DIRECTION COSINES
&
DIRECTION RATIOS

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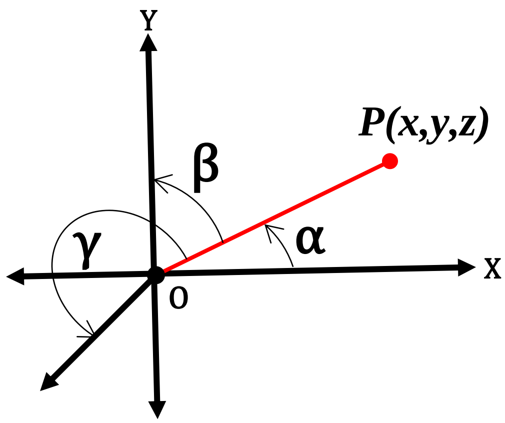
DIRECTION COSINES & DIRECTION RATIOS

Line:
A line is a straight (no curves) one-dimensional figure having no thickness and extending infinitely in both directions. A line is sometimes called a straight line or a right line. A line with direction is called directed line.

Line Segment:
A line segment is a part of a line that is bounded/connected by two distinct end points, and contains every point on the line between its endpoints.

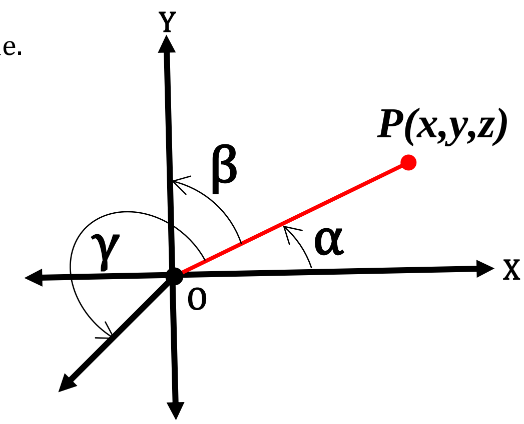
Directed Angle:
A line passing through the origin and makes positive angle with coordinate axes is called Directed Angles.



If the line **OP** makes positive angles α , β and γ with the coordinate axes x, y and z respectively then directed angles of that line **OP** is generally denoted by the symbol $\left[\alpha,\beta,\gamma\right]$ where $0 \le \alpha$ and $\beta,\gamma < \pi$. Also the directed angles of line **PO** is $\left[\pi-\alpha,\pi-\beta,\pi-\gamma\right]$. Some mathematician writes the directed angles of line **PO** as $\left[\pi+\alpha,\pi+\beta,\pi+\gamma\right]$.

Note:
The directed angles of the parallel lines are same.

Direction Cosines of a line:



If α,β,γ are the angles that a given line **OP** makes with the positive directions of the coordinate axes x, y and z respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines(d.c.'s) of the given line. The direction cosines of a line are usually denoted by the letters l, m, n where $l=\cos\alpha, m=\cos\beta, n=\cos\gamma$ .Direction cosines are generally put in square bracket as like $\left[l,m,n\right]$.

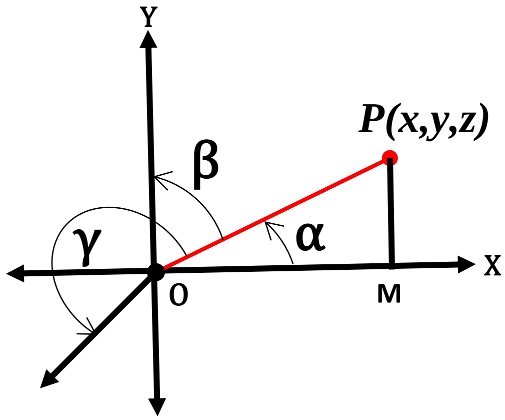
Theorem: (Fundamental theorem of direction cosine):

If l, m, n are the direction cosines of a line, then prove that $l^2+m^2+n^2=1$.

Or

The sum of the squares of the direction cosines of every line is one.

Proof:



Suppose P (x, y, z) be a point in a space and O is origin such that $OP = r$ .Draw PM $\perp$OX.
From the right angle triangle OPM we can write

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

And

$$\frac{OM}{OP} = \cos\alpha$$

$$\frac{x}{r} = l$$

$$x = lr$$

Similarly, we have

$y = mr$ and $z = nr$

Now squaring and adding x, y and z we get

$$x^2 + y^2 + z^2 = r^2 l^2 + r^2 m^2 + r^2 n^2$$

$$x^2 + y^2 + z^2 = r^2 \left(l^2 + m^2 + n^2\right)$$

$$r^2 = r^2 \left(l^2 + m^2 + n^2\right)$$

$$l^2 + m^2 + n^2 = 1$$

(Proved)

Problem 01: If a line makes α, β, γ with the axes show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Solution:

From the fundamental law of direction cosine, we have

$$l^2 + m^2 + n^2 = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$3 - \left(\sin^2\alpha + \sin^2\beta + \sin^2\gamma\right) = 1$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

(Showed)

Problem 02: Find the direction cosines of a line that makes equal angles with the axes.

Solution:

Assume that the directed angles of the line are $[\alpha, \beta, \delta]$ and according to the question $\alpha = \beta = \delta$ that implies $\cos\alpha = \cos\beta = \cos\delta$ i.e. $l = m = n$.

From the fundamental law of direction cosine, we have

$$l^2+m^2+n^2=1$$
$$l^2+l^2+l^2=1$$
$$3l^2=1$$
$$l^2=\frac{1}{3}$$
$$l=\pm\frac{1}{\sqrt{3}}$$

Therefore $l=m=n=\pm\frac{1}{\sqrt{3}}$ **(As desired)**

H.W:

1. Prove that $\sin\theta=\pm\sqrt{\frac{2}{3}}$ where directed angles of a line is $[\theta,\theta,\theta]$.

Problem 03: Can the members $\frac{1}{2\sqrt{2}}$, $\frac{1}{2^2\sqrt{2}}$ & $\frac{1}{2^3\sqrt{2}}$ be the direction cosines of any directed line? Give reason for your Answer.

Solution:

Given that,

$$l=\frac{1}{2\sqrt{2}}\ ,m=\frac{1}{2^2\sqrt{2}}=\frac{1}{4\sqrt{2}}\ \&n=\frac{1}{2^3\sqrt{2}}=\frac{1}{8\sqrt{2}}\qquad [Say]$$

Now

$$l^2+m^2+n^2=\left(\frac{1}{2\sqrt{2}}\right)^2+\left(\frac{1}{4\sqrt{2}}\right)^2+\left(\frac{1}{8\sqrt{2}}\right)^2$$
$$l^2+m^2+n^2=\left(\frac{1}{4.2}\right)+\left(\frac{1}{16.2}\right)+\left(\frac{1}{64.2}\right)$$
$$l^2+m^2+n^2=\frac{1}{8}+\frac{1}{32}+\frac{1}{128}$$
$$l^2+m^2+n^2=\frac{16+4+1}{128}$$
$$l^2+m^2+n^2=\frac{21}{128}\neq1$$
$$l^2+m^2+n^2\neq1$$

Because of violating the fundamental law direction cosines given members are not representing the direction cosines of a line.

(Justified)

Problem 04: The direction cosines of two lines is described by two equations $l-5m+3n=0$ and $7l^2+5m^2-3n^2=0$ .Find their direction cosines.

Solution:

Given equations are as follows

$$l-5m+3n=0$$
$$l=5m-3n\ \cdots\cdots\cdots\cdots\cdots(i)$$
$$7l^2+5m^2-3n^2=0\ \cdots\cdots\cdots(ii)$$

Putting the value of l in the equation (ii) we get

$$7(5m-3n)^2+5m^2-3n^2=0$$

$$7\left(25m^2-30mn+9n^2\right)+5m^2-3n^2=0$$
$$\left(175m^2-210mn+63n^2\right)+5m^2-3n^2=0$$
$$175m^2-210mn+63n^2+5m^2-3n^2=0$$
$$180m^2-210mn+60n^2=0$$
$$18m^2-21mn+6n^2=0$$
$$6m^2-7mn+2n^2=0$$
$$6m^2-4mn-3mn+2n^2=0$$
$$2m\left(3m-2n\right)-n\left(3m-2n\right)=0$$
$$2m\left(3m-2n\right)-n\left(3m-2n\right)=0$$
$$\left(3m-2n\right)\left(2m-n\right)=0$$

Therefore, $\left(3m-2n\right)=0 \quad or \quad \left(2m-n\right)=0$

$$3m-2n=0 \cdots\cdots\cdots\cdots(iii)$$

And $2m-n=0 \cdots\cdots\cdots\cdots(iv)$

Solution of equations

$$l-5m+3n=0 \cdots\cdots\cdots\cdots(i)$$
$$0.l+3m-2n=0 \cdots\cdots\cdots\cdots(iii)$$
$$\therefore \frac{l}{10-9}=\frac{m}{0+2}=\frac{n}{3-0}$$
$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}$$
$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+2^2+3^2}}$$
$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{1}{\sqrt{1+4+9}}$$
$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{1}{\sqrt{14}}$$

Therefore,

$$l=\frac{1}{\sqrt{14}}\ ,\ m=\frac{2}{\sqrt{14}}\quad \&\ n=\frac{3}{\sqrt{14}}$$

Again,

Solution of equations

$$l-5m+3n=0 \cdots\cdots\cdots\cdots(i)$$
$$0.l+2m-n=0 \cdots\cdots\cdots\cdots(iv)$$
$$\therefore \frac{l}{5-6}=\frac{m}{0+1}=\frac{n}{2-0}$$
$$\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}$$
$$\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{\left(-1\right)^2+1^2+2^2}}$$
$$\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}=\frac{1}{\sqrt{1+1+4}}$$

$$\frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{1}{\sqrt{6}}$$

Therefore,

$$l = \frac{-1}{\sqrt{6}} \;,\; m = \frac{1}{\sqrt{6}} \;\;\&\; n = \frac{2}{\sqrt{6}}$$ **(As desired)**

H.W:

1. The direction cosines of two lines are connected by relations $l+m+n=0$ and $2lm+2\ln-mn=0$. Find their direction cosines.

2. The direction cosines of two lines are determined by relations $l+m-n=0$ and $mn+6\ln-12lm=0$. Find their direction cosines.

3. The direction cosines of two lines are determined by relations $l+m+n=0$ and $l^2+m^2-n^2=0$. Find their direction cosines.

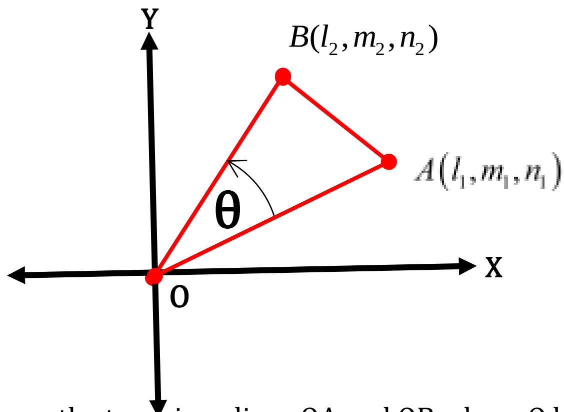
4. The direction cosines of two lines are determined by relations $3.lm+4.\ln+mn=0$ and $l+2m+3n=0$. Find their direction cosines.

5. The direction cosines of two lines are determined by relations $l-5m+3n=0$ and $7l^2+5m^2-3n^2=0$. Find their direction cosines.

6.The direction cosines of a moving line in two adjacent positions are l,m,n and $l+\delta l, m+\delta m, n+\delta n$ .Show that the small angle $\delta\theta$ between the positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Theorem (Angle between two lines): Prove that $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ where θ is the angle the lines whose direction cosines are $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$.

Proof:



Suppose θ be the angle between the two given lines OA and OB where O be the origin.

Let OA = OB=1, therefore the coordinates of A and B are $A(l_1, m_1, n_1)$ and $B(l_2, m_2, n_2)$.

From the distance formula we have

$$AB = \sqrt{(l_1-l_2)^2 + (m_1-m_2)^2 + (n_1-n_2)^2}$$

$$AB^2 = (l_1-l_2)^2 + (m_1-m_2)^2 + (n_1-n_2)^2$$

$$AB^2 = l_1^{\,2} + l_2^{\,2} - 2l_1 l_2 + m_1^{\,2} + m_2^{\,2} - 2m_1 m_2 + n_1^{\,2} + n_2^{\,2} - 2n_1 n_2$$

$$AB^2 = l_1^{\,2} + m_1^{\,2} + n_1^{\,2} + l_2^{\,2} + +m_2^{\,2} + +n_2^{\,2} - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$AB^2 = 1+1-2(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$$AB^2 = 2-2(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

Applying the cosine formula for triangle in the triangle OAB we have

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2.OA.OB}$$

$$\cos\theta = \frac{1^2 + 1^2 - \{2 - 2(l_1l_2 + m_1m_2 + n_1n_2)\}}{2.1.1}$$

$$\cos\theta = \frac{1 + 1 - \{2 - 2(l_1l_2 + m_1m_2 + n_1n_2)\}}{2}$$

$$\cos\theta = \frac{2 - \{2 - 2(l_1l_2 + m_1m_2 + n_1n_2)\}}{2}$$

$$\cos\theta = \frac{2 - 2 + 2(l_1l_2 + m_1m_2 + n_1n_2)}{2}$$

$$\cos\theta = \frac{2(l_1l_2 + m_1m_2 + n_1n_2)}{2}$$

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \qquad \textit{(Proved)}$$

Note:

1. *Two line be perpendicular if* $l_1l_2 + m_1m_2 + n_1n_2 = 0$
2. *Two lines be parallel if* $l_1 = l_2$, $m_1 = m_2$ & $n_1 = n_2$ *or* $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Problem05: Find the angle between the lines whose direction cosines are $\left[\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$ and $\left[\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$.

Solution:

Given direction cosines of two lines are $[l_1, m_1, n_1] = \left[\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$ and $[l_2, m_2, n_2] = \left[\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$.

If θ be the angle between two lines, then we have

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

$$\cos\theta = \frac{2}{6} - \frac{2}{6} + \frac{1}{6}$$

$$\cos\theta = \frac{1}{6}$$

$$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Therefore, the angle between the lines is $\cos^{-1}\left(\frac{1}{6}\right)$. **(As desired)**

H.W:

1. Find the angle between the lines whose direction cosines are $\left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$ and $\left[\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right]$.
2. Find the angle between the lines whose direction cosines are $\left[\frac{-3}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right]$ and $\left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right]$.

Direction Ratios of a line:

Quantities proportional to direction cosines of a line are called direction ratios of that line.

If the direction cosine of a line is $[l, m, n]$ and direction ratios of that lines is $[a, b, c]$ then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

Using the property of ratios, we can write

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{1}}{\sqrt{a^2+b^2+c^2}} \qquad \left[\because l^2+m^2+n^2=1\right]$$

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{1}{\sqrt{a^2+b^2+c^2}}$$

Therefore,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}} \;,m=\frac{b}{\sqrt{a^2+b^2+c^2}} \;\&\; n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Above equations shows the relation between the direction cosines and direction ratios of a line.
Direction cosine also be negative so

$$l=\frac{-a}{\sqrt{a^2+b^2+c^2}} \;,m=\frac{-b}{\sqrt{a^2+b^2+c^2}} \;\&\, n=\frac{-c}{\sqrt{a^2+b^2+c^2}}$$

Therefore,

$$l=\frac{\pm a}{\sqrt{a^2+b^2+c^2}} \;,m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}} \;\&\, n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

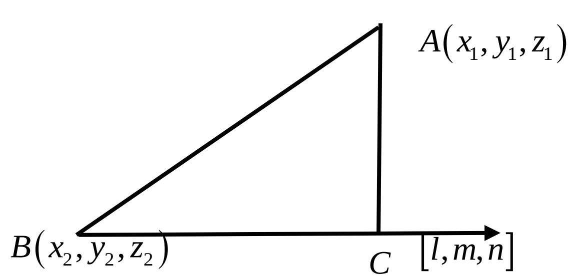
Theorem (Angle between two lines): Prove that $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^{\,2}+b_1^{\,2}+c_1^{\,2}}.\sqrt{a_2^{\,2}+b_2^{\,2}+c_2^{\,2}}}$ where θ is the angle the lines whose direction ratios are $\left[a_1,b_1,c_1\right]$ and $\left[a_2,b_2,c_2\right]$.

Theorem: The direction cosines of a line joining to the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $\left[\frac{x_2-x_1}{PQ},\frac{y_2-y_1}{PQ},\frac{z_2-z_1}{PQ}\right]$ where $PQ=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$.

Theorem: The direction ratios of a line joining to the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are $\left[x_2-x_1\,,y_2-y_1\,,z_2-z_1\right]$.

Theorem: The direction ratios of a line is $\left[l\,,m,n\right]$ and direction ratios of another line is $\left[a,b,c\right]$ then both be perpendicular iff $la+mb+nc=0$.

Theorem: The projection $BC=\left(x_2-x_1\right)l+\left(y_2-y_1\right)m+\left(z_2-z_1\right)n$ of joining of two points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ on another line whose direction cosine is given $\left[l\,,m,n\right]$.



Mathematical Problem on direction cosine and direction ratios

Problem01: Direction ratios of a line are $\left[3,4,12\right]$.What are its direction cosines?

Solution:

Given direction ratios are $\left[a,b,c\right]=\left[3,4,12\right]$.

Now,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{3}{\sqrt{3^2+4^2+12^2}}=\frac{3}{13}$$

$$m=\frac{b}{\sqrt{a^2+b^2+c^2}}=\frac{4}{\sqrt{3^2+4^2+12^2}}=\frac{4}{13}$$

And

$$n=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{12}{\sqrt{3^2+4^2+12^2}}=\frac{12}{13}$$

(As desired)

Problem02: Show that the direction cosine of a line is $\left[3,4,12\right]$ whose direction ratios are $\left[a,a,a\right]$.

Solution:

Given direction ratios are $\left[a,b,c\right]=\left[a,a,a\right]$.

Now,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{a}{\sqrt{a^2+a^2+a^2}}=\frac{1}{\sqrt{3}}$$

$$m=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{a}{\sqrt{a^2+a^2+a^2}}=\frac{1}{\sqrt{3}}$$

And

$$n=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{a}{\sqrt{a^2+a^2+a^2}}=\frac{1}{\sqrt{3}}$$

Therefore, the direction ratios are $\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$. **(As desired)**

H.W:

1. $\left[6,2,3\right]$ are proportional to the direction cosines of a line. What are their actual values?
2. Find two-sided direction cosines of a line whose direction ratios are $\left[1,2,3\right]$.
3. Find one-sided direction cosines of a line whose direction ratios are $\left[2,-1,2\right]$.

Problem02: Find the direction cosines of the line joining the points $\left(4,3,-5\right)$ and $\left(-2,1,-8\right)$.

Solution:

The coordinates of the given points are $P(x_1,y_1,z_1)=P\left(4,3,-5\right)$ and $Q(x_2,y_2,z_2)=Q\left(-2,1,-8\right)$

Here,

$$PQ=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

$$PQ=\sqrt{\left(-2-4\right)^2+\left(1-3\right)^2+\left(-8+5\right)^2}$$

$$PQ=\sqrt{\left(-2-4\right)^2+\left(1-3\right)^2+\left(-8+5\right)^2}$$

$$PQ=\sqrt{36+4+9}$$

$$PQ = \sqrt{49} = 7$$

Now the direction cosines of the line PQ are as follows

$$l = \frac{x_2 - x_1}{PQ} = \frac{-2-4}{7} = \frac{-6}{7}$$

$$m = \frac{y_2 - y_1}{PQ} = \frac{1-3}{7} = \frac{-2}{7}$$

And

$$n = \frac{z_2 - z_1}{PQ} = \frac{-8+5}{7} = \frac{-3}{7}$$

Therefore, the direction cosines are $[l, m, n] = \left[\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}\right] \text{ or } \left[\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right]$ **(As desired)**

Problem03: Find the direction cosines of the line which is perpendicular to the line with direction cosines proportional to $[6, 4, -4]$ and $[-6, 2, 1]$.

Solution:

Let $[l, m, n]$ be the direction cosines of the line perpendicular to the given lines with direction ratios $[6, 4, -4]$ and $[-6, 2, 1]$.

Therefore,

$$6l + 4m - 4n = 0$$
$$3l + 2m - 2n = 0 \cdots\cdots\cdots\cdots\cdots (i)$$

And

$$-6l + 2m + n = 0$$
$$6l - 2m - n = 0 \cdots\cdots\cdots\cdots\cdots (ii)$$

Using cross multiplication method, we get

$$\frac{l}{-2-4} = \frac{m}{-12+3} = \frac{n}{-6-12}$$

$$\frac{l}{-6} = \frac{m}{-9} = \frac{n}{-18}$$

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6}$$

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{2^2 + 3^2 + 6^2}}$$

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{1}{\sqrt{4+9+36}} \qquad \left[\because l^2 + m^2 + n^2 = 1\right]$$

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{1}{\sqrt{49}}$$

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{1}{7}$$

Therefore $l = \frac{2}{7}$, $m = \frac{3}{7}$ & $n = \frac{6}{7}$. **(As desired)**

H.W:

1. The coordinates of a point P is $(3, 12, 4)$. Find the direction cosines of the line OP where O is origin.

2. Find the direction ratios of the line joining to the points $(4, 3, -5)$ and $(-2, 1, -8)$.

3. P and Q are $(1,-5,7)$ and $(-3,6,-2)$.Find direction cosine of OP, OQ &PQ.

4.Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $[-2,-2,1]$ and $[2,1,0]$.

5. Find the direction ratios of the line joining to the points $(1,2,-3)$ and $(-2,3,1)$.

6. Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $[3,-1,1]$ and $[-3,2,4]$.

7. Find the direction cosines of the line which is perpendicular to the lines whose direction ratios are $[1,-2,-2]$ and $[0,2,1]$.

Problem04: The direction cosines of two lines are connected by relations $l+m+n=0$ and $l^2+m^2-n^2=0$. Find the angle between them.

Solution:

Given equations are as follows

$$l+m+n=0$$
$$l=-(m+n) \cdots\cdots\cdots\cdots(i)$$

And $$l^2+m^2-n^2=0$$

Putting the value of l in the above equation we get

$$\{-(m+n)\}^2+m^2-n^2=0$$
$$(m+n)^2+m^2-n^2=0$$
$$m^2+n^2+2mn+m^2-n^2=0$$
$$2m^2+2mn=0$$
$$m^2+mn=0$$
$$m(m+n)=0$$

Therefore $m=0\cdots\cdots\cdots\cdots(ii)$ and $m+n=0\cdots\cdots\cdots\cdots(iii)$

From equations (i) and (ii) we get

$$l+m+n=0\cdots\cdots\cdots\cdots(i)$$
$$0.l+m+0.n=0\cdots\cdots\cdots\cdots(ii)$$
$$\therefore \frac{l}{0-1}=\frac{m}{0-0}=\frac{n}{1-0}$$
$$\frac{l}{-1}=\frac{m}{0}=\frac{n}{1}$$

Now the direction ratios of one line are $[a_1,b_1,c_1]=[-1,0,1]$

Again,

From equations (i) and (iii) we get

$$l+m+n=0\cdots\cdots\cdots\cdots(i)$$
$$0.l+m+n=0\cdots\cdots\cdots\cdots(iii)$$
$$\therefore \frac{l}{1-1}=\frac{m}{0-1}=\frac{n}{1-0}$$
$$\frac{l}{0}=\frac{m}{-1}=\frac{n}{1}$$

Now the direction ratios of another line are $[a_2, b_2, c_2] = [0, -1, 1]$

If θ is the angle between two lines, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{0+0+1}{\sqrt{1+0+1}.\sqrt{0+1+1}}$$

$$\cos\theta = \frac{1}{\sqrt{2}.\sqrt{2}} = \frac{1}{2}$$

$$\cos\theta = \cos\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$ **(As desired)**

H.W:

1.Find the angle between the lines whose direction cosines are given by the equation $3l + m + 5n = 0$ and $6mn - 2nl - 5ml = 0$.

2.Show that the pair of lines whose direction cosines are given by $3lm - 4\ln + mn = 0$, $l + 2m + 3n = 0$ are perpendicular.

3. Find the angle between the lines whose direction cosines the equation $l + 2m - n = 0$ and $l^2 - 4m^2 - 3n^2 = 0$.

4. Find the angle between the lines whose direction cosines l, m, n satisfy the equations $l + m + n = 0$ and $2lm + 2\ln - mn = 0$.

5. Prove that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

6.A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

7.Find the angle between the lines whose direction ratios are $[2, -1, 3]$ and $[-1, 3, 4]$.

8. Find the angle between the lines whose direction cosines are proportional to $[1, 2, 4]$ and $[-2, 1, 5]$.

9. Find the angle between the lines whose direction ratios are $[1, 2, 1]$ and $[2, -3, 6]$.

10. Find the angle between the lines whose direction ratios are $[1, 1, 2]$ and $\left[\sqrt{3} - 1, -\sqrt{3} - 1, 4\right]$.

Problem05: The direction cosines of two lines are connected by relations $2l + 2m - n = 0$ and $mn + nl + lm = 0$. Show that two lines are perpendicular or at right angle.

Solution:

Given equations are as follows

$$2l + 2m - n = 0$$

$$n = 2l + 2m \cdots\cdots\cdots\cdots (i)$$

And $$mn + nl + lm = 0 \cdots\cdots\cdots\cdots\cdots (ii)$$

Putting the value of n in the above equation we get

$$m(2l + 2m) + (2l + 2m)l + lm = 0$$

$$(2lm + 2m^2) + (2l^2 + 2ml) + lm = 0$$

$$2lm + 2m^2 + 2l^2 + 2ml + lm = 0$$

$$5lm + 2m^2 + 2l^2 = 0$$

$$2l^2 + 5lm + 2m^2 = 0$$

$$2\frac{l^2}{m^2} + 5\frac{l}{m} + 2 = 0 \quad [\textit{dividing by } m^2]$$

$$2\left(\frac{l}{m}\right)^2 + 5\frac{l}{m} + 2 = 0$$

Assuming the direction cosines of the lines are $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ then the roots of the above quadratic equation are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

Therefore, product of the roots is $\frac{l_1}{m_1}.\frac{l_2}{m_2} = \frac{2}{2} = 1$

$$\frac{l_1 l_2}{m_1 m_2} = 1$$

$$\frac{l_1 l_2}{1} = \frac{m_1 m_2}{1}$$

Again,

From the equation (i), we have $l = \frac{n-2m}{2}$ and putting this value in above equation (ii) we find,

$$mn + n\left(\frac{n-2m}{2}\right) + \left(\frac{n-2m}{2}\right)m = 0$$

$$2\,mn + n(n-2m) + (n-2m)m = 0$$

$$2mn + (n^2 - 2mn) + (mn - 2m^2) = 0$$

$$2mn + n^2 - 2mn + mn - 2m^2 = 0$$

$$n^2 - 2m^2 + mn = 0$$

$$2m^2 - mn - n^2 = 0$$

$$2\frac{m^2}{n^2} - \frac{m}{n} - 1 = 0$$

$$2\left(\frac{m}{n}\right)^2 - \frac{m}{n} - 1 = 0$$

Above equation is a quadratic equation in $\frac{m}{n}$ so two roots of this equations are $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$.

Therefore, product of the roots is $\frac{m_1}{n_1}.\frac{m_2}{n_2} = \frac{-1}{2}$

$$\frac{m_1 m_2}{n_1 n_2} = \frac{1}{-2}$$

$$\frac{n_1 n_2}{-2} = \frac{m_1 m_2}{1}$$

Considering above relation we can write

$$\frac{n_1 n_2}{-2} = \frac{m_1 m_2}{1} = \frac{l_1 l_2}{1}$$

$$\frac{n_1 n_2}{-2} = \frac{m_1 m_2}{1} = \frac{l_1 l_2}{1} = k \; [\textit{Say}]$$

That implies $l_1l_2 = k$, $m_1m_2 = k$ & $n_1n_2 = -2k$

Now, $l_1l_2 + m_1m_2 + n_1n_2 = k + k - 2k$

$$l_1l_2 + m_1m_2 + n_1n_2 = 2k - 2k$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Consequently, given that two lines are perpendicular. **(As desired)**

H.W:

1. The direction cosines of two lines are connected by relations $3lm - 4\ln + mn = 0$ and $l + +2m + 3n = 0$. Show that two lines are perpendicular.

2.Prove that the straight lines whose direction cosines are given by $al + bm + cn = 0$ & $mn + nl + lm = 0$ are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ and parallel if $a^2 f^2 + b^2 g^2 + c^2 h^2 - 2bcgh - 2cahf - 2abfg = 0$ or $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$.

3. Show that the straight lines whose direction cosines are given by $al + bm + cn = 0$ & $fmn + gnl + hlm = 0$ are at right angle if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ and parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

4. Show that the straight lines whose direction cosines are given by $l + m + n = 0$ & $al^2 + bm^2 + cn^2 = 0$ are at right angle if $a + b + c = 0$ and parallel if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

5. Show that the straight lines whose direction cosines are given by $al + bm + cn = 0$ & $ul^2 + vm^2 + \omega n^2 = 0$ are at right angle if $a^2(v + \omega) + b^2(\omega + u) + c^2(u + v) = 0$ and parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$.

Problem06: If P, Q, A, B are $(1,2,5)$, $(-2,1,3)$, $(4,4,2)$ & $(2,1,-4)$ respectively then find the projection of PQ on AB.

Solution:

The coordinates of the given points are $P(1,2,5)$, $Q(-2,1,3)$, $A(4,4,2)$ & $B(2,1,-4)$.

Here,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$AB = \sqrt{(2-4)^2 + (1-4)^2 + (-4-2)^2}$$

$$AB = \sqrt{4 + 9 + 36}$$

$$AB = \sqrt{49} = 7$$

Now, the direction cosine of the line AB are as follows

$$l = \frac{x_1 - x_2}{AB} = \frac{4-2}{7} = \frac{2}{7}$$

$$m = \frac{y_1 - y_2}{AB} = \frac{4-1}{7} = \frac{3}{7}$$

And

$$n = \frac{z_1 - z_2}{AB} = \frac{2+4}{AB} = \frac{6}{7}$$

Therefore, the direction cosines are $[l, m, n] = \left[\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right]$ $or \left[\frac{-2}{7}, \frac{-3}{7}, \frac{-6}{7}\right]$

Projection of PQ on AB $=(x_1-x_2)l+(y_1-y_2)m+(z_1-z_2)n$

$$=(1+2)\frac{2}{7}+(2-1)\frac{3}{7}+(5-3)\frac{6}{7}$$

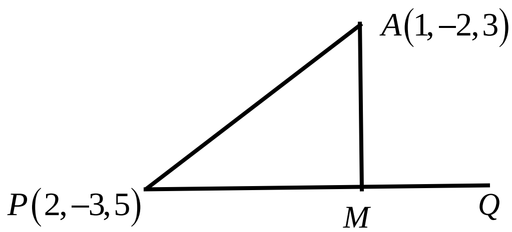
$$=\frac{6}{7}+\frac{3}{7}+\frac{12}{7}$$

$$=\frac{6+3+12}{7}$$

$$=\frac{21}{7}=3$$ **(As desired)**

Problem07: Find the distance of $A(1,-2,3)$ from the line PQ through $P(2,-3,5)$ which makes equal angles with the axes.

Solution:



According to the condition the line PQ makes equal angles with the axes.

Assume that the directed angles of the line PQ are $[\alpha,\beta,\delta]$ and according to the question $\alpha=\beta=\delta$ that implies $\cos\alpha=\cos\beta=\cos\delta$ i.e. $l=m=n$.

From the fundamental law of direction cosine, we have

$$l^2+m^2+n^2=1$$

$$l^2+l^2+l^2=1$$

$$3l^2=1$$

$$l^2=\frac{1}{3}$$

$$l=\pm\frac{1}{\sqrt{3}}$$

Therefore $[l,m,n]=\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$ or $\left[\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right]$

Now Projection PM $=(x_1-x_2)l+(y_1-y_2)m+(z_1-z_2)n$

$$=(2-1)\frac{1}{\sqrt{3}}+(-3+2)\frac{1}{\sqrt{3}}+(5-3)\frac{1}{\sqrt{3}}$$

$$=\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}$$

From the triangle PAM, we get

$$AP=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$=\sqrt{(2-1)^2+(-3+2)^2+(5-3)^2}$$

$$=\sqrt{1+1+4}=\sqrt{6}$$

Applying Pythagorean theorem on the triangle PAM we find the desired distance AM.

$$AP^2 = AM^2 + PM^2$$
$$AM^2 = AP^2 - PM^2$$
$$\therefore\ AM = \sqrt{AP^2 - PM^2}$$
$$\therefore\ AM = \sqrt{6 - \frac{4}{3}} = \sqrt{\frac{18-4}{3}} = \sqrt{\frac{14}{3}} \text{ units}$$ **(As desired)**

H.W:

1. If A, B, C, D are $(3,6,4)$, $(2,5,2)$, $(6,4,4)$ & $(0,2,1)$ respectively then find the projection of AB on CD.

2.Find the length of a line having its projections on the coordinate axes to be 6,-2,3.

3.A line passing through the point $(-3,-2,8)$ and having direction ratios $[3,2,-2]$.Find the perpendicular distance of that line from the point $(1,-1,-2)$.

4. Find the projection of the join of points $(-1,-1,3)$ and $(2,0,1)$ on the line through the points $(-7,5,3)$ and $(2,6,8)$.