3D

Coordinates Transformation

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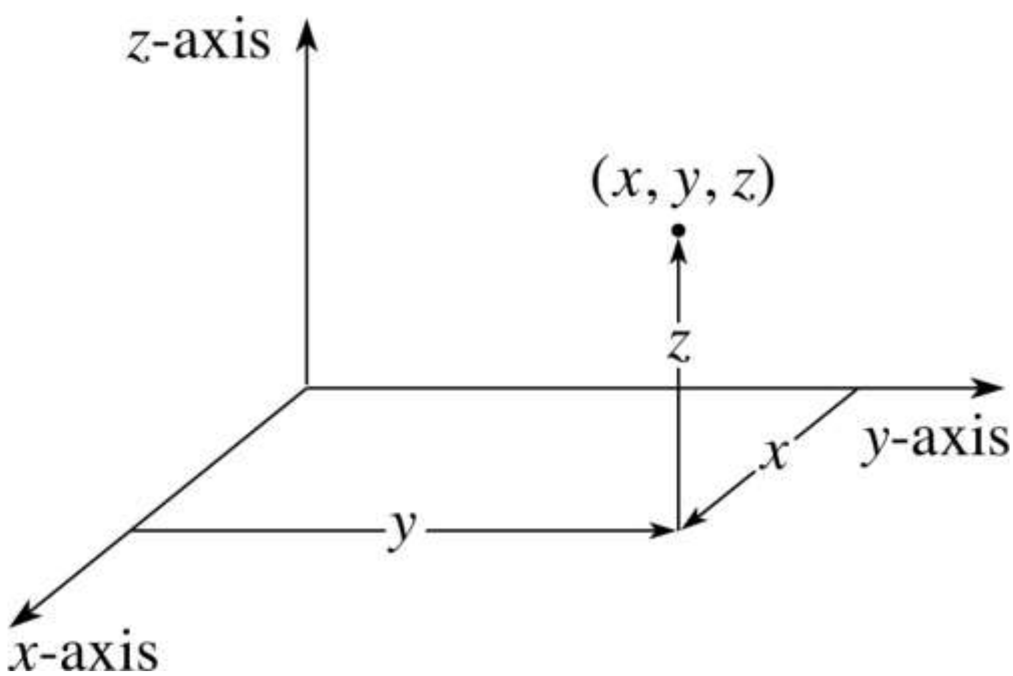
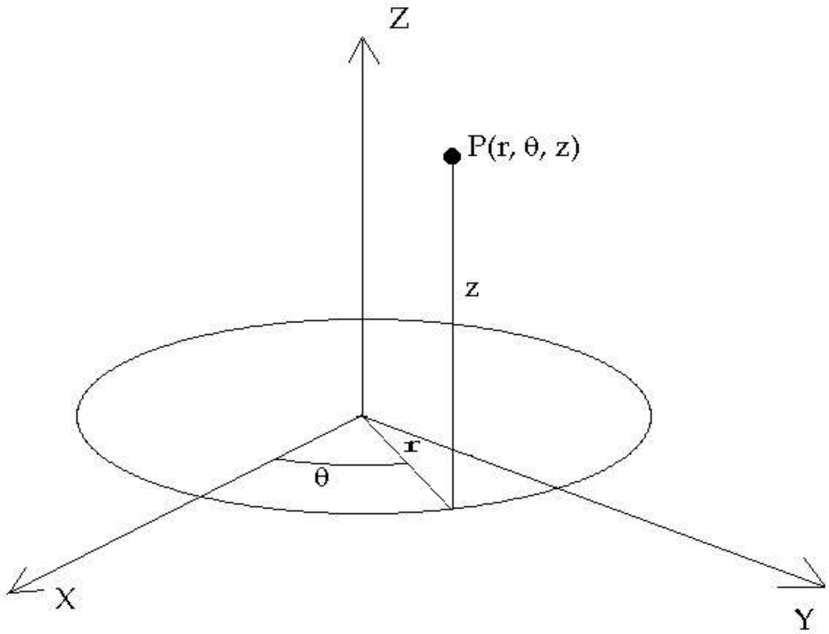
In three dimensional systems there are three coordinate systems such as

1. Cartesian /Rectangular coordinate System (**RS**)
2. Cylindrical coordinate System (**CS**)
3. Spherical coordinate System **(SS)**

Cartesian /Rectangular coordinate System:
In the Cartesian coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (x, y, z) where x is the distance on x axis, y is the distance on y axis and z is the distance on z axis of the point (x, y, z).
Figure:



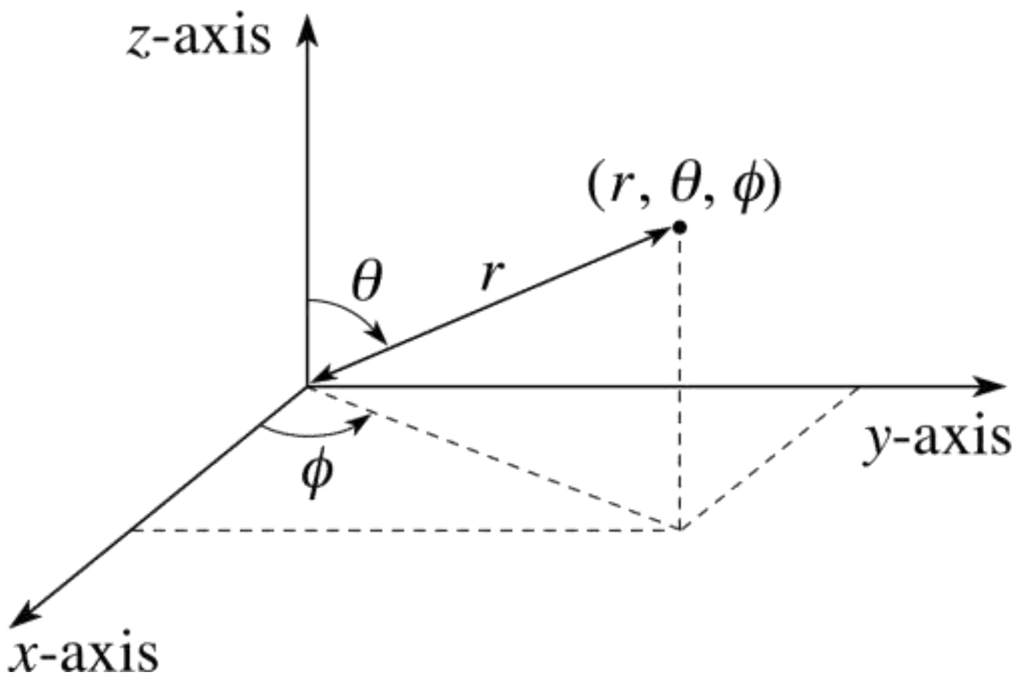
Cylindrical coordinate System:
In the Cylindrical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (r, θ, z) where r is the distance of the point from origin or length of radial line, θ is the angle between radial line and x axis and z is the distance of the point (r, θ, z) from the xy plane.
Many mathematicians use the symbol ' ρ ' in place of ' r '.



Spherical coordinate system:

In the Spherical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (ρ,θ,φ) where r is the distance of the point from origin or length of radial line, θ is the angle between radial line and z axis and φ the angle between radial line (joining with the foot point of the perpendicular from the given point on the xy plane) and the x axis.

Many mathematicians use the symbol ‘r’ in place of ‘ ρ ’.



⊡⊡ Relation between Cartesian and Cylindrical System:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z \qquad z = z$$

⊡⊡ Relation between Cartesian and Spherical System:

$$x = \rho\sin\theta\cos\varphi \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

$$y = \rho\sin\theta\sin\varphi \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \quad or\ \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}\left(\frac{z}{\rho}\right)$$

$$z = \rho\cos\theta \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

⊡⊡ Relation between Cylindrical and Spherical System:

$$r = \rho\sin\theta \qquad \rho = \sqrt{z^2 + r^2}$$

$$\theta = \varphi \qquad \theta = \tan^{-1}\left(\frac{r}{z}\right)$$

$$z = \rho\cos\theta \qquad \varphi = \theta$$

Restriction:

$$x, y, z \in (-\infty, \infty); \quad \rho, r \in [0, \infty); \quad \theta \in \left[0^o, 180^o\right] \ and\ \varphi \in \left[0^o, 360^o\right]$$

Mathematical Problems

Problem no: 01

Convert $\left(3, \frac{\pi}{3}, -4\right)$ from Cylindrical to Cartesian Coordinates.

Solution:

Given that,

Cylindrical coordinates of a point is $(r\ ,\ \theta,\ z) = \left(3, \frac{\pi}{3}, -4\right)$

We know that,

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ \ and\ \ z = z$$

Now, $x = r\cos\theta = 3\cos\frac{\pi}{3} = 3\times\frac{1}{2} = \frac{3}{2}$

$$y = 3\sin\frac{\pi}{3} = 3\times\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$
$$z = z = -4$$

Therefore the Cartesian coordinates of the given point is $(x\,,\, y,\, z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{3}, -4\right)$.

H.W:
Convert the followings cylindrical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3}, \frac{\pi}{4}, -4\right)$
2. $\left(4\sqrt{3}, 0, 5\right)$

Problem no: 02
Convert $(-2,2,3)$ from Cartesian to Cylindrical Coordinates.

Solution:
Given that,
Cartesian coordinates of a point is $(x\,,\, y,\, z) = (-2,2,3)$
We know that,

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Now,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{2}{-2} = -1$$

Here

$$\tan\theta = -1$$
$$\tan\theta = -\tan\frac{\pi}{4}$$
$$\tan\theta = \tan(\pi - \frac{\pi}{4})$$
$$\tan\theta = \tan(\frac{3\pi}{4})$$
$$\theta = \frac{3\pi}{4}$$

And

$$z = z = 3$$

Therefore the Cylindrical coordinates of the given point is $(r\ ,\ \theta,\ z) = \left(2\sqrt{2}, \frac{3\pi}{4}, 3\right)$.

H.W:

Convert the followings Cartesian coordinates to the Cylindrical Coordinates system:

1. $(4\sqrt{3}, 4, -4)$
2. $(-\sqrt{3}, -4, 4)$
3. $(-\sqrt{3}, 4, 2)$
4. $(4\sqrt{2}, -1, -4)$
5. $(\sqrt{3}, 0, 0)$
6. $(0, 4, 9)$
7. $(4\sqrt{3}, 0, 5)$

Problem no: 03

Convert $\left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from Spherical to Cartesian Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho\ ,\ \theta,\ \varphi) = \left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$

We know that,

$x = \rho \sin\theta \cos\varphi\ ,\ y = \rho \sin\theta \sin\varphi$ *and* $z = \rho \cos\theta$

Now,

$$x = \rho \sin\theta \cos\varphi = 8 \sin\frac{\pi}{4} \cos\frac{\pi}{6} = 8 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}$$

$$y = \rho \sin\theta \sin\varphi = 8 \sin\frac{\pi}{4} \sin\frac{\pi}{6} = 8 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

And

$$z = \rho \cos\theta = 8 \cos\frac{\pi}{4} = 8 \times \frac{1}{\sqrt{2}} = 4\sqrt{2}$$

Therefore the Cartesian coordinates of the given point is $(x\ ,\ y,\ z) = \left(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2}\right)$

H.W:

Convert the followings Spherical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3},\frac{\pi}{4},\frac{\pi}{4}\right)$
2. $\left(-\sqrt{3},124^{o},75^{o}\right)$
3. $\left(\sqrt{3},140^{o},140^{o}\right)$

Problem no: 04

Convert $\left(2\sqrt{3},6,-4\right)$ from Cartesian to Spherical Coordinates.

Solution:
Given that,
Cartesian coordinates of a point is $(x\ ,\ y,\ z)=\left(2\sqrt{3},6,-4\right)$

We know that,

$$\rho=\sqrt{x^2+y^2+z^2}$$

$$\theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\varphi=\tan^{-1}\left(\frac{y}{x}\right)$$

Now ,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{(2\sqrt{3})^2+6^2+(-4)^2}=\sqrt{12+36+16}=\sqrt{64}=8$$

$$\theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\Rightarrow\tan\theta=\frac{\sqrt{x^2+y^2}}{z}\Rightarrow\tan\theta=\frac{\sqrt{(2\sqrt{3})^2+6^2}}{-4}$$

$$\Rightarrow\tan\theta=\frac{\sqrt{12+36}}{-4}=\frac{\sqrt{48}}{-4}=\frac{4\sqrt{3}}{-4}=-\sqrt{3}$$

Here

$$\tan\theta=-\sqrt{3}$$

$$\tan\theta=-\tan\frac{\pi}{3}$$

$$\tan\theta=\tan\left(\pi-\frac{\pi}{3}\right)$$

$$\tan\theta=\tan\left(\frac{2\pi}{3}\right)$$

$$\therefore\theta=\frac{2\pi}{3}$$

And $\varphi = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \tan\varphi = \frac{y}{x} \Rightarrow \tan\varphi = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

Here

$$\tan\varphi = \sqrt{3}$$

$$\tan\varphi = \tan\frac{\pi}{3}$$

$$\therefore \varphi = \frac{\pi}{3}$$

Therefore the Spherical coordinates of the given point is $(\rho\ ,\ \theta,\ \varphi) = \left(8, \frac{2\pi}{3}, \frac{\pi}{3}\right)$.

H.W:
Convert the followings Cartesian coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3}, 4, -4\right)$
2. $\left(-\sqrt{3}, -4, 4\right)$
3. $\left(-\sqrt{3}, 4, 2\right)$
4. $\left(4\sqrt{2}, -1, -4\right)$
5. $\left(\sqrt{3}, 0, 0\right)$
6. $(0, 4, 9)$
7. $\left(4\sqrt{3}, 0, 5\right)$

Problem no: 05

Convert $\left(1\ ,\ \frac{\pi}{2}\ ,\ 1\right)$ from Cylindrical to Spherical Coordinates.

Solution:
Given that,

Cylindrical coordinates of a point is $(r\ ,\ \theta,\ z) = \left(1\ ,\ \frac{\pi}{2}\ ,\ 1\right)$

We know that,

$$\rho = \sqrt{z^2 + r^2}$$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right)$$

$$\varphi = \theta$$

Now,

$$\rho = \sqrt{z^2 + r^2} = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right) \Rightarrow \tan\theta = \frac{r}{z} = \frac{1}{1} = 1$$

Here

$$\tan\theta = 1$$

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\therefore\ \theta = \frac{\pi}{4}$$

And

$$\varphi = \theta = \frac{\pi}{2}$$

Therefore the Spherical coordinates of the given point is $(\rho\ ,\ \theta,\ \varphi) = \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right)$.

H.W:

Convert the followings Cylindrical coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3}, 42^o, -4\right)$
2. $\left(4\sqrt{3}, 0^o, 5\right)$
3. $\left(-\sqrt{3}, 134^o, -4\right)$

Problem no: 06

Convert $\left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$ from Spherical to Cylindrical Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho\ ,\ \theta,\ \varphi) = \left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$

We know that,

$$r = \rho\sin\theta$$

$$\theta = \varphi$$

$$z = \rho\cos\theta$$

Now,

$$r = \rho\sin\theta = 4\sqrt{3}\sin\frac{\pi}{4} = 4\sqrt{3}\times\frac{1}{\sqrt{2}} = 2\sqrt{6}$$

$$\theta = \varphi = \frac{\pi}{4}$$

And $$z = \rho\cos\theta = 4\sqrt{3}\times\cos\frac{\pi}{4} = 4\sqrt{3}\times\frac{1}{\sqrt{2}} = 2\sqrt{6}$$

Therefore the Cylindrical coordinates of the given point is $(r\ ,\ \theta,\ z) = \left(2\sqrt{6}, \frac{\pi}{4}, 2\sqrt{6}\right)$.

H.W:
Convert the followings Spherical coordinates to the Cylindrical Coordinates system:

1. $\left(-\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{3}\right)$

Transformation of Equations

Mathematical problems

Problem: 01

Express Cartesian Equation $x^2 - y^2 = 25$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 - y^2 = 25$

we have

$$x = r\cos\theta\ \ ,\ y = r\sin\theta\ and\ z = z$$

Replacing x and y from the given equation we get desired Cylindrical equation as follows,

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 25$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 25$$
$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 25$$
$$r^2\cos(2\theta) = 25$$
$$r^2 = 25Sec(2\theta)$$

(As desired)

Problem: 02

Express Cartesian Equation $x^2 + y^2 + z^2 = 0$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 + y^2 + z^2 = 0$

We have,

$$x = r\cos\theta\ \ ,\ y = r\sin\theta\ and\ z = z$$

Replacing x , y and z from the given equation we get desired Cylindrical equation as follows,

$$(r\cos\theta)^2 + (r\sin\theta)^2 + z^2 = 0$$
$$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 0$$
$$r^2\left(\cos^2\theta + \sin^2\theta\right) + z^2 = 0$$
$$r^2 + z^2 = 0$$

(As desired)

H.W:
Transform the following Cartesian equations into the Cylindrical Equations:

1. $x^2 - y^2 + 2z^2 = 3x$
2. $x^2 + y^2 + z^2 = 2z$
3. $z^2 = y^2 - x^2$

4. $x+y+z=1$

Problem: 03

Transform Cartesian Equation $x^2+y^2-z^2=1$ to Spherical Equation.

Solution:

Given Cartesian Equation is $x^2+y^2-z^2=1$

We have,

$$x=\rho\sin\theta\cos\varphi$$
$$y=\rho\sin\theta\sin\varphi$$
$$z=\rho\cos\theta$$

Replacing x, y and z from the given equation we get desired Cylindrical equation as follows,

$$(\rho\sin\theta\cos\varphi)^2+(\rho\sin\theta\sin\varphi)^2-(\rho\cos\theta)^2=1$$
$$\rho^2\sin^2\theta\cos^2\varphi+\rho^2\sin^2\theta\sin^2\varphi-\rho^2\cos^2\theta=1$$
$$\rho^2\sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)-\rho^2\cos^2\theta=1$$
$$\rho^2\sin^2\theta-\rho^2\cos^2\theta=1$$
$$\rho^2\left(\sin^2\theta-\cos^2\theta\right)=1$$
$$-\rho^2\left(\cos^2\theta-\sin^2\theta\right)=1$$
$$-\rho^2\cos(2\theta)=1$$
$$\rho^2\cos(2\theta)=-1$$
$$\rho^2=-\sec(2\theta)$$ **(As desired)**

H.W:

Transform the following Cartesian equations into the Spherical Equations:

1. $x^2-y^2+2z^2=3x$
2. $x^2+y^2+z^2=2z$
3. $z^2=y^2-x^2$
4. $x^2+y^2+z^2=0$
5. $x+y+z=1$

Problem: 04

Transform Spherical Equation $\rho=2\cos\varphi$ to Cylindrical Equation.

Solution:

Given Spherical Equation is $\rho=2\cos\varphi$

We have,

$$\rho = \sqrt{z^2 + r^2}$$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right)$$

$$\varphi = \theta$$

Replacing $\rho \, and \, \varphi$ from the given equation we get desired Cylindrical equation as follows,

$$\sqrt{z^2 + r^2} = 2\cos\theta$$

$$\sqrt{z^2 + r^2} = 2 \times \frac{z}{\rho} \qquad [\because z = \rho\cos\theta]$$

$$\sqrt{z^2 + r^2} = 2 \times \frac{z}{\sqrt{z^2 + r^2}}$$

$$z^2 + r^2 = 2z \qquad \textbf{(As desired)}$$

H.W:

Transform the following Spherical Equations into the Cylindrical Equations:

1. $\varphi = \frac{\pi}{4}$
2. $\rho = 2\sec\theta$
3. $\rho = \cos ec\theta$

Problem: 04

Transform Cylindrical Equation $r^2 \cos 2\theta = z$ to Cartesian/Rectangular Equation.

Solution:

Given Cylindrical Equation is $r^2 \cos 2\theta = z$

We have,

$$x = r\cos\theta \text{ , } y = r\sin\theta \text{ and } z = z$$

Given equation is

$$r^2 \cos 2\theta = z$$

$$r^2(\cos^2\theta - \sin^2\theta) = z$$

$$r^2(\cos^2\theta - \sin^2\theta) = z$$

$$r^2\cos^2\theta - r^2\sin^2\theta = z$$

$$(r\cos\theta)^2 - (r\sin\theta)^2 = z$$

$$(x)^2 - (y)^2 = z \qquad \text{[Putting values]} \quad \textbf{(As desired)}$$

H.W:

Transform the following Cylindrical Equations into the Cartesian/Rectangular Equations:

1. $r = 2\sin\theta$
2. $z = 5\sin\theta$

Problem: 05

Transform Spherical Equation $\rho\sin\theta = 1$ to Cartesian/Rectangular Equation.

Solution:

Given Spherical Equation is $\rho\sin\theta = 1$

We have,

$$x = \rho\sin\theta\cos\varphi, y = \rho\sin\theta\sin\varphi, z = \rho\cos\theta \; and \, \rho = \sqrt{x^2 + y^2 + z^2}$$

Now,

$$\rho\sin\theta = 1$$

$$\rho^2\sin^2\theta = 1$$

$$\rho^2(1-\cos^2\theta) = 1$$

$$\rho^2 - \rho^2\cos^2\theta = 1$$

$$\rho^2 - \left(\rho\cos\theta\right)^2 = 1$$

$$\rho^2 - \left(\rho\cos\theta\right)^2 = 1$$

$$x^2 + y^2 + z^2 - z^2 = 1$$

$$x^2 + y^2 = 1$$ **(As desired)**

H.W:

Transform the following Spherical Equations into the Cartesian/Rectangular Equations:

1. $\rho\sin\varphi = 1$
2. $\rho = 2\sec\varphi$
3. $\rho = \csc\varphi$
4. $\rho\sin\varphi = 2\cos\theta$