement. Let this force **–ay** , where **a** is the force constant.

The other one is the resistive or damping force which is proportional to the velocity of the oscillating body. The damping or resistive force may be represented by

$$F = -b\,v = -b\frac{dy}{dt}$$

The differential equation may be written as,

$$m\frac{d^2y}{dt^2} = -ay - b\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ay = 0$$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{a}{m}y = 0$$

$$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = 0$$

This equation is referred as the differential equation of a damped harmonic oscillator.