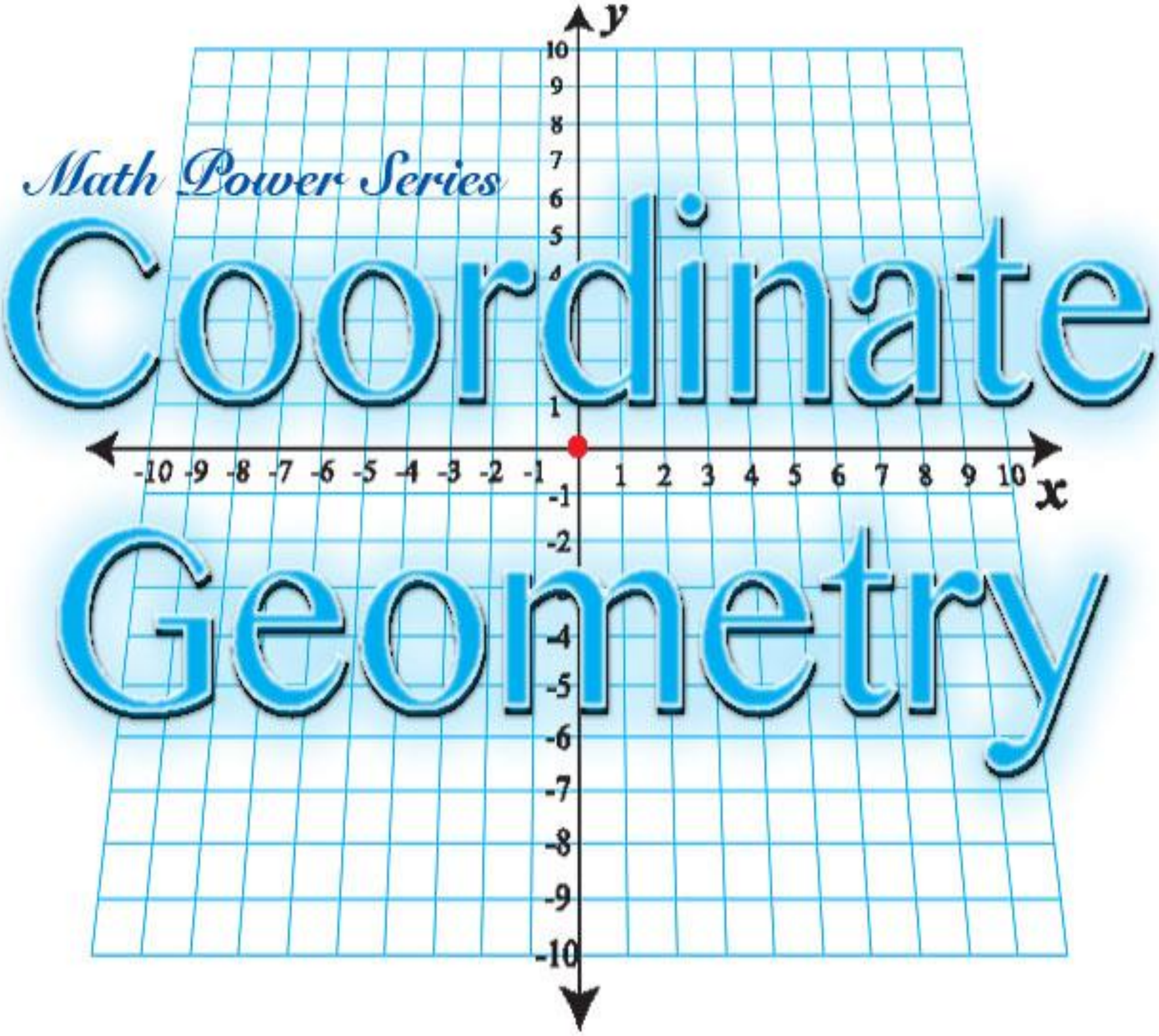

This lecture note is being provided as a partial fulfillment of Mathematics-II offered in DIU.

It is made by,

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Change of Axes

Transformation of coordinates:

The process of changing the coordinates of point or the equation of the curves is called transformation of coordinates.

Transformation of coordinates is of three types such as follows

1. Translation of axes:

In this process the position of the origin is changed but the direction of coordinate axes is being parallel to the old system.

When origin $(0,0)$ shifted to the new point (h,k) and keeping the direction of coordinate axes fixed then the pair of equations

$$x = x' + h \ , y = y' + k$$

represents the relation between new system(X', Y') and old system (X, Y) and is called the translation of axes.

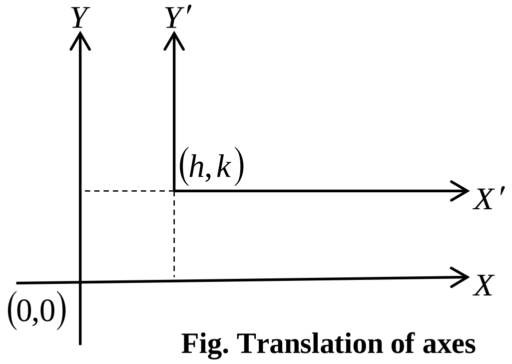


Fig. Translation of axes

2. Rotation of axes :

In this process the position of the origin is not changed but the direction of coordinate axes is being changed through a fixed angle with the x-axis.

When the position of the origin is not changed and the direction of coordinate axes is being changed through a fixed angle θ with the X-axis then then this is called rotation of axes and the relation between new (X', Y') system and old system (X, Y) are given below.

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$or$$

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta$$

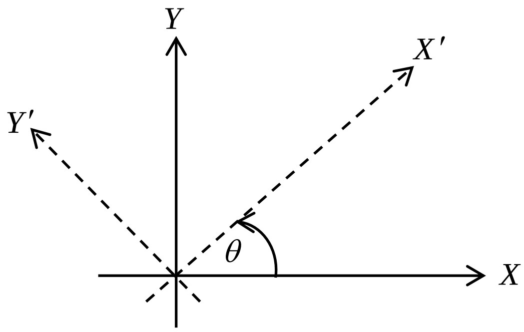
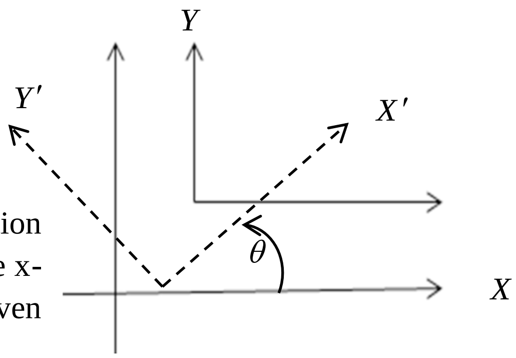


Fig. Rotation of axes



3. Translation-Rotation:

In this process the position of the origin is changed and the direction of coordinate axes is being changed through a fixed angle with the x-axis. The relation between new system and old system are given below.

$$x = x'\cos\theta - y'\sin\theta + h \qquad y = x'\sin\theta + y'\cos\theta + k$$

Mathematical problem

Mathematical problem on Translation of Axis

Problem 01: Determine the equation of the curve $2x^2+3y^2-8x+6y-7=0$ when the origin is transferred to the point $(2,-1)$.

Solution:

Given Equation of the curve is,

$$2x^2+3y^2-8x+6y-7=0 \quad \cdots\cdots\cdots\cdots\cdots(i)$$

Origin is transferred to the point $(h,k)=(2,-1)$ so as the transformed relations are $x=x'+h=x'+2$ and $y=y'+k=y'-1$.

Using the above transformation given equation (i) becomes

$$2(x'+2)^2+3(y'-1)^2-8(x'+2)+6(y'-1)-7=0$$
$$\Rightarrow\ 2(x'^2+4x'+4)+3(y'^2-2y'+1)-8(x'+2)+6(y'-1)-7=0$$
$$\Rightarrow\ (2x'^2+8x'+8)+(3y'^2-6y'+3)-(8x'+16)+(6y'-6)-7=0$$
$$\Rightarrow\ 2x'^2+8x'+8+3y'^2-6y'+3-8x'-16+6y'-6-7=0$$
$$\Rightarrow\ 2x'^2+3y'^2-18=0$$
$$\Rightarrow\ 2x'^2+3y'^2=18$$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$2x^2+3y^2=18$$

Thich is the required equation that represents an ellipse.

Problem 02: What does the equation $x^2+y^2-4x-6y+6=0$ becomes when the origin is transferred to the point $(2,3)$ and the direction of axes remain unaltered.

Solution:

Given Equation of the curve is,

$$x^2+y^2-4x-6y+6=0 \quad \cdots\cdots\cdots\cdots\cdots(i)$$

Origin is transferred to the point $(h,k)=(2,3)$ so as the transformed relations are $x=x'+h=x'+2$ and $y=y'+k=y'+3$.

Using the above transformation given equation (i) reduces to

$$(x'^2+4x'+4)+(y'^2+6y'+9)-4(x'+2)-6(y'+3)+6=0$$
$$\Rightarrow\ (x'^2+4x'+4)+(y'^2+6y'+9)-(4x'+8)-(6y'+18)+6=0$$
$$\Rightarrow\ x'^2+4x'+4+y'^2+6y'+9-4x'-8-6y'-18+6=0$$
$$\Rightarrow\ x'^2+y'^2-17=0$$
$$\Rightarrow\ x'^2+y'^2-17=0$$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$x^2+y^2=17$$

Which is the required equations that represents a circle.

H.W.

1. Transform to parallel axes through the point $(3,5)$ the equation $x^2+y^2-6x-10y-2=0$.
2. Transform $x^2+2y^2-6x+7=0$ to parallel axes through the point $(3,1)$.
3. Transform the equation $3x-25y+41=6$ to parallel axes through $(-3,2)$.
4. Transform the equation $x^2-3y^2+4x+6y=0$ by transferring the origin to the point $(-2,1)$, coordinate axes remaining parallel.
5. Transform the equation $3x^2+14xy-24y^2-22x+110y-121=0$ shifting the origin to the point $(-1,2)$ and keeping the direction of axes fixed.

Mathematical problem on Rotation of Axis

Problem 03: Transform the equation $3x^2+5y^2-3=0$ to axes turned through 45°.

Solution: Given that,

$$3x^2+5y^2-3=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since the axes rotated are an angle 45° and origin be unchanged.

So, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$= x'\cos45^\circ - y'\sin45^\circ$, $= x'\sin45^\circ + y'\cos45^\circ$

$= \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$, $= \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$

Using this value in equation (i), we get,

$$3\left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)^2 + 5\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)^2 - 3 = 0$$

Or, $3\{\frac{1}{\sqrt{2}}(x'-y')\}^2 + 5\{\frac{1}{\sqrt{2}}(x'+y')\}^2 - 3 = 0$

Or, $\frac{3}{2}(x'^2 - 2x'y' + y'^2) + \frac{5}{2}(x'^2 + 2x'y' + y'^2) - 3 = 0$

Or, $3x'^2 - 6x'y' + 3y'^2 + 5x'^2 + 10x'y' + 5y'^2 - 6 = 0$

$\therefore 8x'^2 + 4x'y' + 8y'^2 - 6 = 0$

Now removing suffixes, we can write,

$4x^2 + 2xy + 4y^2 - 3 = 0.$

This is the required equation.

Problem 04: If the axes be turned through an angle $\tan^{-1}2$, what does the equation $4xy-3x^2=a^2$ become?

Solution:

Given Equation of the curve is,

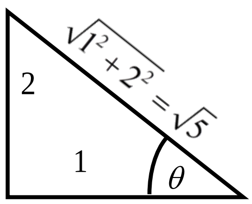
$$4xy-3x^2=a^2 \cdots\cdots\cdots\cdots\cdots (i)$$

The coordinate axes turned through an angle $\theta=\tan^{-1}2$ that implies $\tan\theta=2$.

Now

$\sin\theta = \frac{2}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}}$



Considering the new coordinate of the point is (x', y') and rotating the axes through an angle $\theta = \tan^{-1} 2$ and origin be unchanged as the transformed equations are as follows

$$x = x'\cos\theta - y'\sin\theta$$

$$= \frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}} \quad = \frac{1}{\sqrt{5}}(x' - 2y')$$

And,

$$y = x'\sin\theta + y'\cos\theta$$

$$= \frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}} \quad = \frac{1}{\sqrt{5}}(2x' + y')$$

Putting the value of x and y the above equation (i) becomes

$$4\left\{\frac{1}{\sqrt{5}}(x'-2y')\right\}\left\{\frac{1}{\sqrt{5}}(2x'+y')\right\} - 3\left\{\frac{1}{\sqrt{5}}(x'-2y')\right\}^2 = a^2$$

Or, $4.\frac{1}{5}(x'-2y')(2x'+y') - \frac{3}{5}(x'-2y')^2 = a^2$

Or, $4(x'-2y')(2x'+y') - 3(x'-2y')^2 = 5a^2$

Or, $4\left(2x'^2 + x'y' - 4x'y' - 2y'^2\right) - 3\left(x'^2 - 4x'y' + 4y'^2\right) = 5a^2$

Or, $4\left(2x'^2 - 3x'y' - 2y'^2\right) - 3\left(x'^2 - 4x'y' + 4y'^2\right) = 5a^2$

Or, $\left(8x'^2 - 12x'y' - 8y'^2\right) - \left(3x'^2 - 12x'y' + 12y'^2\right) = 5a^2$

Or, $8x'^2 - 12x'y' - 8y'^2 - 3x'^2 + 12x'y' - 12y'^2 = 5a^2$

Or, $5x'^2 - 20y'^2 = 5a^2$

Or, $x'^2 - 4y'^2 = a^2$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$x^2 - 4y^2 = a^2$$

H.W:

1. Transformed the equation $7x^2 - 2xy + y^2 + 5 = 0$ to the axes turned through an angle $\tan^{-1}(1)$.
2. Transformed the equation $7x^2 - 2xy + y^2 + 1 = 0$ to the axes turned through an angle $\tan^{-1}(\frac{1}{2})$.
3. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ after rotating of axes through $\frac{\pi}{4}$.

Mathematical problem on Translation-Rotation

Problem 05: Determine the transform equation of $3x - 2y + 5 = 0$ when the origin is transferred to the point $(-2, -1)$ and the axes turned through an angle 45^0.

Solution:
Given Equation is,

$$3x - 2y + 5 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Origin is transferred to the point $(h, k) = (-2, -1)$ so as the transformed relations are $x = x' - 2$ and $y = y' - 1$

Using the above transformation given equation (i) becomes

$$3(x' - 2\,) - 2(y' - 1) + 5 = 0$$
$$\Rightarrow 3x' - 6 - 2y' + 2 + 5 = 0$$
$$\Rightarrow 3x' - 2y' + 1 = 0$$

Now removing suffixes, we can write,

$$3x - 2y + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Again the axes rotated are an angle 45°

So, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$= x'\cos 45^\circ - y'\sin 45^\circ$, $\quad = x'\sin 45^\circ + y'\cos 45^\circ$

$= \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$, $\quad = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$

Using this value in equation (ii), we get,

$$3\left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right) - 2\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) + 1 = 0$$
$$\Rightarrow 3(x' - y') - 2(x' + y') + \sqrt{2} = 0$$
$$\Rightarrow 3x' - 3y' - 2x' - 2y' + \sqrt{2} = 0$$
$$\Rightarrow x' - 5y' + \sqrt{2} = 0$$

Now removing suffixes, we can write,

$$x - 5y + \sqrt{2} = 0$$

This is the required transform equation.

H.W:

1. Find transform equation of $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ when the origin is transferred to the point (2,3) and the axes turned through an angle 45^0.
2. Find transform equation of $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ when the origin is transferred to the point $(-2,1)$ and the axes turned through an angle 60^0.
3. Find transform equation of $4x^2 + xy - y^2 - 8x + 2y + 5 = 0$ when the origin is transferred to the point $(-1, -2)$ and the axes turned through an angle 55^0.

Equation & its Geometry

◙ Equation in 2 variables:

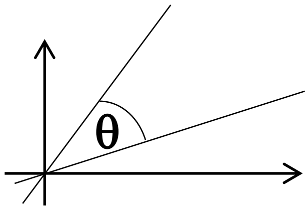
- **1st degree General equation:**

$$ax + by + c = 0$$

It always represents a straight line in plane, provided at least $a \neq 0$ or, $b \neq 0$

❑ **Homogeneous equation:**

An equation in which degree of each term in it is equal is called Homogeneous equation. Such as $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of degree or order 2 because degree of its each term is two. It is noted that homogeneous equation always represents straight lines passing through the origin.



- **2nd degree homogenous equation:**

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{----------------(*)}$$

In plane, it always represents two straight lines passing through the origin (0, 0).

- ❖ Lines be perpendicular if $a + b = 0$
- ❖ Lines be parallel/coincident if $h^2 = ab$
  Since two lines passes through the point $(0,0)$, so lines must be coincident.
- ❖ Lines be real and different if $h^2 > ab$.
- ❖ Lines be imaginary if $h^2 < ab$. But passes through the point $(0,0)$.

❑ **Non-homogeneous equation:**

An equation in which degree of each term in it is not equal is called Non-homogeneous equation. Such as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a non- homogeneous equation of degree or order 2.

- **2nd degree General equation :**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{------------------(**)}$$

Descartes found that the graphs of second-degree equations in two variables in plane always fall into one of seven categories: [1] single point, [2] pair of straight lines, [3] circle, [4] parabola, [5] ellipse, [6] hyperbola, and [7] no graph at all.

Note:

- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, represents pair of straight lines if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

(1) Two parallel lines if $\Delta = 0, h^2 = \mathrm{ab}$.
(2) Two perpendicular lines if $\Delta = 0, \mathrm{a} + \mathrm{b} = 0$.

- Otherwise, (**) will represent:
    1. A circle if $a = b, h = 0$.
    2. A parabola if $\Delta \neq 0, h^2 = \mathrm{ab}$
    3. An ellipse if $\Delta \neq 0, h^2 - \mathrm{ab} < 0$.
    4. A hyperbola if $\Delta \neq 0, h^2 - \mathrm{ab} > 0$.
    5. A rectangular hyperbola if $a + b = 0, h^2 - ab > 0, \Delta \neq 0$.

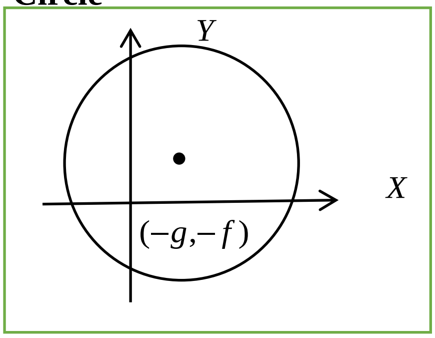
If none of the above conditions are satisfied then (**) represents [1] or [7].

Also,

- ✓ if $c = 0$, then (**) always passes through the origin(0,0).
- ✓ term containing xy can be transformed by a suitably chosen rotation into a form which does not contain xy term. The standard equation is easily obtained.
- ✓ the rotation angle 'θ' that will eliminate the 'xy' term is given by $\cot 2\theta = \frac{a-b}{2h}$.

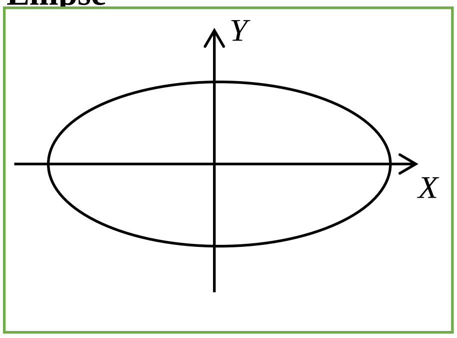
Some figure and standard form:-

Circle

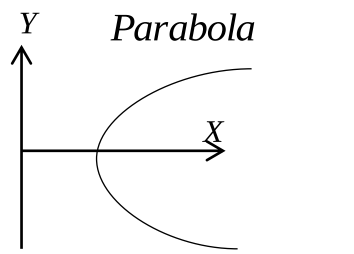


Std.form. $\boldsymbol{x^2 + y^2 + 2gx + 2fy + c = 0}$

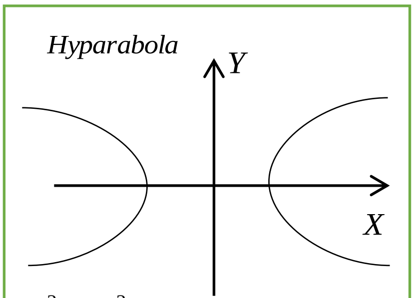
Ellipse



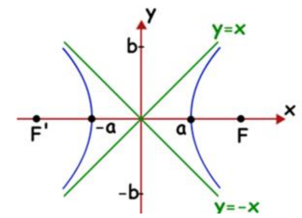
Std.form. $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$



Std.form. $y^2 = 4ax$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Std.form. $x^2 - y^2 = 1$

$Rectangular-Hyparabula$

Angle 'Θ' between 2 (real) straight lines represented by (*) or (**):

We know that 2 straight lines always cut at an angle (real or imaginary). If 'Θ' be that angle, we have to use following formula to find 'Θ'.

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

Case 1 : If $h^2 < ab$, then angle 'Θ' is **imaginary** and we can't view it.

Case 2 : When $h^2 \nless ab$, then angle 'Θ' is not **imaginary**.

Some condition for angle 'Θ':-

- If $\Theta = 0^0$, we say, the straight lines are either parallel or, coincident (*i.e.* same).
- If $a + b = 0$, then $\Theta = 90^0$, and we say, the straight lines are perpendicular.
- If Θ is +ve, we accept it & say that the angle is acute.
- If Θ is -ve, we add 180^0 , and then get an obtuse angle.

Separation of equation of each line from (*) or (**):

To determine the equation of each line separately, we need to solve (*) or (**).*i.e.* we write the equation as $\mathbf{Ax^2 + B\,x + C = 0}$ & then solve it.

Mathematical problem

Problem 01:- If the equation is $3x^2 - 16xy + 5y^2 = 0$, then find, **(a)** angle between the lines **(b)** equation of line.

Solution:

(a)

Given homogeneous equation is as follows

$$3x^2 - 16xy + 5y^2 = 0$$

Comparing the given equation with the general homogeneous equation $ax^2 + 2hxy + by^2 = 0$ we have $a = 3, h = -8$ *and* $b = 5$.

Let an angle between the lines is θ.

Then we have $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$

$$\text{Or, } \tan\theta = \frac{2\sqrt{(-8)^2 - 3.5}}{3+5}$$

$$\text{Or, } \tan\theta = \frac{2\sqrt{64-15}}{8}$$

$$\text{Or, } \tan\theta = \frac{2\sqrt{49}}{8} = \frac{2.7}{8} = \frac{14}{8}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{14}{8}\right) = 60.26^0$$

Therefore the angle between the lines is 60.26^0.

(b)

Given homogeneous equation is as follows

$$3x^2 - 16xy + 5y^2 = 0$$

We expressed the given equation as

$$3x^2 - 16xy + 5y^2 = 0$$

$$\text{Or, } 3x^2 - 16y.x + 5y^2 = 0$$

$$\text{Or, } x = \frac{16y \pm \sqrt{(-16y)^2 - 4.3.5y^2}}{2.3}$$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Or, } x = \frac{16y \pm \sqrt{256y^2 - 60y^2}}{6}$$

$$\text{Or, } x = \frac{16y \pm \sqrt{196y^2}}{6}$$

$$\text{Or, } x = \frac{16y \pm 14y}{6}$$

Taking positive sign we get $x = \frac{16y + 14y}{6} = \frac{30y}{6} = 5y$

Therefore $x = 5y \Rightarrow x - 5y = 0$

And taking negative sign we get $x = \frac{16y - 14y}{6} = \frac{2y}{6} = \frac{y}{3}$

Therefore, $x = \frac{y}{3} \Rightarrow 3x = y\ \therefore 3x - y = 0$

Therefore $x - 5y = 0$ and $3x - y = 0$

These are the straight lines passing through the origin.

Problem 02:- Show that $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents pair of straight lines.

Solution:

Given equation is,

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad \cdots\cdots\cdots\cdots (i)$$

Comparing this above equation with the standard equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$$a = 6,\ h = -\frac{5}{2}, b = -6, g = 7, f = \frac{5}{2}\ \&\ c = 4$$

Now,

$$\Delta = \begin{vmatrix} 6 & -\frac{5}{2} & 7 \\ -\frac{5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} = 6(-24 - \frac{25}{4}) - \left(-\frac{5}{2}\right)\left(-10 - \frac{35}{2}\right) + 7\left(-\frac{25}{4} + 42\right)$$

$$=6(-24-\frac{25}{4})-\left(-\frac{5}{2}\right)\left(-10-\frac{35}{2}\right)+7\left(-\frac{25}{4}+42\right)$$

$$=6(-24-\frac{25}{4})+\frac{5}{2}\left(-10-\frac{35}{2}\right)+7\left(-\frac{25}{4}+42\right)$$

$$=(-144-\frac{150}{4})+\left(-25-\frac{175}{4}\right)+\left(-\frac{175}{4}+294\right)$$

$$=\frac{-576-150}{4}+\left(\frac{-100-175}{4}\right)+\left(\frac{-175+1176}{4}\right)$$

$$=\frac{-726}{4}+\left(\frac{-275}{4}\right)+\left(\frac{1001}{4}\right)$$

$$=-\frac{1001}{4}+\frac{1001}{4}=0$$

Since $\Delta=0$ so the given equation represents a pair of straight lines.

Another process,

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0 \quad \cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=6\ ,h=-\frac{5}{2},b=-6,g=7,\ f=\frac{5}{2}\ \&c=4$$

We know that, $\Delta = abc+2fgh-af^2-bg^2-ch^2$

$$=6*(-6)*4+2*\left(\frac{5}{2}\right)*7*\left(-\frac{5}{2}\right)-6*\left(\frac{5}{2}\right)^2-(-6)*(7)^2-4*\left(-\frac{5}{2}\right)^2$$

$$=0$$

Since $\Delta=0$ so the given equation represents a pair of straight lines.

Problem 03:- Find the angle between the straight lines represented by the equation

$$6x^2-5xy-6y^2+14x+5y+4=0.$$

Solution:

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=6\ ,h=-\frac{5}{2},b=-6,g=7,\ f=\frac{5}{2}\ \&c=4$$

Assume that θ be the angle between the straight lines then we have the followings

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow \quad \tan\theta=\frac{2\sqrt{\frac{25}{4}+36}}{6-6}$$

$$\Rightarrow \quad \tan\theta=\infty$$

$$\Rightarrow \quad \theta=\tan^{-1}(\infty) \qquad \therefore\ \theta=\frac{\pi}{2}$$

Problem 04:- Find the equation of the straight lines represented by the equation
$x^2+6xy+9y^2+4x+12y-5=0$.

Solution:

Given equation is,

$$x^2+6xy+9y^2+4x+12y-5=0$$

Arrange the above equation as a quadratic equation in x we get

$$x^2+6xy+9y^2+4x+12y-5=0$$

$$x^2+(6y+4)x+9y^2+12y-5=0$$

$$\therefore\ x=\frac{-(6y+4)\pm\sqrt{(6y+4)^2-4.1.(9y^2+12y-5)}}{2.1}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm\sqrt{(6y+4)^2-4(9y^2+12y-5)}}{2}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-(36y^2+48y-20)}}{2}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-36y^2-48y+20}}{2}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm\sqrt{16+20}}{2}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm\sqrt{36}}{2}$$

$$\Rightarrow\ x=\frac{-(6y+4)\pm 6}{2}$$

Taking positive we get $x=\frac{-(6y+4)+6}{2}$

$$\Rightarrow\ x=\frac{-6y-4+6}{2}$$

$$\Rightarrow\ x=\frac{-6y+2}{2}$$

$$\Rightarrow\ 2x=-6y+2$$

$$\Rightarrow\ x=-3y+1$$

$$\Rightarrow\ x+3y-1=0$$

Taking negative we get $x=\frac{-(6y+4)-6}{2}$

$$\Rightarrow\ x=\frac{-6y-4-6}{2}$$

$$\Rightarrow\ x=\frac{-6y-10}{2}$$

$$\Rightarrow\ 2x=-6y-10$$

$$\Rightarrow\ x=-3y-5$$

$$\Rightarrow\ x+3y+5=0$$

Therefore, required equations of the straight lines $x+3y-1=0$ and $x+3y+5=0$.

Problem 05:- What are represented by the following equations?

(1) $12x^2+7xy-10y^2+13x+45y-35=0$

(2) $x^2+2xy+y^2+2x-1=0$

(A) If it will be a straight line then find,

(i) The angle in between the lines; (ii) find the equation of each line.

(B) If it will be a curve then find,

(i) Find a rotation angle by which the xy-term will be eliminated; (ii) find the standard form.

Solution:- (1)

Given equation is,

$$12x^2+7xy-10y^2+13x+45y-35=0$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=12,\quad b=-10,\quad c=-35,\quad g=\frac{13}{2},\quad f=\frac{45}{2},\quad h=\frac{7}{2}$$

Now we have, $\Delta = abc+2fgh-af^2-bg^2-ch^2$

$$=12*(-10)*(-35)+2*\left(\frac{45}{2}\right)*\left(\frac{13}{2}\right)*\left(\frac{7}{2}\right)-12*\left(\frac{45}{2}\right)^2-(-10)*\left(\frac{13}{2}\right)^2-(-35)*\left(\frac{7}{2}\right)^2$$

$$=0$$

Since $\Delta=0$ so the given equation represents a pair of straight lines.

The angle between two lines:-

If θ be the angle between the straight lines then we know that

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow \tan\theta=\frac{2\sqrt{\left(\frac{7}{2}\right)^2-12*(-10)}}{12+(-10)}$$

$$\Rightarrow \tan\theta=11.5$$

$$\Rightarrow \theta=\tan^{-1}(11.5)$$

$$\Rightarrow \theta=85^0 1' 48.93''$$

Separation of lines:-

Given equation is,

$$12x^2+7xy-10y^2+13x+45y-35=0$$

$$\Rightarrow 12x^2+(7y+13)x+(-10y^2+45y-35)=0$$

$$\Rightarrow x=\frac{-(7y+13)\pm\sqrt{(7y+13)^2-4\cdot(-10y^2+45y-35)\cdot 12}}{2\cdot 12}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow 24x=-(7y+13)\pm\sqrt{(23y-43)^2}$$

$$\Rightarrow 24x+7y+13=\pm(23y-43)$$

Taking positive,

$$24x+7y+13=(23y-43)$$
$$\Rightarrow\ 3x-2y+7=0$$

Taking negative,

$$24x+7y+13=-(23y-43)$$
$$4x+5y-5=0$$

Hence, $3x-2y+7=0$ and $4x+5y-5=0$ are the required equations of two straight lines represented by $12x^2+7xy-10y^2+13x+45y-35=0$.

Solution:- (2)

Given general equation of second degree is

$$x^2+2xy+y^2+2x-1=0\cdots\cdots\cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=1,h=1,b=1,g=1,f=0\ \&\ c=-1$$

Now,

$$\Delta=\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & -1\end{vmatrix}=1\neq 0$$

And

$$h^2-ab=1-1=0$$

Since $\Delta\neq 0$ and $h^2-ab=0$. So the equation represents parabola.

To remove 'xy' term we need a rotation of 'θ' where

$$\cot 2\theta=\frac{a-b}{2h}$$
$$\text{Or, }\cot 2\theta=\frac{1-1}{2\cdot 1}=0$$
$$\text{Or, }2\theta=90^0$$
$$\theta=45^0$$

Thich is the required angle.

Now we know that,

$$x=x'\cos\theta-y'\sin\theta \qquad \text{and} \qquad y=x'\sin\theta+y'\cos\theta$$
$$=x'\cos45^\circ-y'\sin45^\circ \qquad\qquad =x'\sin45^\circ+y'\cos45^\circ$$
$$=\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y' \qquad\qquad =\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'$$
$$=\frac{1}{\sqrt{2}}(x'-y') \qquad\qquad =\frac{1}{\sqrt{2}}(x'+y')$$

Putting this in (i) we get,

$$x^2+2xy+y^2+2x-1=0$$
$$\text{Or, }(x+y)^2+2x-1=0$$
$$\text{Or, }\left\{\frac{1}{\sqrt{2}}(x'-y')+\frac{1}{\sqrt{2}}(x'+y')\right\}^2+2\cdot\frac{1}{\sqrt{2}}(x'-y')-1=0$$
$$\text{Or, }\left\{\frac{1}{\sqrt{2}}(x'-y'+x'+y')\right\}^2+2\cdot\frac{1}{\sqrt{2}}(x'-y')-1=0$$

Or, $2(x')^2+\sqrt{2}x'-\sqrt{2}y'-1=0$

Or, $2(x')^2+\sqrt{2}x'=\sqrt{2}y'+1$ Thich can be written as

Or, $2(x)^2+\sqrt{2}x=\sqrt{2}y+1$

Standard form,

$$2(x)^2+\sqrt{2}x=\sqrt{2}y+1$$

Or, $(x)^2+\frac{1}{\sqrt{2}}x=\frac{1}{\sqrt{2}}y+\frac{1}{2}$ (Both side multiply by 2)

Or, $(x)^2+2\cdot x\frac{1}{2\sqrt{2}}+(\frac{1}{2\sqrt{2}})^2-\frac{1}{8}=\frac{1}{\sqrt{2}}y+\frac{1}{2}$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}y+\frac{5}{8}$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}(y+\sqrt{2}\cdot\frac{5}{8})$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}(y+\cdot\frac{5}{4\sqrt{2}})$

Or, $(x+\frac{1}{2\sqrt{2}})^2=4\frac{1}{4\sqrt{2}}(y+\cdot\frac{5}{4\sqrt{2}})$

$X^2=4AY$ where $X=(x+\frac{1}{2\sqrt{2}}), A=\frac{1}{4\sqrt{2}}\ \&\ Y=(y+\cdot\frac{5}{4\sqrt{2}})$

That is the standard form of Parabola.

H.W

What are represented by the following equations?

1. $x^2+xy+y^2+x+y=0$.
2. $6x^2-5xy-6y^2=0$;
3 $4x^2-15xy-4y^2-46x+14y+60=0$;
4. $x^2-5xy+y^2+8x-20y+15=0$;
3. $34x^2+24xy+41y^2+48x+14y-108=0$;
5. $9x^2-24xy+16y^2-18x-101y+19=0$;
6. $x^2+2xy-y^2+2x+4y-3=0$;
7. $12x^2+7xy+32x+2y=0$;

(A) If it will be a straight line then find,

(i) The angle in between the lines ; (ii) find the equation of each line.

(B) If it will be a curve then find,

(i) Find a rotation angle by which the xy-term will be eliminated; (ii) find the standard form.

Equation in 3 variables:

- **1st degree general equation :** $ax + by + cz + d = 0$

  It always represents a plane.

- **The 2nd degree general equation :**

  It is given by: $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0$.

  It represents a quadratic surface (in brief, quadrics).

  Some Surfaces in standard forms:

Equations	**Represents**
(a second order polynomial containing quadratic terms in x, y, and z) = constant	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	an elliptic cone
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	an ellipsoid.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	a hyperboloid of one sheet
$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	a hyperboloid of two sheets.
(Equations containing two quadratic terms and one linear term)	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	an elliptic paraboloid.
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	a hyperbolic paraboloid.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	an elliptic cylinder
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	a hyperbolic cylinder
$x^2 + 2kz = 0$	a parabolic cylinder

Coordinates Transformation

■ **TWO DIMENSIONAL (2D):**

- Rectangular/Cartesian (x,y) system
- Polar (r, θ) system.

■ **THREE DIMENSIONAL (3D):**

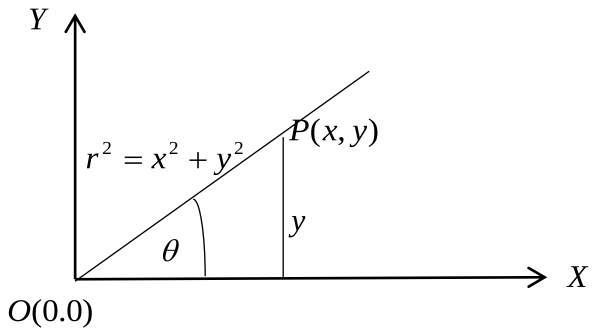
- Rectangular/Cartesian (x,y,z) system
- Cylindrical (r,θ,h) system.
- Spherical (ρ, φ, θ) system. Θ is called the azimuthal angle, φ the zenith angle.

■ TWO DIMENSIONAL SYSTEM (2D):

Cartesian /Rectangular coordinate System:-

In the Cartesian coordinate system in 2D, the point in a space or in three dimensional systems be represented by the symbol (x, y) where x is the distance on X axis and y is the distance on Y axis of the point (x, y).

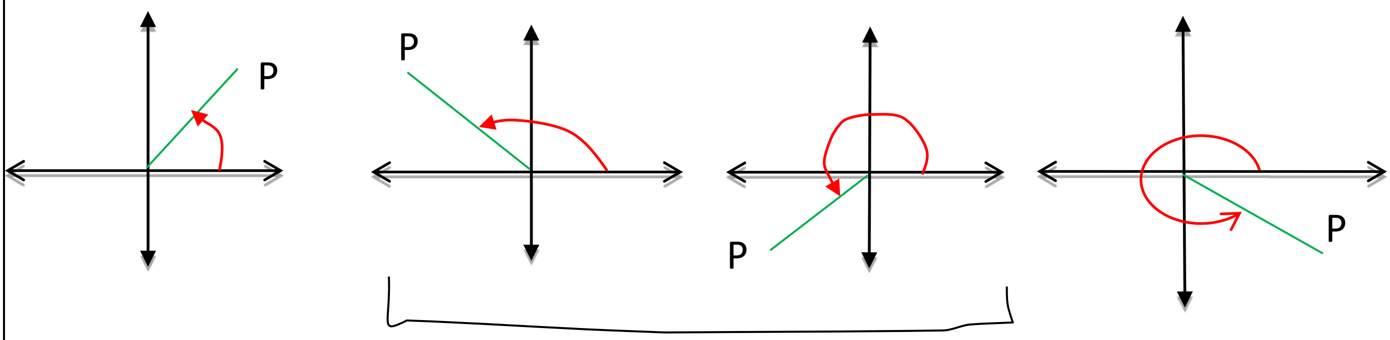
Figure:



■ Mutual relation (2D Cases)

$x = r\cos\theta$, $y = r\sin\theta$ **;** $r^2 = x^2 + y^2$,

There are 3 formulas to find θ for some given point P(x,y). These are:



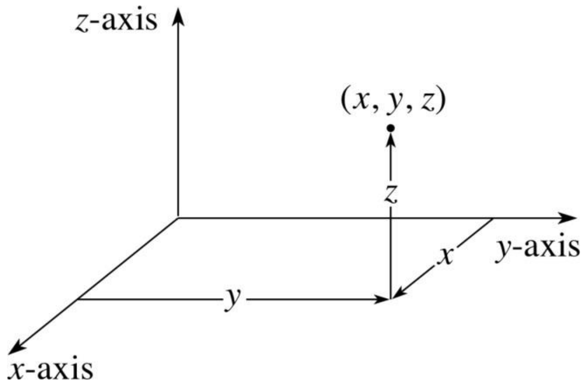
$\theta = \tan^{-1}(y/x)$ $\theta = \tan^{-1}(y/x) + 180^0$, $\theta = \tan^{-1}(y/x) + 360^0$

■ THREE DIMENSIONAL SYSTEM (3D):

❖ Cartesian /Rectangular coordinate System (RS):

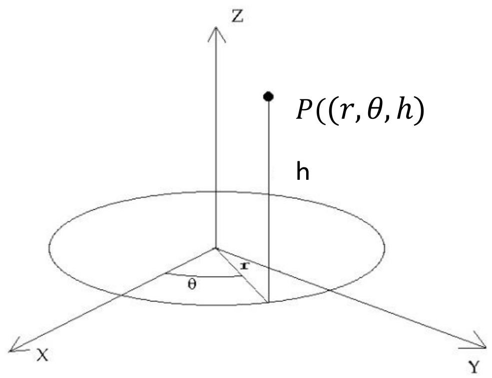
In the Cartesian coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (x, y, z) where x is the distance on x axis, y is the distance on y axis and z is the distance on z axis of the point (x, y, z).

Figure:



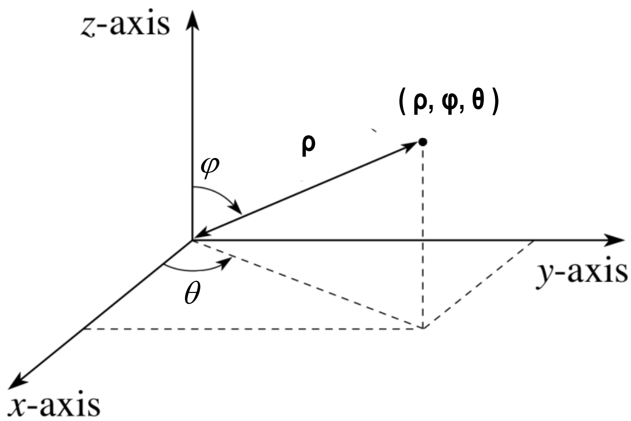
❖ Cylindrical coordinate System (CS):

In the Cylindrical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (r, θ, h) where r is the distance of the point from origin or length of radial line, θ is the angle between radial line and x axis and h is the distance of the point (r, θ, h) from the xy plane.



❖ Spherical coordinate system (SS):

In the Spherical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (ρ, φ, θ) where ρ is the distance of the point from origin or length of radial line, φ is the angle between radial line and z axis and θ the angle between radial line (joining with the foot point of the perpendicular from the given point on the xy plane) and the x axis.



◘◘ Relation between Cartesian/Rectangular and Cylindrical System:

$$CS \rightarrow RS$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = h$$

$$RS \rightarrow CS$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(\frac{y}{x})$$
$$h = z$$

◘◘ Relation between Cartesian/Rectangular and Spherical System:

$$SS \rightarrow RS$$
$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$

$$RS \rightarrow SS$$
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\varphi = \cos^{-1}(\frac{z}{\sqrt{x^2 + y^2 + z^2}}) = \cos^{-1}(\frac{z}{\varphi})$$
$$\theta = \tan^{-1}(\frac{y}{x})$$

◘◘ Relation between Cylindrical and Spherical System:

$$CS \rightarrow SS$$
$$r = \rho\sin\varphi$$
$$\theta = \theta$$
$$h = \rho\cos\varphi$$

$$SS \rightarrow CS$$
$$\rho = \sqrt{r^2 + h^2}$$
$$\varphi = \tan^{-1}(\frac{r}{h})$$
$$\theta = \theta$$

Restriction:

$$(x, y, z, h) \in (-\infty, \infty); \quad (\rho, r) \in [0, \infty);$$
$$\theta \in [0, 360^0); \quad \varphi \in [0, 180^0];$$

Mathematical Problems

Problem no: 01

Convert $\left(3, \frac{\pi}{3}, -4\right)$ from Cylindrical to Cartesian Coordinates.

Solution:

Given that,

Cylindrical coordinates of a point is $(r, \theta, h) = \left(3, \frac{\pi}{3}, -4\right)$

We know that,

$$x = r\cos\theta, y = r\sin\theta, z = h$$

Now, $x = r\cos\theta = 3\cos\frac{\pi}{3} = 3\times\frac{1}{2} = \frac{3}{2}$

$$y = 3\sin\frac{\pi}{3} = 3\times\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$z = h = -4$$

Therefore the Cartesian coordinates of the given point is $(x\ ,\ y,\ z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{3}, -4\right)$.

H.W:

Convert the followings cylindrical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3}, \frac{\pi}{4}, -4\right)$ **2.** $\left(4\sqrt{3}, 0,\ 5\right)$ **3.** $(\sqrt{5}, 55^0, -3)$ **4.** $(3, 70^0, 2)$

Problem no: 02

Convert $(-2, 2, 3)$ from Cartesian to Cylindrical Coordinates.

Solution:

Given that,

Cartesian coordinates of a point is $(x\ ,\ y,\ z) = (-2, 2, 3)$

We know that,

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$h = z$$

Now,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{2}{-2} = -1$$

Here

$$\tan\theta = -1$$

$$\tan\theta = -\tan\frac{\pi}{4}$$

$$\tan\theta = \tan(\pi - \frac{\pi}{4})$$

$$\tan\theta = \tan(\frac{3\pi}{4})$$

$$\theta = \frac{3\pi}{4}$$

And $h = z = 3$

Therefore the Cylindrical coordinates of the given point is $(r\ ,\ \theta,\ z) = \left(2\sqrt{2}, \frac{3\pi}{4}, 3\right)$.

H.W:

Convert the followings Cartesian coordinates to the Cylindrical Coordinates system:

1. $\left(4\sqrt{3}, 4, -4\right)$ 2. $\left(-\sqrt{3}, -4, 4\right)$ 3. $\left(-\sqrt{3}, 4, 2\right)$ 4. $\left(4\sqrt{2}, -1, -4\right)$
5. $\left(4\sqrt{3}, 0, 5\right)$ 6. $\left(0, 4, 9\right)$

Problem no: 03

Convert $\left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from Spherical to Cartesian Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho, \varphi, \theta) = \left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$

We know that,

$$x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$$

Now,

$$x = \rho\sin\varphi\cos\theta \ = 8\sin\frac{\pi}{4}\cos\frac{\pi}{6} = 8\times\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}$$

$$y = \rho\sin\varphi\sin\theta \ = 8\sin\frac{\pi}{4}\sin\frac{\pi}{6} = 8\times\frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

And

$$z = \rho\cos\varphi \ = 8\cos\frac{\pi}{4} = 8\times\frac{1}{\sqrt{2}} = 4\sqrt{2}$$

Therefore the Cartesian coordinates of the given point is $(x\ ,\ y,\ z) = \left(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2}\right)$

H.W:

Convert the followings Spherical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ 2. $\left(-\sqrt{3}, 124^o, 75^o\right)$ 3. $\left(\sqrt{3}, 140^o, 140^o\right)$

Problem no: 04

Convert $\left(2\sqrt{3},6,-4\right)$ from Cartesian to Spherical Coordinates.

Solution:

Given that,

Cartesian coordinates of a point is $(x\ ,\ y,\ z)=\left(2\sqrt{3},6,-4\right)$

We know that,

$$\rho=\sqrt{x^2+y^2+z^2}$$

$$\varphi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\cos^{-1}\left(\frac{z}{\rho}\right)\qquad Or,\varphi=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

Now ,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{(2\sqrt{3})^2+6^2+(-4)^2}=\sqrt{12+36+16}=\sqrt{64}=8$$

$$\varphi=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\Rightarrow\tan\varphi=\frac{\sqrt{x^2+y^2}}{z}=\frac{\sqrt{(2\sqrt{3})^2+6^2}}{-4}$$

$$=\frac{\sqrt{12+36}}{-4}=\frac{\sqrt{48}}{-4}=\frac{4\sqrt{3}}{-4}=-\sqrt{3}$$

Here

$$\tan\varphi=-\sqrt{3}$$

$$\tan\varphi=-\tan\frac{\pi}{3}$$

$$\tan\varphi=\tan\left(\pi-\frac{\pi}{3}\right)$$

$$\tan\varphi=\tan\left(\frac{2\pi}{3}\right)$$

$$\varphi=\frac{2\pi}{3}$$

And $\theta=\tan^{-1}\left(\frac{y}{x}\right)\Rightarrow tan\theta=\frac{y}{x}=\frac{6}{2\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}$

Here

$$tan\theta=\sqrt{3}$$

$$tan\theta=\tan\frac{\pi}{3}$$

$$\theta=\frac{\pi}{3}$$

Therefore the Spherical coordinates of the given point is $(\rho,\varphi,\theta)=\left(8,\frac{2\pi}{3},\frac{\pi}{3}\right)$.

H.W:

Convert the followings Cartesian coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3},4,-4\right)$ 2. $\left(-\sqrt{3},-4,4\right)$ 3. $\left(-\sqrt{3},4,2\right)$ 4. $\left(4\sqrt{2},-1,-4\right)$

5. $\left(\sqrt{3},0,0\right)$ 6. $\left(0,4,9\right)$ 7. $\left(4\sqrt{3},0,5\right)$

Problem no: 05

Convert $\left(1,\frac{\pi}{2},1\right)$ from Cylindrical to Spherical Coordinates.

Solution:

Given that,

Cylindrical coordinates of a point is $(r,\theta,h)=\left(1,\frac{\pi}{2},1\right)$

We know that,

$$\rho=\sqrt{r^2+h^2}$$

$$\varphi=\tan^{-1}(\frac{r}{h})$$

$$\theta=\theta$$

Now,

$$\rho=\sqrt{r^2+h^2}=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$$

$$\varphi=\tan^{-1}(\frac{r}{h})\Rightarrow\tan\varphi=\frac{r}{z}=\frac{1}{1}=1$$

Here

$$\tan\varphi=1$$

$$\tan\varphi=\tan\frac{\pi}{4}$$

$$\varphi=\frac{\pi}{4}$$

And

$$\theta=\theta=\frac{\pi}{2}$$

Therefore the Spherical coordinates of the given point is $(\rho,\varphi,\theta)=(\sqrt{2},\frac{\pi}{4},\frac{\pi}{2})$.

H.W:

Convert the followings Cylindrical coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3},42^o,-4\right)$ 2. $\left(4\sqrt{3},0^o,5\right)$ 3. $\left(-\sqrt{3},134^o,-4\right)$

Problem no: 06

Convert $\left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$ from Spherical to Cylindrical Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho,\varphi,\theta)=\left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$

We know that,

$$r=\rho\sin\varphi$$
$$\theta=\theta$$
$$h=\rho\cos\varphi$$

Now,

$$r=\rho\sin\varphi \quad =4\sqrt{3}\sin\frac{\pi}{4}=4\sqrt{3}\times\frac{1}{\sqrt{2}}=2\sqrt{6}$$

$$\theta=\theta \quad =\frac{\pi}{4}$$

And $$h=\rho\cos\varphi=4\sqrt{3}\times\cos\frac{\pi}{4}=4\sqrt{3}\times\frac{1}{\sqrt{2}}=2\sqrt{6}$$

Therefore the Cylindrical coordinates of the given point is $(r,\theta,h)=\left(2\sqrt{6},\frac{\pi}{4},2\sqrt{6}\right)$.

H.W:

Convert the followings Spherical coordinates to the Cylindrical Coordinates system:

1. $(\sqrt{5},\frac{\pi}{4},\frac{\pi}{3})$ 2. $(-\sqrt{5},140^0,75^0)$ 3. $(-3,40^0,15^0)$

Example:- Convert each of (i) $(8,-3,-7)$ (ii) $(5,120^0,330^0)$ to the other two system.

Solution:-

(i)

Hear the given coordinate system is $(8,-3,-7)$ which is RS, so we need to convert it to CS and SS.

In CS:

$$r=\sqrt{x^2+y^2}=\sqrt{8^2+(-3)^2}=8.54$$
$$\theta=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{-3}{8})=-\tan^{-1}(\frac{3}{8})=360^0-\tan^{-1}(\frac{3}{8})=339^0 26'38''$$
$$h=z=-7$$

Hence the point is, $(r,\theta,h)=(8.54,339^0 26'38'',-7)$

In SS:

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{8^2 + (-3)^2 + (-7)^2} = 11.045$$

$$\varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-7}{11.045}\right) = 129^0 18' 27''$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-3}{8}\right) = -\tan^{-1}\left(\frac{3}{8}\right) = 360^0 - \tan^{-1}\left(\frac{3}{8}\right) = 339^0 26' 38''$$

Hence the point is, (ρ, φ, θ) =$(11.045, 129^0 18'27'', 339^0 26'38'')$.

(ii)

Hear the given coordinate system is $(5, 120^0, 330^0)$ which is SS, so we need to convert it to CS and RS.

In RS:

$$x = \rho \sin\varphi \cos\theta = 5\sin 120^0 \cos 330^0 = 3.75$$

$$y = \rho \sin\varphi \sin\theta = 5\sin 120^0 \sin 330^0 = -2.17$$

$$z = \rho \cos\varphi = 5\cos 120^0 = -2.5$$

Hence the point is, (x, y, z) =($3.75, -2.17, -2.5$).

In CS:

$$r = \rho \sin\varphi = 5\sin 120^0 = 4.43$$

$$\theta = \theta = 330^0$$

$$h = z = -2.5$$

Hence the point is, (r, θ, h) = ($4.43, 330^0, -2.5$).

H.W:

Convert each of from below to other 2 systems.

(i) (4√3, 4, -4) ; (ii) (-√3, 4, 2) ; (iii) (-√3, -4, 4);

(iv) (5, 120^0 , 330^0) ; (v) (4√2, -1, -4); (vi)(0, 4, 9) ;

(vii) (√3, 0, 0) ; (viii) (4√3, 42^0 , -4) ; (ix) (4√3, 0^0, 5) ;

(x) (-√3, 134^0, -4); (xi) (4√3, 45^0,45^0) ; (xii)(-√3, 124^0,75^0) ;

(xiii) (√3, 140^0,140^0) ;

Transformation of Equations

Mathematical problems

Problem 01:- Express Cartesian Equation $x^2 - y^2 = 25$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 - y^2 = 25$

We have

$$x = r\cos\theta \ , \ y = r\sin\theta \ and \quad z = h$$

Replacing x and y from the given equation we get desired cylindrical equation as follows,

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 25$$

$$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 25$$

$$\Rightarrow \quad r^2\left(\cos^2\theta - \sin^2\theta\right) = 25$$

$$\Rightarrow \quad r^2\cos(2\theta) = 25$$

$$r^2 = 25Sec(2\theta)$$

This is the required cylindrical equation.

Problem 02:- Express Cartesian Equation $x^2 + y^2 + z^2 = 0$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 + y^2 + z^2 = 0$

We have,

$$x = r\cos\theta \ , \ y = r\sin\theta \ and \quad z = h$$

Replacing x, y and z from the given equation we get desired Cylindrical equation as follows,

$$(r\cos\theta)^2 + (r\sin\theta)^2 + z^2 = 0$$

$$\Rightarrow \quad r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 0$$

$$\Rightarrow \quad r^2\left(\cos^2\theta + \sin^2\theta\right) + z^2 = 0$$

$$r^2 + z^2 = 0$$

This is the required cylindrical equation.

H.W:

Transform the following Cartesian equations into the Cylindrical Equations:

1. $x^2 - y^2 + 2z^2 = 3x$ 2. $x^2 + y^2 + z^2 = 2z$ 3. $z^2 = y^2 - x^2$ 4. $x + y + z = 1$

Problem 03:- Transform Cartesian Equation $x^2+y^2-z^2=1$ to Spherical Equation.

Solution:

Given Cartesian Equation is $x^2+y^2-z^2=1$

We have,

$$x=\rho\sin\varphi\cos\theta$$
$$y=\rho\sin\varphi\sin\theta$$
$$z=\rho\cos\varphi$$

Replacing x, y and z from the given equation we get desired Cylindrical equation as follows,

$$(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2-(\rho\cos\varphi)^2=1$$
$$\Rightarrow\quad \rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta-\rho^2\cos^2\varphi=1$$
$$\Rightarrow\quad \rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)-\rho^2\cos^2\varphi=1$$
$$\Rightarrow\quad \rho^2\sin^2\varphi-\rho^2\cos^2\varphi=1$$
$$\Rightarrow\quad \rho^2(\sin^2\varphi-\cos^2\varphi)=1$$
$$\Rightarrow\quad -\rho^2(\cos^2\varphi-\sin^2\varphi)=1$$
$$\Rightarrow\quad -\rho^2\cos(2\varphi)=1$$
$$\rho^2=-\sec(2\varphi)$$

This is the required Spherical Equation.

H.W:

Transform the following Cartesian equations into the Spherical Equations:

1. $x^2-y^2+2z^2=3x$ 2. $x^2+y^2+z^2=2z$ 3. $z^2=y^2-x^2$ 4. $x+y+z=1$

Problem 04:- Transform Spherical Equation $\rho=2\cos\varphi$ to Cylindrical Equation.

Solution:

Given Spherical Equation is $\rho=2\cos\varphi$

We have,

$$\rho=\sqrt{r^2+h^2},\varphi=\tan^{-1}(\frac{r}{h}),\theta=\theta$$

Replacing ρ and φ from the given equation we get desired Cylindrical equation as follows,

$$\sqrt{r^2+h^2}=2\cos\varphi$$
$$\Rightarrow\quad \sqrt{r^2+h^2}=2\times\frac{h}{\rho} \qquad [\because h=\rho\cos\varphi]$$
$$\Rightarrow\quad \sqrt{r^2+h^2}=2\times\frac{h}{\sqrt{r^2+h^2}}$$
$$\therefore r^2+h^2=2h$$

This is the required Cylindrical equation.

H.W:

Transform the following Spherical Equations into the Cylindrical Equations:

1. $\varphi = \frac{\pi}{4}$ 2. $\rho = 2\sec\theta$ 3. $\rho = \cos ec\theta$

Problem 05:- Transform Cylindrical Equation $r^2 \cos 2\theta = h$ to Cartesian/Rectangular Equation.

Solution:

Given Cylindrical Equation is $r^2 \cos 2\theta = h$

We have,

$$x = r\cos\theta \text{ , } y = r\sin\theta \text{ and } z = h$$

Given equation is

$$r^2 \cos 2\theta = h$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = z$$

$$\Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = z$$

$$\Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = z$$

$$\therefore (x)^2 - (y)^2 = z \quad \text{[Putting values]}$$

This is the required Cartesian/Rectangular Equation.

H.W:

Transform the following Cylindrical Equations into the Cartesian/Rectangular Equations:

1. $r = 2\sin\theta$ **2.** $z = 5\sin\theta$

Problem 05:- Transform Spherical Equation $\rho\sin\varphi = 1$ to Cartesian/Rectangular Equation.

Solution:

Given Spherical Equation is $\rho\sin\varphi = 1$

We have, $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$ and $\rho = \sqrt{x^2 + y^2 + z^2}$

Now,

$$\rho\sin\varphi = 1$$

$$\Rightarrow \rho^2\sin^2\varphi = 1$$

$$\Rightarrow \rho^2(1 - \cos^2\varphi) = 1$$

$$\Rightarrow \rho^2 - (\rho\cos\varphi)^2 = 1$$

$$\Rightarrow x^2 + y^2 + z^2 - z^2 = 1$$

$$\therefore x^2 + y^2 = 1$$

H.W:

Transform the following Spherical Equations into the Cartesian/Rectangular Equations:

1. $\rho\sin\varphi = 1$ 2. $\rho = 2\sec\varphi$ 3. $\rho = \csc\varphi$ 4. $\rho\sin\varphi = 2\cos\theta$