Data Structures – LECTURE 14

Strongly connected components

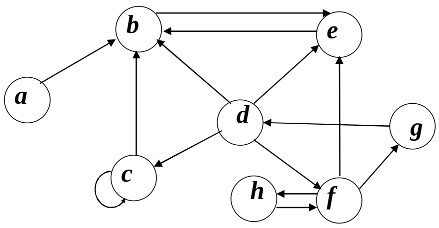
- Definition and motivation
- Algorithm

Chapter 22.5 in the textbook (pp 552—557).

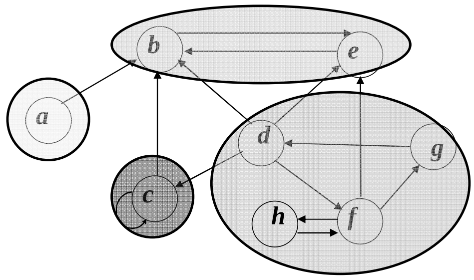
Connected components

- Find the largest components (sub-graphs) such that there is a path from any vertex in it to any other vertex.
- Applications: networking, communications.
- Undirected graphs: apply BFS/DFS (inner function) from a vertex, and mark vertices as *visited*. Upon termination, repeat for every unvisited vertex.
- Directed graphs: strongly connected components, not just connected: a path from u to v AND from v to u, which are not necessarily the same!

Example: strongly connected components



Example: strongly connected components



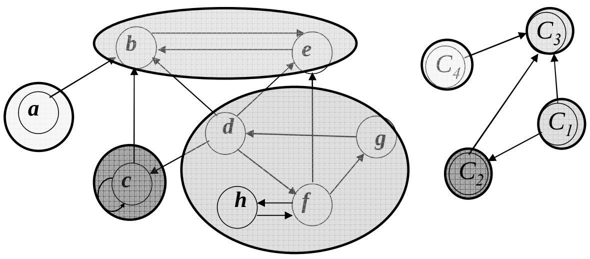
Strongly connected components

- Definition: the strongly connected components (SCC) $C_1, \ldots, C_k$ of a directed graph $G = (V,E)$ are the *largest* disjoint sub-graphs (no common vertices or edges) such that for any two vertices u and v in C_i, there is a path from u to v and from v to u.
- Equivalence classes of the binary *path*(u,v) relation, denoted by $u \sim v$. The relation is not symmetric!
- Goal: compute the strongly connected components of G in linear time $\Theta(|V|+|E|)$.

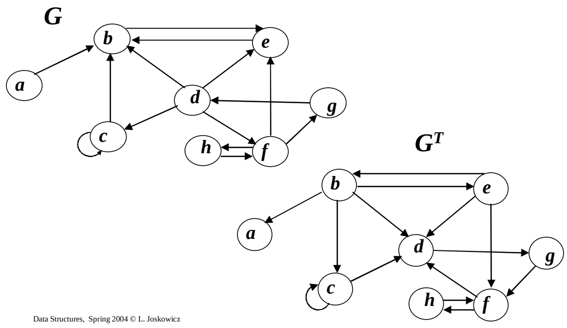
Strongly connected components graph

- Definition: the *SCC graph* $G^{\sim} = (V^{\sim},E^{\sim})$ of the graph $G = (V,E)$ is as follows:
  - $V^{\sim} = \{C_1, \ldots, C_k\}$. Each SCC is a vertex.
  - $E^{\sim} = \{(C_i,C_j) \mid i \neq j$ and $(x,y) \in E$, where $x \in C_i$ and $y \in C_j\}$. A directed edge between components corresponds to a directed edge between them from any of their vertices.
- $\boldsymbol{G^{\sim}}$ is a directed acyclic graph (no directed cycles)!
- Definition: the *transpose graph* $G^T = (V,E^T)$ of the graph $G = (V,E)$ is G with its edge directions reversed: $E^T = \{(u,v) \mid (v,u) \in E\}$.

Example: SCC graph



Example: transpose graph G^T



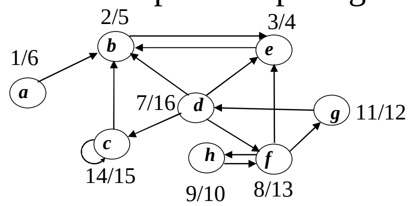
SCC algorithm

Idea: compute the SCC graph $G^{\sim} = (V^{\sim}, E^{\sim})$ with two DFS, one for G and one for its transpose G^T, visiting the vertices in reverse order.

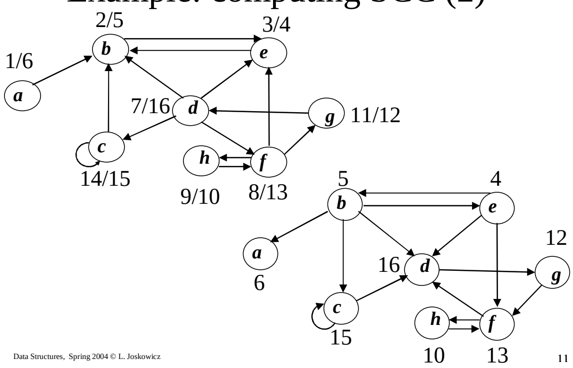
SCC(G)

1. DFS(G) to compute finishing times $f[v]$, $\forall v \in V$
2. Compute G^T
3. DFS(G^T) in the order of decreasing $f[v]$
4. Output the vertices of each tree in the DFS forest as a separate SCC.

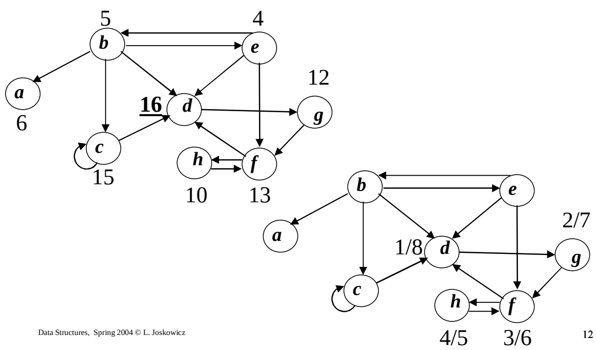
Example: computing SCC (1)



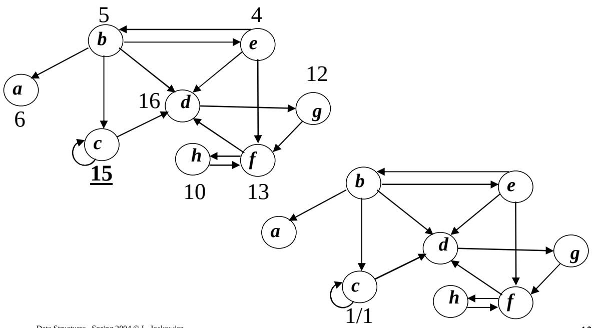
Example: computing SCC (2)



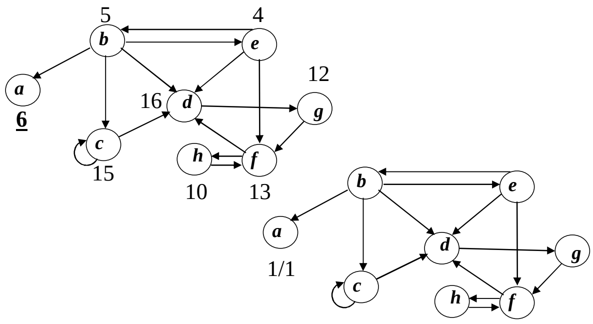
Example: computing SCC (3)



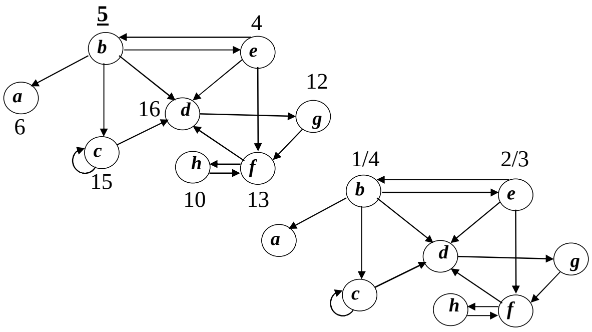
Example: computing SCC (4)



Example: computing SCC (5)

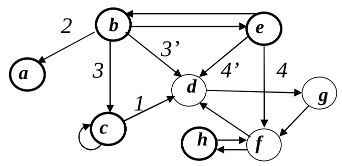


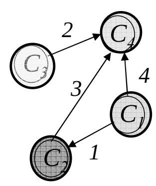
Example: computing SCC (6)



Example: computing SCC (2)

Labeled transpose graph G^{T}





Proof of correctness: SCC (1)

Lemma 1: Let C and C' be two distinct SCC of $G = (V,E)$, let $u,v \in C$ and $u',v' \in C'$. If there is a path from u to u', then there cannot be a path from v to v'.

Definition: the start and finishing times of a set of vertices $U \subseteq V$ is:

$$d[U] = min_{u\in U}\{d\,[U]\}$$

$$f\,[U] = min_{u\in U}\{f\,[U]\}$$

Proof of correctness: SCC (2)

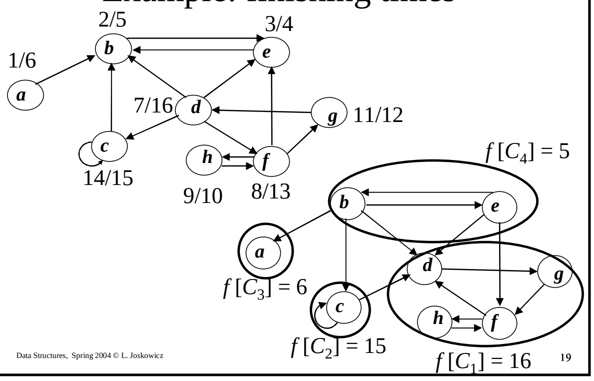
Lemma 2: Let C and C' be two distinct SCC of G, and let $(u,v)\in E$ where and $u\in C$ and $v\in C'$. Then, $f\,[C] > f\,[C']$

Proof: there are two cases, depending on which strongly connected component, C or C' is discovered first.

1. C was discovered before C': $d(C) < d(C')$

Let x be the first vertex discovered in C. There is a path in G from x to each vertex of C which has not yet been discovered. Because $(u,v)\in E$, for any vertex $w\in C'$, there is also a path at time $d[x]$ from x to w in G consisting only of unvisited vertices: $x \rightarrow u \rightarrow v \rightarrow w$. Thus, all vertices in C and C' become descendants of x in the depth-first tree. Therefore, $f\,[x] = f\,[C] > f\,[C']$.

Example: finishing times



Proof of correctness: SCC (3)

2. <u>$d(C) > d(C')$</u>

Let y be the first vertex discovered in C'. At time $d[y]$, all vertices in C' are unvisited. There is a path in G from y to each vertex of C' which has only vertices not yet discovered. Thus, all vertices in C' will become descendants of y in the depth-first tree, and so $f[y] = f[C']$. At time $d[y]$, all vertices in C are unvisited. Since there is an edge (u,v) from C to C', there cannot, by Lemma 1, be a path from C' to C. Hence, no vertex in C is reachable from y. At time $f[y]$, therefore, all vertices in C are unvisited. Thus, no vertex in C is reachable from y. At time $f[y]$, therefore, all vertices in C are still unvisited. Thus, for anuy vertex w in C:

$$f[w] > f[y] \rightarrow f[C] > f[C'].$$

Proof of correctness: SCC (4)

<u>Corollary</u>: for edge $(u,v) \in E^T$, and $u \in C$ and $v' \in C'$
$f[C] < f[C']$

This provides the clue to what happens during the second DFS.

The algorithm starts at x with the SCC C whose finishing time $f[C]$ is maximum. Since there are no vertices in G^T from C to any other SCC, the search from x will not visit any other component!

Once all the vertices have been visited, a new SCC is constructed as above.

Proof of correctness: SCC (4)

<u>Theorem</u>: The SCC algorithm computes the strongly connected components of a directed graph G.

<u>Proof</u>: by induction on the number of depth-first trees found in the DFS of G^T: the vertices of each tree form a SCC. The first k trees produced by the algorithm are SCC.

<u>Basis</u>: for $k = 0$, this is trivially true.

<u>Inductive step</u>: The first k trees produced by the algorithm are SCC. Consider the $(k+1)^{st}$ tree rooted at u in SCC C. By the lemma, $f[u] = f[C] > f[C']$ for SCC C' that has not yet been visited.

Proof of correctness: SCC (3)

When u is visited, all the vertices v in its SCC have not been visited. Therefore, all vertices v are descendants of u in the depth-first tree.

By the inductive hypothesis, and the corollary, any edges in G^T that leave C must be in SCC that have already been visited. Thus, no vertex in any SCC other than C will be a descendant of u during the depth first search of G^T. Thus, the vertices of the depth-first search tree in G^T that is rooted at u form exactly one connected component.

Uses of the SCC graph

- <u>Articulation</u>: a vertex whose removal disconnects G.
- <u>Bridge</u>: an edge whose removal disconnects G.
- <u>Euler tour</u>: a cycle that traverses all edges of G exactly once (vertices can be visited more than once)

All can be computed in $O(|E|)$ on the SCC.