# Coordinate Geometry 2D


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Coordinate geometry is the branch of mathematics in which geometry is studied with the help of algebra. The great France mathematician and philosopher Rene Descartes (1596-1650) first applied algebraic formulae in geometry. A system of geometry where the position of points on the plane is described using two numbers called an ordered pair of numbers or coordinates. The first element of the ordered pair represents the distance of that point on x-axis called abscissa and second element on y-axis called ordinate. This abscissa and ordinate makes coordinates of that point.

René Descartes

The method of describing the location of points in this way was proposed by the French mathematician René Descartes (1596 - 1650). (Pronounced "day CART"). He proposed further that curves and lines could be described by equations using this technique, thus being the first to link algebra and geometry. In honor of his work, the coordinates of a point are often referred to as its Cartesian coordinates and the coordinate plane as the Cartesian Coordinate Plane and coordinate geometry sometimes called Cartesian geometry.

Classification of geometry :

Generally, geometry is classified into two following categories such as

***1’Ordinary Geometry:***

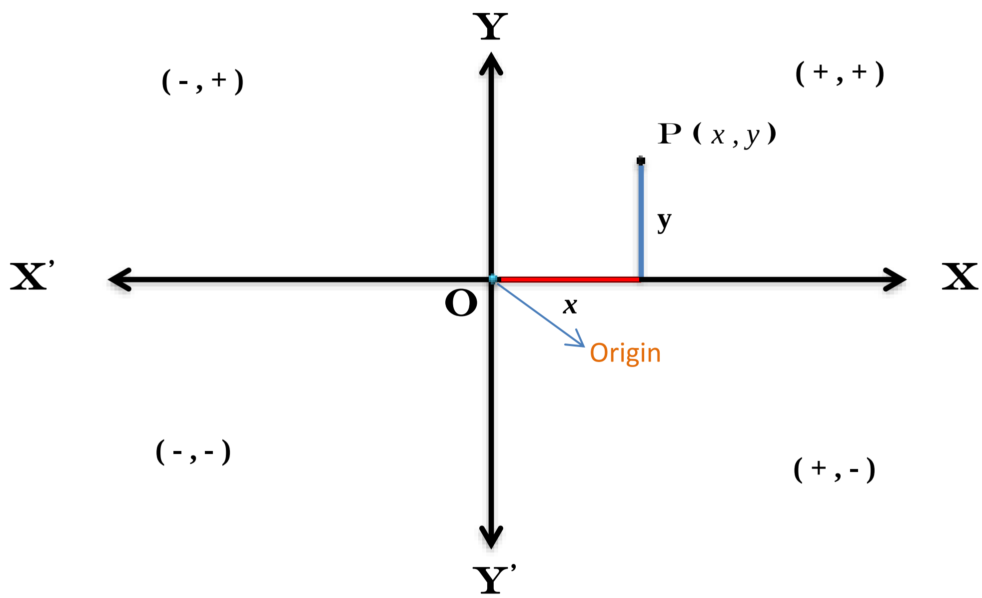
Ordinary geometry is a branch of mathematics concerned with questions of shape, size and relative position of figures and the properties of space. A mathematician who works in the field of geometry is called a geometer. Geometry arose independently in a number of early cultures as a body of practical knowledge concerning lengths, areas, and volumes etc. The father of ordinary Geometry is Euclid who is famous Greek mathematician for his writing elements book which has 13 volumes.

**2′Coordinate/Analytical Geometry:**

Coordinate geometry gives us a way to describe exactly where the point is located by using two numbers, called coordinates. The father of coordinate/Analytical Geometry is French mathematician René Descartes who also invents the rule of sign by this we can guess how many positive, negative ,imaginary roots a polynomial equation have?

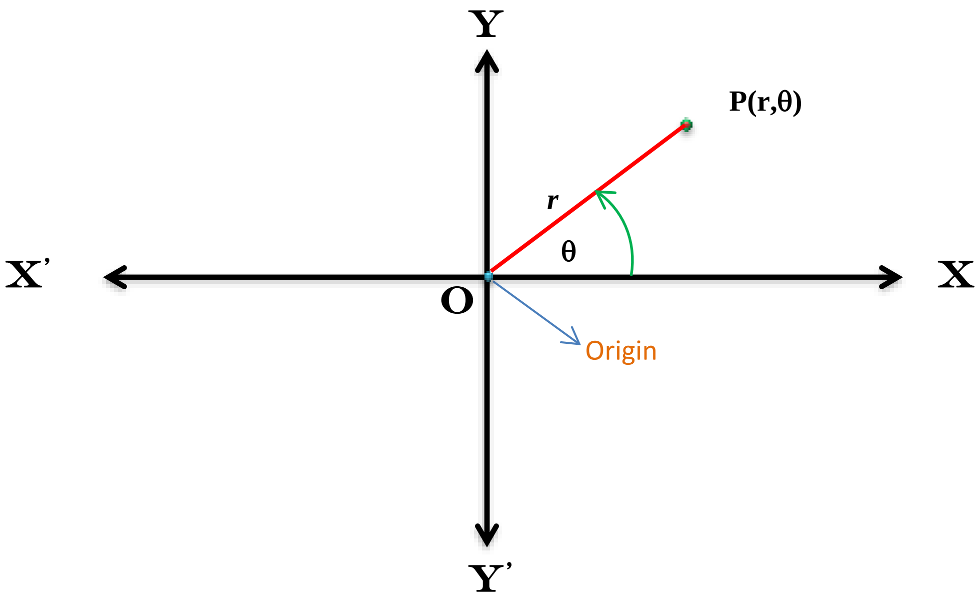
**Cartesian/Rectangular Coordinates System:**

Every plane is two dimensional so to locate the position of a point in a plane there is needed two coordinates. The Mathematician Rene Descartes first considered two perpendicular intersecting fixed straight lines in a plane as axes of coordinates. These two straight lines are named as rectangular axes and intersecting point as the origin denotes by the symbol **O** and the symbol **O** comes from the first letter of the word origin. The Cartesian coordinate system is named after the inventor name Rene Descartes. The Cartesian coordinate system is also known as Rectangular coordinates system as the axes are in right angle (Rec means right).In Cartesian coordinate system position of a point measured by the distance on both axes. First one is on x-axis called abscissa or x-coordinate denoted by the symbol $x$ and second one is on y-axis called ordinate or y-coordinate denoted by the symbol $y$.We express the coordinate of a point **P** in Cartesian plane by the ordered pair *P(x , y) or P(abscissa, ordinate).*




The horizontal line XOX′ is called x-axis and the vertical line YOY′ is called y-axis .Both axes divide the whole plane into four parts called Quadrants .Four Quadrants XOY, X′OY, X′OY′ and XOY′ are called anti-clock-wisely 1st ,2nd,3rd and 4th quadrant respectively. The coordinate of the origin is **O**(0,0) because all distances measured considering origin as starting point.

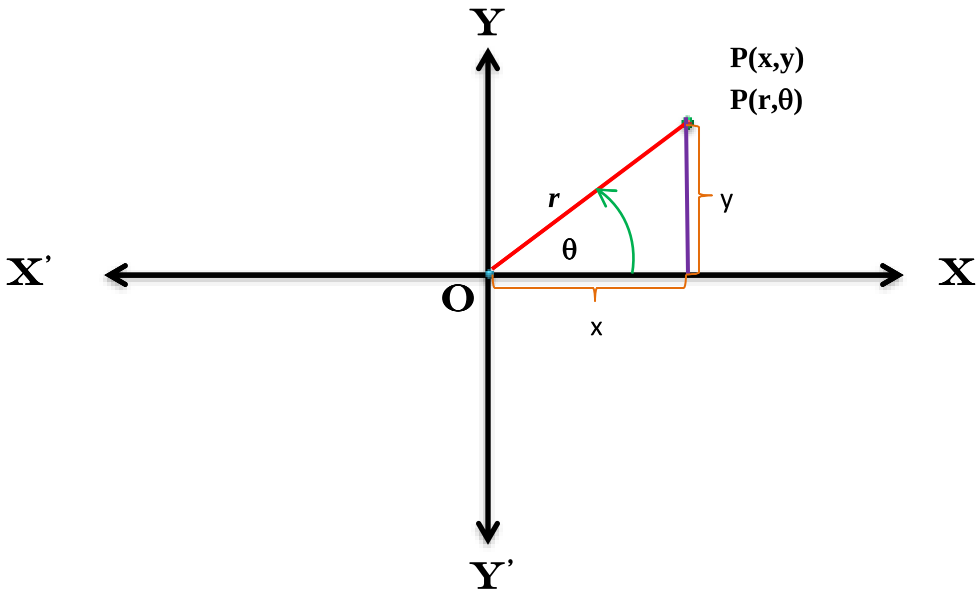
**Polar coordinates System:**




In similar manner of Cartesian system for fixing or locating the point **P** in a plane we take a fixed point **O** called the pole and a fixed straight line **OX** called the initial line. Joining line of the points P and O is called radius vector and length of radius vector
**OP** = r and the positive angle $\angle XOP = \theta$ is called vectorial angle. It is sometimes convenient to locate the position of a point **P** in terms of its distances from a fixed point and its direction from a fixed line through this point. So the coordinates of locating points in this system is called Polar coordinates system. The coordinates of point in this system are called Polar coordinates. The polar coordinates of the point P are expressed as $P(r,\theta)$ .In expressing the polar coordinates of the point P the radius vector is always written as the first coordinate. It is considered positive if measured from the pole along the line bounding the vectorial angle otherwise negative.
In a polar system the same point has an infinite number of representation and it is the demerits of polar coordinate system to Cartesian system.
For example: The point P has the coordinates $(r,\theta),(-r,\theta+\pi),(-r,\theta-\pi),(r,\theta-2\pi)$etc.

**Relation between Cartesian and Polar Coordinate System:**

Suppose that the coordinates of the point P in Cartesian system is $P(x,y)$ and in Polar system is $P(r,\theta)$ .Our target here to establish the relation between two coordinates systems. From the triangle with the help of trigonometry we can find the relation between the Cartesian system & the polar system.




From the pictorial triangle
We get,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \qquad \text{and} \quad \sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

Again
Applying Pythagorean Theorem from geometry we have a relation $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.

Therefore, the relations are:

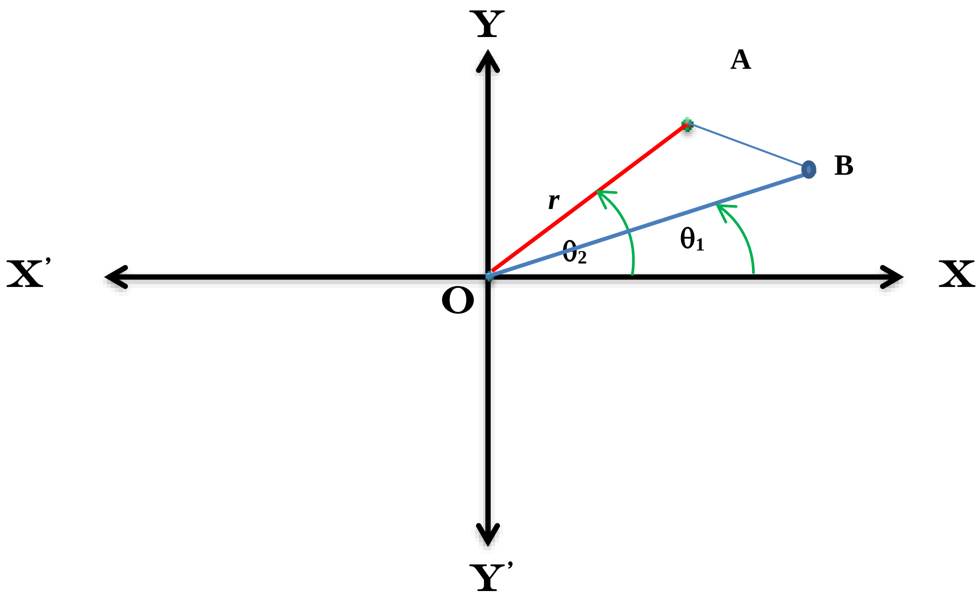
$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

**Distance between two Points:**

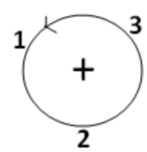
If the coordinates of two points in Cartesian system are $A(x_1, y_1)$ and $B(x_2, y_2)$, then the distance between two points is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.




**Area of a triangle:**

If the coordinates of the vertices of the triangle in Cartesian system are respectively $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the area of the triangle is

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{Sq. Units}$$




$$\Delta \boldsymbol{ABC} = \frac{1}{2}\{\boldsymbol{x_1}(\boldsymbol{y_2} - \boldsymbol{y_3}) + \boldsymbol{x_2}(\boldsymbol{y_3} - \boldsymbol{y_1}) + \boldsymbol{x_3}(\boldsymbol{y_1} - \boldsymbol{y_2})\}$$

Area by Sarrus Diagram Method:

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}$$

An alternative representation of Sarrus diagram Method:

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}$$

Note:

1. In the determinant point must be choose in anti-clock-wise direction.
2. In Sarrus Diagram method first point is repeated.
3. Applying Sarrus Diagram Method we find the area of a polygon.

**Mathematical Problem**

**Problem 01:** Determine the polar coordinates of the point $\left(\sqrt{3}, -1\right)$.

Solution:

We have given $(x, y) = \left(\sqrt{3}, -1\right)$.

Therefore $x = \sqrt{3}$ and $y = -1$

We know that,

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$r = \sqrt{3+1} = \sqrt{4} = 2$$

And

$$\tan\theta = \frac{-1}{\sqrt{3}}$$

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\tan\theta = -\tan\frac{\pi}{6}$$

$$\tan\theta = \tan\left(2\pi - \frac{\pi}{6}\right)$$

$$\tan\theta = \tan\left(\frac{12\pi - \pi}{6}\right)$$

$$\tan\theta = \tan\left(\frac{11\pi}{6}\right)$$

$$\theta = \frac{11\pi}{6}$$

Therefore, the polar form of the given point is $(r,\theta) = \left(2, \frac{11\pi}{6}\right)$ or $(r,\theta) = \left(2, -\frac{\pi}{6}\right)$.

**Problem 02:** Determine the Cartesian coordinates of the point $\left(2\sqrt{2}, \frac{5\pi}{4}\right)$.

Solution:

We have given $(r,\theta) = \left(2\sqrt{2}, \frac{5\pi}{4}\right)$.

Therefore $r = 2\sqrt{2}$ and $\theta = \frac{5\pi}{4}$

We know that,

$$x = r\cos\theta = 2\sqrt{2}\cos\frac{5\pi}{4} = 2\sqrt{2}\cos\left(\pi + \frac{\pi}{4}\right) = -2\sqrt{2}\cos\frac{\pi}{4} = -2\sqrt{2} \times \frac{1}{\sqrt{2}} = -2$$

And

$$y = r\sin\theta = 2\sqrt{2}\sin\frac{5\pi}{4} = 2\sqrt{2}\sin\left(\pi + \frac{\pi}{4}\right) = -2\sqrt{2}\sin\frac{\pi}{4} = -2\sqrt{2} \times \frac{1}{\sqrt{2}} = -2$$

Therefore, the Cartesian form of the given point is $(x, y) = (-2, -2)$.

**H.W:**

**1. Convert the following points to the polar form:**

**i)** $\left(1, -\sqrt{3}\right)$ **ii)** $\left(2\sqrt{3}, -2\right)$ **iii)** $(-1, -1)$ **iv)** $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

**v)** $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ **vi)** $\left(a, a\sqrt{3}\right)$ **vii)** $\left(\frac{5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2}\right)$

**2. Convert the following points to the Cartesian form:**

**i)** $\left(3, \frac{\pi}{6}\right)$ **ii)** $\left(5, -\frac{\pi}{4}\right)$ **iii)** $\left(-2a, -\frac{2\pi}{3}\right)$ **iv)** $\left(2, \frac{2\pi}{3}\right)$ **v)** $\left(1, \frac{\pi}{6}\right)$

**vi)** $\left(2, \frac{\pi}{3}\right)$ **vii)** $\left(3, \frac{\pi}{2}\right)$ **viii)** $\left(2, -\frac{\pi}{6}\right)$ **ix)** $\left(4, \frac{11\pi}{6}\right)$ **x)** $\left(\sqrt{2}, \frac{5\pi}{4}\right)$

**Problem 03:** Transform the equation $x^3 + y^3 = 3axy$ to Polar equation.

Solution:

Given Cartesian Equation is $x^3 + y^3 = 3axy$.

We have

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Now replacing x and y from the above equation by its values given equation reduces to the following form

$$r^3\cos^3\theta + r^3\sin^3\theta = 3a.r\cos\theta.r\sin\theta$$
$$r^3\left(\cos^3\theta+\sin^3\theta\right) = 3a.r^2\cos\theta\sin\theta$$
$$r\left(\cos^3\theta+\sin^3\theta\right) = 3a\cos\theta\sin\theta$$
$$r\left(\cos^3\theta+\sin^3\theta\right) = \frac{3}{2}a\times 2\cos\theta\sin\theta$$
$$r\left(\cos^3\theta+\sin^3\theta\right) = \frac{3}{2}a\cos 2\theta \qquad \textbf{(As desired)}$$

**Problem 04:** If x, y be related by means of the equation $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, find the corresponding relation between $r$ and $\theta$.

Solution:

Given Cartesian Equation is $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$.

We have

$x = r\cos\theta$ and $y = r\sin\theta$

Putting $x = r\cos\theta$ and $y = r\sin\theta$ the above relation is transformed into the following form

$$\left(r^2\cos^2\theta + r^2\sin^2\theta\right)^2 = a^2\left(r^2\cos^2\theta - r^2\sin^2\theta\right)$$
$$r^4\left(\cos^2\theta+\sin^2\theta\right)^2 = a^2r^2\left(\cos^2\theta-\sin^2\theta\right)$$
$$r^2(1)^2 = a^2\left(\cos^2\theta-\sin^2\theta\right)$$
$$r^2 = a^2\cos 2\theta \qquad \textbf{(As desired)}$$

**H.W:**

**Convert the followings to the polar form:**

1. $x^2\left(x^2+y^2\right) = a^2\left(x^2-y^2\right)$
2. $xy^3 + yx^3 = a^2$
3. $\left(x^2-y^2\right)^2 - y^2(2x+1) + 2x^3 = 0$
4. $4\left(x^3-y^3\right) - 3(x-y)\left(x^2+y^2\right) = 5kxy$
5. $x^3 = y^2(2a-x)$
6. $x^4 + x^2y - (x+y)^2 = 0$

**Problem 05:** Transform the equation $2a\sin^2\theta - r\cos\theta = 0$ to Cartesian equation.

Solution:

Given Polar Equation is $2a\sin^2\theta - r\cos\theta = 0$.

We have

$x = r\cos\theta, y = r\sin\theta$ *and* $x^2+y^2 = r^2$

Putting $x = r\cos\theta, y = r\sin\theta$ *and* $x^2+y^2 = r^2$ the above relation is transformed into the following form

$$2a\sin^2\theta - r\cos\theta = 0$$
$$\frac{2a\,r^2\sin^2\theta}{r^2} - r\cos\theta = 0$$

$$\frac{2a\left(r\sin\theta\right)^2}{r^2} - r\cos\theta = 0$$

$$\frac{2a\,y^2}{x^2+y^2} - x = 0$$

$$2a\,y^2 - x\left(x^2+y^2\right) = 0$$

$$2a\,y^2 = x^3 + xy^2$$

$$x^3 = 2a\,y^2 - xy^2$$

$$x^3 = y^2\left(2a-x\right)$$ **(As desired)**

**H.W:**

**Convert the followings to the Cartesian form:**

1. $r^2\cos^2\theta = a^2\cos 2\theta$
2. $r^4 = 2a^2\cos ec 2\theta$
3. $r\cos 2\theta = 2\sin^2\frac{\theta}{2}$
4. $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$
5. $2a\sin^2\theta = r\cos\theta$
6. $r = \pm\left(1+\tan\theta\right)$

**Broad Questions:**

1. Find the distances among the points $\left(1,\frac{\pi}{6}\right)$, $\left(2,\frac{\pi}{3}\right)$ and $\left(3,\frac{\pi}{2}\right)$.
2. Find the area of the triangle formed by the points $\left(1,\frac{\pi}{6}\right)$, $\left(2,\frac{\pi}{3}\right)$ and $\left(3,\frac{\pi}{2}\right)$.
3. If three points $\left(-1,2\right)$, $\left(2,-1\right)$ and $\left(h,3\right)$ are collinear then show that $h=-2$.
4. Find the area of a polygon whose vertices are given $\left(1,3\right)$, $\left(4,1\right)$, $\left(5,3\right)$, $\left(3,2\right)$ and $\left(2,4\right)$.
5. Show that the three points $\left(1,-1\right)$, $\left(-\frac{1}{2},\frac{1}{2}\right)$ and $\left(1,2\right)$ form a right angled triangle.
6. Find locus of the parametric coordinates point $\left(a\cos\theta, b\sin\theta\right)$ where $\theta$ is a parameter.

**Problem 06:** **Find the Area of the triangle formed by the points (1 , 2) ,(2,6) & (3 , 4)**

Solution: Given points are **(1 , 2) ,(2,6) & (3 , 4)**

We know that ,

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 6 & 4 & 2 \end{vmatrix}$$

$$= \frac{1}{2}\{(1\times6+2\times4+3\times2)-(2\times2+6\times3+4\times1)\}$$

$$= \frac{1}{2}\{(6+8+6)-(4+18+4)\} = \frac{1}{2}\{20-26\} = -3$$

**Therefore , the Area of the triangle is = 3 Sq. Units** (Ans).

**Problem 07:** **Find the points (1 , 2) ,(1,3) & (1 , 4) are colinear.**

Solution: Given points are **(1 , 2) ,(1,3) & (1 , 4)**

We know that ,

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 2 \end{vmatrix}$$

$$= \frac{1}{2}\{(1\times3+1\times4+1\times2)-(2\times1+3\times1+4\times1)\}$$

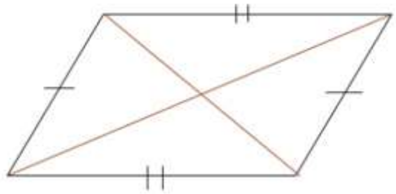
$$= \frac{1}{2}\{(3+4+2)-(2+3+4)\} \quad = \frac{1}{2}\{9-9\} = 0$$

**Therefore , given points are co-linear.** (Ans).

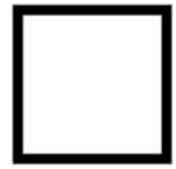
**H.W : For which value of "k" the points (1 , 2) ,(1,3) & (k , 4) are colinear.**

# Application of distance Formula

Parallelogram

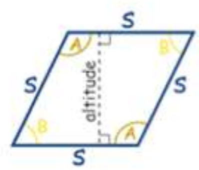


Square



Rectangle

Rhombus




**Problem 08:** **Find the points (1 , 2) ,(3,2) ,(1, 3) & (3,3) form a rectangle.**

**Solution:** Let the points are A(1,2) , B(1,3) ,C(3,2) & D(3,3).

Now, the sides are

$$AB = \sqrt{(1-1)^2 + (2-3)^2} = \sqrt{0+1} = 1$$

$$BD = \sqrt{(1-3)^2 + (3-3)^2} = \sqrt{4+0} = 2$$

$$CD = \sqrt{(3-3)^2 + (2-3)^2} = \sqrt{0+1} = 1$$

$$AC = \sqrt{(1-3)^2 + (2-2)^2} = \sqrt{4+0} = 2$$

Now, the diagonals are

$$AD = \sqrt{(1-3)^2 + (2-3)^2} = \sqrt{4+1} = \sqrt{5}$$

$$BC = \sqrt{(1-3)^2 + (3-2)^2} = \sqrt{4+1} = \sqrt{5}$$

Since AB=CD,BD=AC & AD=BC , So the points A,B,C & D form a rectangle.