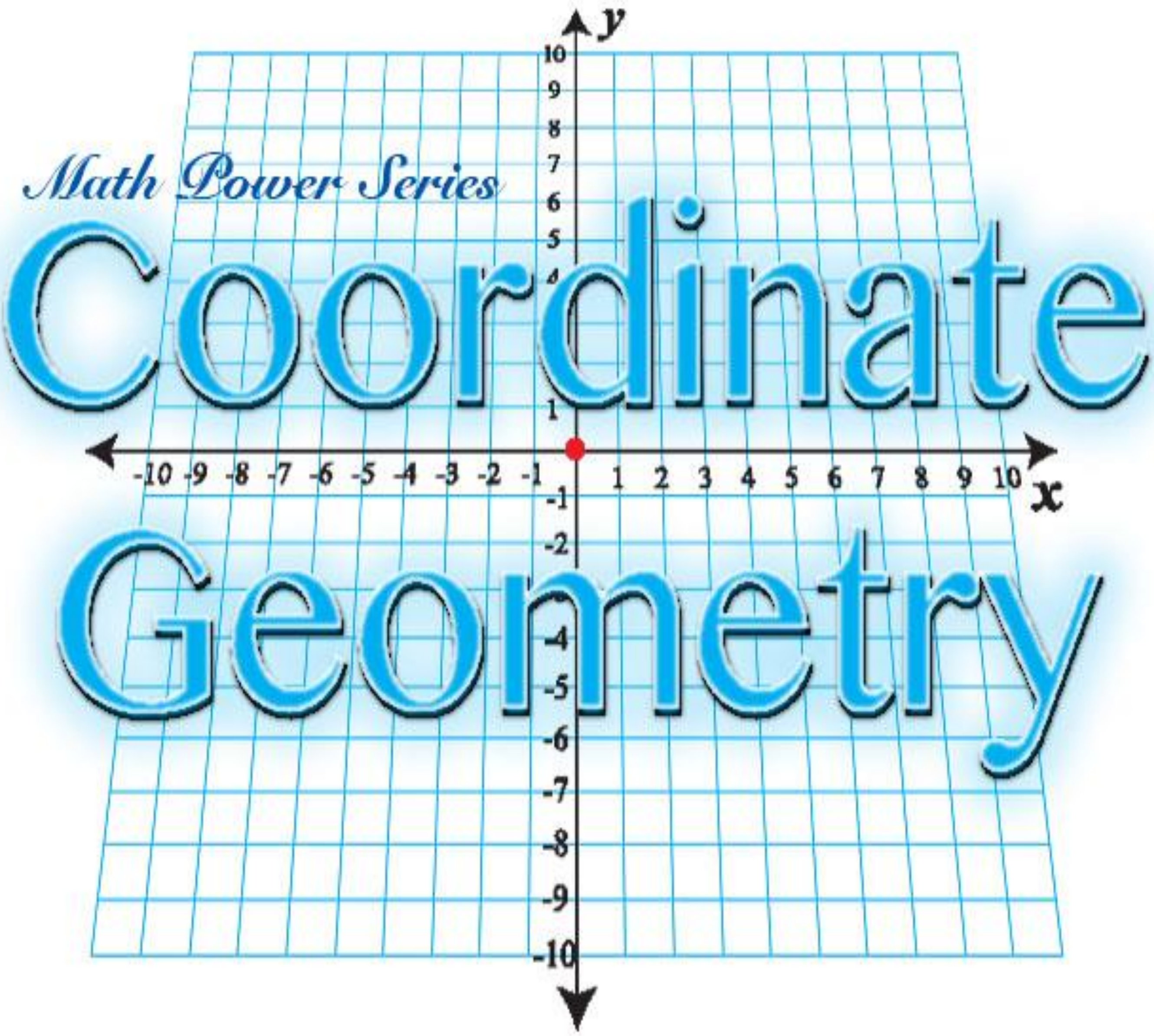

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History of Coordinate Geometry

Coordinate geometry is the branch of mathematics in which geometry is studied with the help of algebra. The great France mathematician and philosopher ***Rene Descartes*** (1596-1650) first applied algebraic formulae in geometry. A system of geometry where the position of points on the plane is described using two numbers called an ordered pair of numbers or coordinates. The first element of the ordered pair represents the distance of that point on x-axis called abscissa and second element on y-axis called ordinate. This abscissa and ordinate makes coordinates of that point.

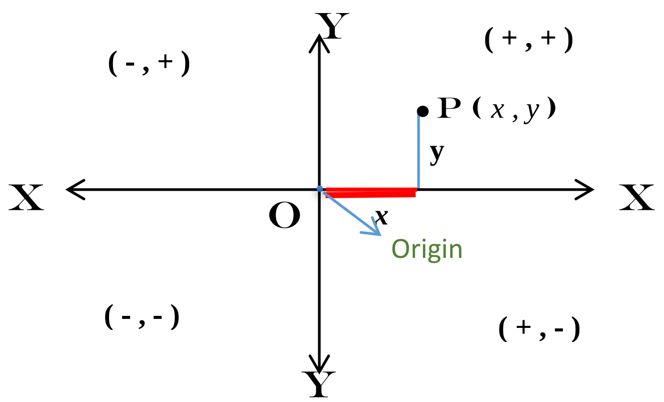
René Descartes

The method of describing the location of points in this way was proposed by the French mathematician René Descartes (1596 - 1650). (Pronounced "day CART"). He proposed further that curves and lines could be described by equations using this technique, thus being the first to link algebra and geometry. In honor of his work, the coordinates of a point are often referred to as its Cartesian coordinates and the coordinate plane as the Cartesian Coordinate Plane and coordinate geometry sometimes called Cartesian geometry.

Cartesian/Rectangular Coordinates System:

Every plane is two dimensional so to locate the position of a point in a plane there is needed two coordinates. The Mathematician *Rene Descartes* first considered two perpendicular intersecting fixed straight lines in a plane as axes of coordinates. These two straight lines are named as rectangular axes and intersecting point as the origin denotes by the symbol **O** and the symbol **O** comes from the first letter of the word origin. The Cartesian coordinate system is named after the inventor name Rene Descartes. The Cartesian coordinate system is also known as Rectangular coordinates system as the axes are in right angle (Rec means right).In Cartesian coordinate system position of a point measured by the distance on both axes. First one is on x-axis called abscissa or x-coordinate denoted by the symbol ***x*** and second one is on y-axis called ordinate or y-coordinate denoted by the symbol *y*.We express the coordinate of a point **P** in Cartesian plane by the ordered pair *P(x , y) or P(abscissa, ordinate).*

The horizontal line XOX' is called x-axis and the vertical line YOY' is called y-axis .Both axes divide the whole plane into four parts called Quadrants .Four Quadrants XOY, X'OY, X'OY' and XOY' are called anti-clock-wisely 1st ,2nd,3rd and 4th quadrant respectively. The coordinate of the origin is **O**(0,0) because all distances measured considering origin as starting point.



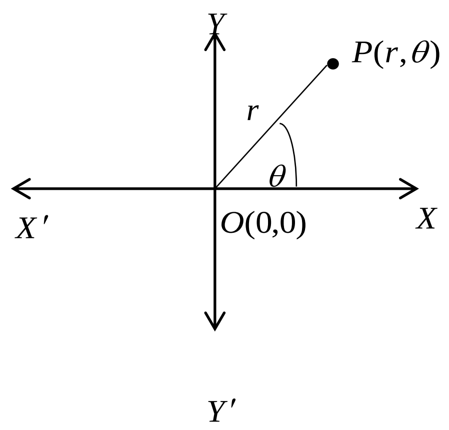
Polar coordinates System:

In similar manner of Cartesian system for fixing or locating the point **P** in a plane we take a fixed point **O** called the pole and a fixed straight line **OX** called the initial line. Joining line of the points P and O is called radius vector and length of radius vector

OP = r and the positive angle $\angle XOP = \theta$ is called vectorial angle. It is sometimes convenient to locate the position of a point **P** in terms of its distances from a fixed point and its direction from a fixed line through this point. So the coordinates of locating points in this system is called Polar coordinates system. The coordinates of point in this system are called Polar coordinates. The polar coordinates of the point P are expressed as $P(r,\theta)$.

In expressing the polar coordinates of the point P the radius vector is always written as the first coordinate. It is considered positive if measured from the pole along the line bounding the vectorial angle otherwise negative. In a polar system the same point has an infinite number of representations and it is the demerits of polar coordinate system to Cartesian system.

For example: The point P has the coordinates $(r,\theta),(-r,\theta+\pi),(-r,\theta-\pi),(r,\theta-2\pi)$ etc.



Relation between Cartesian and Polar Coordinate System:

Suppose that the coordinates of the point P in Cartesian system is $P(x, y)$ and in Polar system is $P(r,\theta)$ .Our target here to establish the relation between two coordinates systems. From the triangle with the help of trigonometry we can find the relation between the Cartesian system & the polar system.

From the pictorial triangle

We get,

$\cos\theta = \frac{x}{r} \quad \Rightarrow x = r\cos\theta$

And, $\sin\theta = \frac{y}{r} \quad \Rightarrow y = r\sin\theta$

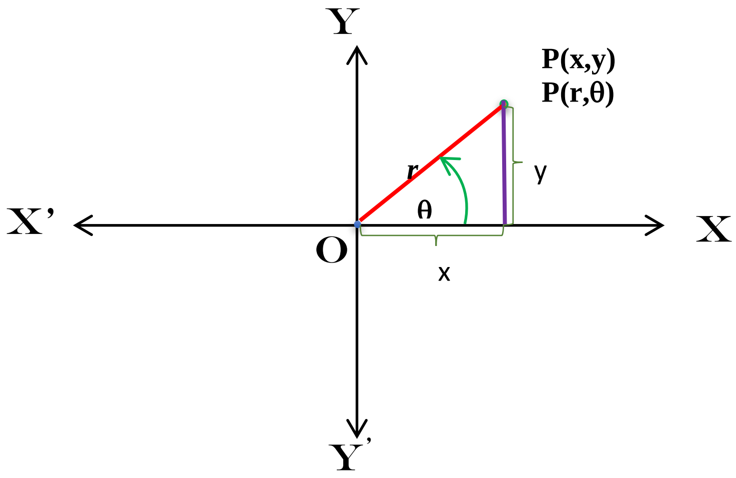
Again

Applying Pythagorean Theorem from geometry we have a relation,

$$r^2 = x^2 + y^2 \quad \Rightarrow r = \sqrt{x^2 + y^2}$$

And, $\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}(\frac{y}{x})$ {Principal Argument}



Therefore the relations are:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

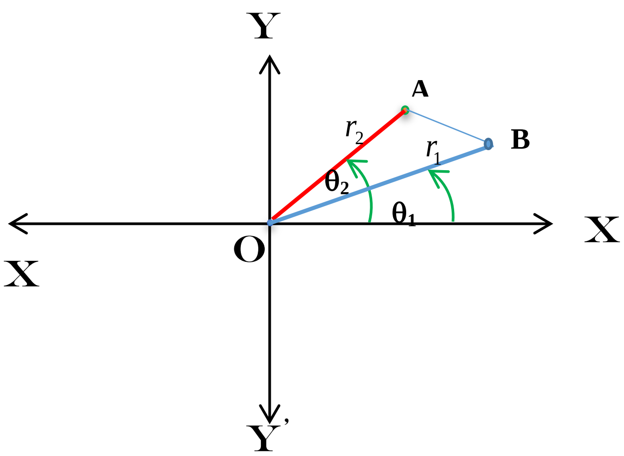
Distance between two Points:

If the coordinates of two points in Cartesian system are $A(x_1, y_1)$ and $B(x_2, y_2)$, then the distance between two points is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Again,

If the coordinates of two points in Cartesian system are $A(r_1, \theta_1)$ *and* $B(r_2, \theta_2)$ then the distance between two points is,

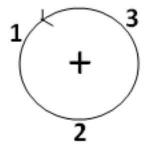
$$\boldsymbol{AB = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}}$$



Area of a triangle:

If the coordinates of the vertices of the triangle in Cartesian system are respectively $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the area of the triangle is

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{Sq. Units}$$



$$\Delta \boldsymbol{ABC} = \frac{\mathbf{1}}{\mathbf{2}}\{\boldsymbol{x_1}(\boldsymbol{y_2} - \boldsymbol{y_3}) + \boldsymbol{x_2}(\boldsymbol{y_3} - \boldsymbol{y_1}) + \boldsymbol{x_3}(\boldsymbol{y_1} - \boldsymbol{y_2})\}$$

Area by Sarrus Diagram Method:

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}$$

An alternative representation of Sarrus diagram Method:

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}$$

Note:

1. In the determinant point must be choose in anti-clock-wise direction.
2. In Sarrus Diagram method first point is repeated.
3. Applying Sarrus Diagram Method we find the area of a polygon.

Again,

If the coordinates of the vertices of the triangle in Cartesian system are respectively $A(r_1,\theta_1)$, $B(r_2,\theta_2)$ and $C(r_3,\theta_3)$, then the area of the triangle is

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} r_1\cos\theta_1 & r_1\sin\theta_1 & 1 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 1 \\ r_3\cos\theta_3 & r_3\sin\theta_3 & 1 \end{vmatrix} \quad \text{Sq. Units}$$

$$\Delta ABC = \frac{1}{2}\{r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)-r_1r_3\sin(\theta_3-\theta_1)\}$$

Mathematical Problem

Problem 01:- Determine the polar coordinates of the point $(\sqrt{3},-1)$.

Solution:

We have given $(x,y)=(\sqrt{3},-1)$.

Therefore $x=\sqrt{3}$ and $y=-1$

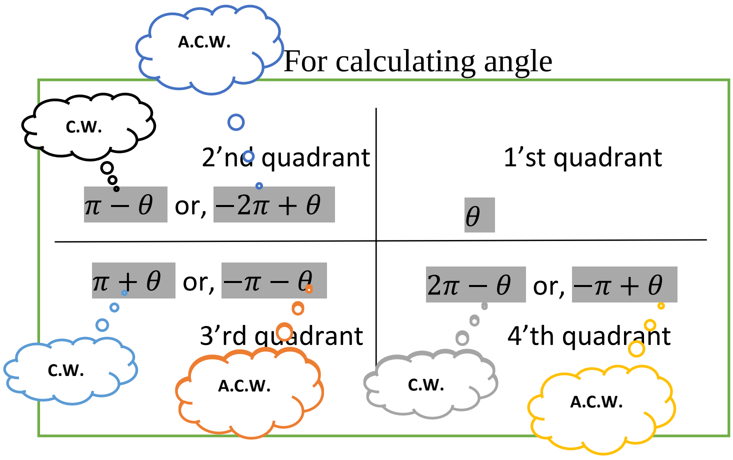
We know that,

$$r=\sqrt{x^2+y^2}$$

$$\Rightarrow r=\sqrt{(\sqrt{3})^2+(-1)^2}$$

$$\Rightarrow r=\sqrt{3+1}=\sqrt{4}=2$$

For calculating angle



And

$$\tan\theta=\frac{-1}{\sqrt{3}}$$

$$\Rightarrow \tan\theta=-\frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan\theta=-\tan\frac{\pi}{6}$$

$$\Rightarrow \tan\theta=\tan\left(2\pi-\frac{\pi}{6}\right)$$

$$\Rightarrow \tan\theta=\tan\left(\frac{12\pi-\pi}{6}\right)$$

$$\Rightarrow \tan\theta=\tan\left(\frac{11\pi}{6}\right)$$

$$\theta=\frac{11\pi}{6}$$

Therefore the polar form of the given point is $(r,\theta)=\left(2,\frac{11\pi}{6}\right)$ or $(r,\theta)=\left(2,-\frac{\pi}{6}\right)$.

Problem 02:- Determine the Cartesian coordinates of the point $\left(2\sqrt{2},\dfrac{5\pi}{4}\right)$.

Solution:

We have given $(r,\theta)=\left(2\sqrt{2},\dfrac{5\pi}{4}\right)$.

Therefore $r=2\sqrt{2}$ and $\theta=\dfrac{5\pi}{4}$

We know that,

$$x=r\cos\theta=2\sqrt{2}\cos\frac{5\pi}{4}=2\sqrt{2}\cos\left(\pi+\frac{\pi}{4}\right)=-2\sqrt{2}\cos\frac{\pi}{4}=-2\sqrt{2}\times\frac{1}{\sqrt{2}}=-2$$

And

$$y=r\sin\theta=2\sqrt{2}\sin\frac{5\pi}{4}=2\sqrt{2}\sin\left(\pi+\frac{\pi}{4}\right)=-2\sqrt{2}\sin\frac{\pi}{4}=-2\sqrt{2}\times\frac{1}{\sqrt{2}}=-2$$

Therefore the Cartesian form of the given point is $(x,y)=(-2,-2)$.

H.W:

1. Convert the following points to the polar form:

i) $\left(1,-\sqrt{3}\right)$ **ii)** $\left(2\sqrt{3},-2\right)$ **iii)** $(-1,-1)$ **iv)** $\left(\dfrac{3\sqrt{3}}{2},\dfrac{3}{2}\right)$

v) $\left(\dfrac{3\sqrt{3}}{2},\dfrac{3}{2}\right)$ **vi)** $\left(a,a\sqrt{3}\right)$ **vii)** $\left(\dfrac{5\sqrt{2}}{2},\dfrac{-5\sqrt{2}}{2}\right)$

2. Convert the following points to the Cartesian form:

i) $\left(3,\dfrac{\pi}{6}\right)$ **ii)** $\left(5,-\dfrac{\pi}{4}\right)$ **iii)** $\left(-2a,-\dfrac{2\pi}{3}\right)$ **iv)** $\left(2,\dfrac{2\pi}{3}\right)$ **v)** $\left(1,\dfrac{\pi}{6}\right)$

vi) $\left(2,\dfrac{\pi}{3}\right)$ **vii)** $\left(3,\dfrac{\pi}{2}\right)$ **viii)** $\left(2,-\dfrac{\pi}{6}\right)$ **ix)** $\left(4,\dfrac{11\pi}{6}\right)$ **x)** $\left(\sqrt{2},\dfrac{5\pi}{4}\right)$

Problem 03:- Transform the equation $x^3+y^3=3axy$ to Polar equation.

Solution:

Given Cartesian Equation is $x^3+y^3=3axy$.

We have

$x=r\cos\theta$ and $y=r\sin\theta$

Now replacing x and y from the above equation by its values given equation reduces to the following form

$$r^3\cos^3\theta+r^3\sin^3\theta=3a.r\cos\theta.r\sin\theta$$

$$\Rightarrow\quad r^3\left(\cos^3\theta+\sin^3\theta\right)=3a.r^2\cos\theta\sin\theta$$

$$\Rightarrow\quad r\left(\cos^3\theta+\sin^3\theta\right)=3a\cos\theta\sin\theta$$

$$\Rightarrow\quad r\left(\cos^3\theta+\sin^3\theta\right)=\frac{3}{2}a\times 2\cos\theta\sin\theta$$

$$\therefore \quad r\left(\cos^3\theta+\sin^3\theta\right)=\frac{3}{2}a\cos 2\theta$$

This required polar equation.

Problem 04:- If x, y be related by means of the equation $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$, find the corresponding relation between r and θ .

Solution:

Given Cartesian Equation is $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$.

We have

$$x=r\cos\theta \text{ and } y=r\sin\theta$$

Putting $x=r\cos\theta$ and $y=r\sin\theta$ the above relation is transformed into the following form

$$\left(r^2\cos^2\theta+r^2\sin^2\theta\right)^2=a^2\left(r^2\cos^2\theta-r^2\sin^2\theta\right)$$

$$\Rightarrow \quad r^4\left(\cos^2\theta+\sin^2\theta\right)^2=a^2r^2\left(\cos^2\theta-\sin^2\theta\right)$$

$$\Rightarrow \quad r^2(1)^2=a^2\left(\cos^2\theta-\sin^2\theta\right)$$

$$r^2=a^2\cos 2\theta$$

This required relation.

H.W:

Convert the followings to the polar form:

i) $x^2\left(x^2+y^2\right)=a^2\left(x^2-y^2\right)$ **ii)** $xy^3+yx^3=a^2$ **iii)** $\left(x^2-y^2\right)^2-y^2(2x+1)+2x^3=0$

iv) $4\left(x^3-y^3\right)-3(x-y)\left(x^2+y^2\right)=5kxy$ **v)** $x^3=y^2(2a-x)$ **vi)** $x^4+x^2y-(x+y)^2=0$

Problem 05:- Transform the equation $2a\sin^2\theta-r\cos\theta=0$ to Cartesian equation.

Solution:

Given Polar Equation is $2a\sin^2\theta-r\cos\theta=0$.

We have

$$x=r\cos\theta,\, y=r\sin\theta \textit{ and } x^2+y^2=r^2$$

Putting $x=r\cos\theta,\, y=r\sin\theta$ *and* $x^2+y^2=r^2$ the above relation is transformed into the following form,

$$2a\sin^2\theta-r\cos\theta=0$$

$$\Rightarrow \quad \frac{2a\,r^2\sin^2\theta}{r^2}-r\cos\theta=0$$

$$\Rightarrow \quad \frac{2a\left(r\sin\theta\right)^2}{r^2}-r\cos\theta=0$$

$$\Rightarrow \quad \frac{2a\,y^2}{x^2+y^2}-x=0$$

$\Rightarrow \quad 2a\,y^2 - x\left(x^2 + y^2\right) = 0$

$\Rightarrow \quad 2a\,y^2 = x^3 + xy^2$

$\Rightarrow \quad x^3 = 2a\,y^2 - xy^2$

$\therefore \quad x^3 = y^2\left(2a - x\right)$ **(As desired)**

This required Cartesian equation.

H.W:

Convert the followings to the Cartesian form:

i) $r^2\cos^2\theta = a^2\cos 2\theta$ **ii)** $r^4 = 2a^2\operatorname{cosec} 2\theta$ **iii)** $r\cos 2\theta = 2\sin^2\dfrac{\theta}{2}$

iv) $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$ **v)** $2a\sin^2\theta = r\cos\theta$ **vi)** $r = \pm\left(1 + \tan\theta\right)$

Broad Questions:

1. Find the distances among the points $\left(1, \dfrac{\pi}{6}\right)$, $\left(2, \dfrac{\pi}{3}\right)$ and $\left(3, \dfrac{\pi}{2}\right)$.
2. Find the area of the triangle formed by the points $\left(1, \dfrac{\pi}{6}\right)$, $\left(2, \dfrac{\pi}{3}\right)$ and $\left(3, \dfrac{\pi}{2}\right)$.
3. If three points $\left(-1, 2\right)$, $\left(2, -1\right)$ and $\left(h, 3\right)$ are collinear then show that $h = -2$.
4. Find the area of a polygon whose vertices are given $\left(1, 3\right)$, $\left(4, 1\right)$, $\left(5, 3\right)$, $\left(3, 2\right)$ and $\left(2, 4\right)$.
5. Show that the three points $\left(1, -1\right)$, $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $\left(1, 2\right)$ form a right angled triangle.
6. Find locus of the parametric coordinates point $\left(a\cos\theta, b\sin\theta\right)$ where θ is a parameter.

CHANGE OF AXES

Transformation of coordinates:

The process of changing the coordinates of point or the equation of the curves is called transformation of coordinates.

Transformation of coordinates is of three types such as follows

1. Translation of axes:

In this process the position of the origin is changed but the direction of coordinate axes is being parallel to the old system.

When origin $(0,0)$ shifted to the new point (h,k) and keeping the direction of coordinate axes fixed then the pair of equations

$$x = x' + h \;,\; y = y' + k$$

represents the relation between new system(X', Y') and old system (X, Y) and is called the translation of axes.

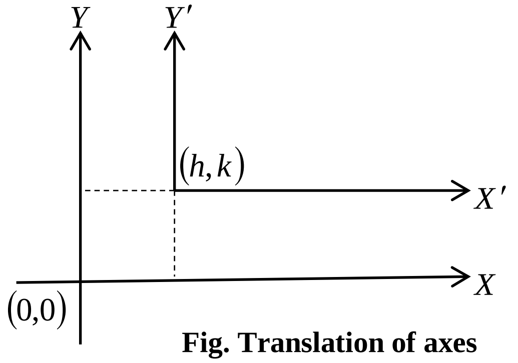


Fig. Translation of axes

2. Rotation of axes :

In this process the position of the origin is not changed but the direction of coordinate axes is being changed through a fixed angle with the x-axis.

When the position of the origin is not changed and the direction of coordinate axes is being changed through a fixed angle θ with the X-axis then then this is called rotation of axes and the relation between new (X', Y') system and old system (X, Y) are given below.

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$or$$

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta$$

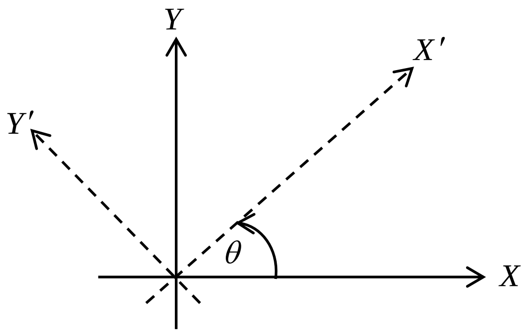
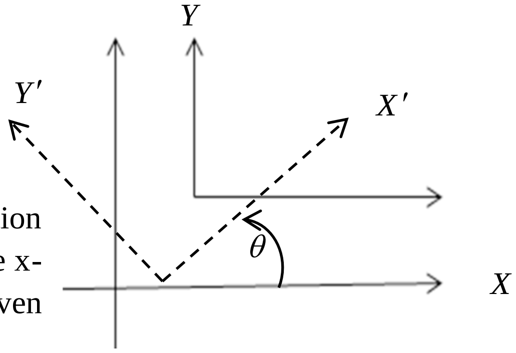


Fig. Rotation of axes

3. Translation-Rotation:



In this process the position of the origin is changed and the direction of coordinate axes is being changed through a fixed angle with the x-axis. The relation between new system and old system are given below.

$$x = x'\cos\theta - y'\sin\theta + h \qquad y = x'\sin\theta + y'\cos\theta + k$$

Mathematical problem

Mathematical problem on Translation of Axis

Problem 01: Determine the equation of the curve $2x^2+3y^2-8x+6y-7=0$ when the origin is transferred to the point $(2,-1)$.

Solution:

Given Equation of the curve is,

$$2x^2+3y^2-8x+6y-7=0 \quad \cdots\cdots\cdots\cdots\cdots(i)$$

Origin is transferred to the point $(h,k)=(2,-1)$ so as the transformed relations are $x=x'+h=x'+2$ and $y=y'+k=y'-1$.

Using the above transformation given equation (i) becomes

$$2(x'+2)^2+3(y'-1)^2-8(x'+2)+6(y'-1)-7=0$$
$$\Rightarrow \quad 2(x'^2+4x'+4)+3(y'^2-2y'+1)-8(x'+2)+6(y'-1)-7=0$$
$$\Rightarrow \quad (2x'^2+8x'+8)+(3y'^2-6y'+3)-(8x'+16)+(6y'-6)-7=0$$
$$\Rightarrow \quad 2x'^2+8x'+8+3y'^2-6y'+3-8x'-16+6y'-6-7=0$$
$$\Rightarrow \quad 2x'^2+3y'^2-18=0$$
$$\Rightarrow \quad 2x'^2+3y'^2=18$$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$2x^2+3y^2=18$$

Thich is the required equation that represents an ellipse.

Problem 02: What does the equation $x^2+y^2-4x-6y+6=0$ becomes when the origin is transferred to the point $(2,3)$ and the direction of axes remain unaltered.

Solution:

Given Equation of the curve is,

$$x^2+y^2-4x-6y+6=0 \quad \cdots\cdots\cdots\cdots\cdots(i)$$

Origin is transferred to the point $(h,k)=(2,3)$ so as the transformed relations are $x=x'+h=x'+2$ and $y=y'+k=y'+3$.

Using the above transformation given equation (i) reduces to

$$(x'^2+4x'+4)+(y'^2+6y'+9)-4(x'+2)-6(y'+3)+6=0$$
$$\Rightarrow \quad (x'^2+4x'+4)+(y'^2+6y'+9)-(4x'+8)-(6y'+18)+6=0$$
$$\Rightarrow \quad x'^2+4x'+4+y'^2+6y'+9-4x'-8-6y'-18+6=0$$
$$\Rightarrow \quad x'^2+y'^2-17=0$$
$$\Rightarrow \quad x'^2+y'^2-17=0$$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$x^2+y^2=17$$

Which is the required equations that represents a circle.

H.W.

1. Transform to parallel axes through the point $(3,5)$ the equation $x^2+y^2-6x-10y-2=0$.
2. Transform $x^2+2y^2-6x+7=0$ to parallel axes through the point $(3,1)$.
3. Transform the equation $3x-25y+41=6$ to parallel axes through $(-3,2)$.
4. Transform the equation $x^2-3y^2+4x+6y=0$ by transferring the origin to the point $(-2,1)$, coordinate axes remaining parallel.
5. Transform the equation $3x^2+14xy-24y^2-22x+110y-121=0$ shifting the origin to the point $(-1,2)$ and keeping the direction of axes fixed.

Mathematical problem on Rotation of Axis

Problem 03: Transform the equation $3x^2+5y^2-3=0$ to axes turned through 45°.

Solution: Given that,

$$3x^2+5y^2-3=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since the axes rotated are an angle 45° and origin be unchanged.

So, $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$= x'\cos 45^\circ - y'\sin 45^\circ$, $\quad = x'\sin 45^\circ + y'\cos 45^\circ$

$= \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$, $\quad = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$

Using this value in equation (i), we get,

$$3\left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)^2 + 5\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)^2 - 3 = 0$$

Or, $3\{\frac{1}{\sqrt{2}}(x'-y')\}^2 + 5\{\frac{1}{\sqrt{2}}(x'+y')\}^2 - 3 = 0$

Or, $\frac{3}{2}(x'^2 - 2x'y' + y'^2) + \frac{5}{2}(x'^2 + 2x'y' + y'^2) - 3 = 0$

Or, $3x'^2 - 6x'y' + 3y'^2 + 5x'^2 + 10x'y' + 5y'^2 - 6 = 0$

$\therefore 8x'^2 + 4x'y' + 8y'^2 - 6 = 0$

Now removing suffixes, we can write,

$4x^2 + 2xy + 4y^2 - 3 = 0.$

This is the required equation.

Problem 04: If the axes be turned through an angle $\tan^{-1}2$, what does the equation $4xy-3x^2=a^2$ become?

Solution:

Given Equation of the curve is,

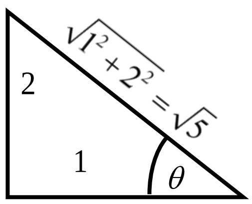
$$4xy-3x^2=a^2 \cdots\cdots\cdots\cdots\cdots(i)$$

The coordinate axes turned through an angle $\theta=\tan^{-1}2$ that implies $\tan\theta=2$.

Now

$$\sin\theta=\frac{2}{\sqrt{5}}$$

$$\cos\theta=\frac{1}{\sqrt{5}}$$



Considering the new coordinate of the point is (x', y') and rotating the axes through an angle $\theta = \tan^{-1} 2$ and origin be unchanged as the transformed equations are as follows

$$x = x'\cos\theta - y'\sin\theta$$
$$= \frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}} \quad = \frac{1}{\sqrt{5}}(x' - 2y')$$

And, $$y = x'\sin\theta + y'\cos\theta$$
$$= \frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}} \quad = \frac{1}{\sqrt{5}}(2x' + y')$$

Putting the value of x and y the above equation (i) becomes

$$4\left\{\frac{1}{\sqrt{5}}(x' - 2y')\right\}\left\{\frac{1}{\sqrt{5}}(2x' + y')\right\} - 3\left\{\frac{1}{\sqrt{5}}(x' - 2y')\right\}^2 = a^2$$

Or, $4.\frac{1}{5}(x' - 2y')(2x' + y') - \frac{3}{5}(x' - 2y')^2 = a^2$

Or, $4(x' - 2y')(2x' + y') - 3(x' - 2y')^2 = 5a^2$

Or, $4\left(2x'^2 + x'y' - 4x'y' - 2y'^2\right) - 3\left(x'^2 - 4x'y' + 4y'^2\right) = 5a^2$

Or, $4\left(2x'^2 - 3x'y' - 2y'^2\right) - 3\left(x'^2 - 4x'y' + 4y'^2\right) = 5a^2$

Or, $\left(8x'^2 - 12x'y' - 8y'^2\right) - \left(3x'^2 - 12x'y' + 12y'^2\right) = 5a^2$

Or, $8x'^2 - 12x'y' - 8y'^2 - 3x'^2 + 12x'y' - 12y'^2 = 5a^2$

Or, $5x'^2 - 20y'^2 = 5a^2$

Or, $x'^2 - 4y'^2 = a^2$

Removing suffices from the above equation we get the transformed equation of the given curve.

$$x^2 - 4y^2 = a^2$$

H.W:

1. Transformed the equation $7x^2 - 2xy + y^2 + 5 = 0$ to the axes turned through an angle $\tan^{-1}(1)$.
2. Transformed the equation $7x^2 - 2xy + y^2 + 1 = 0$ to the axes turned through an angle $\tan^{-1}(\frac{1}{2})$.
3. Determine the equation of the parabola $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ after rotating of axes through $\frac{\pi}{4}$.

Mathematical problem on Translation-Rotation

Problem 05: Determine the transform equation of $3x - 2y + 5 = 0$ when the origin is transferred to the point $(-2, -1)$ and the axes turned through an angle 45^0.

Solution:

Given Equation is,

$$3x - 2y + 5 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Origin is transferred to the point $(h, k) = (-2, -1)$ so as the transformed relations are $x = x' - 2$ and $y = y' - 1$

Using the above transformation given equation (i) becomes

$$3(x' - 2\,) - 2(y' - 1) + 5 = 0$$
$$\Rightarrow 3x' - 6 - 2y' + 2 + 5 = 0$$
$$\Rightarrow 3x' - 2y' + 1 = 0$$

Now removing suffixes, we can write,

$$3x - 2y + 1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Again the axes rotated are an angle 45°

So, $\quad x = x'\cos\theta - y'\sin\theta \quad$ and $\quad y = x'\sin\theta + y'\cos\theta$

$\quad = x'\cos45^\circ - y'\sin45^\circ, \qquad = x'\sin45^\circ + y'\cos45^\circ$

$\quad = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y', \qquad = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$

Using this value in equation (ii), we get,

$$3(\tfrac{1}{\sqrt{2}}x' - \tfrac{1}{\sqrt{2}}y') - 2(\tfrac{1}{\sqrt{2}}x' + \tfrac{1}{\sqrt{2}}y') + 1 = 0$$
$$\Rightarrow 3(x' - y') - 2(x' + y') + \sqrt{2} = 0$$
$$\Rightarrow 3x' - 3y' - 2x' - 2y' + \sqrt{2} = 0$$
$$\Rightarrow x' - 5y' + \sqrt{2} = 0$$

Now removing suffixes, we can write,

$$x - 5y + \sqrt{2} = 0$$

This is the required transform equation.

H.W:

1. Find transform equation of $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ when the origin is transferred to the point (2,3) and the axes turned through an angle 45^0.
2. Find transform equation of $x^2 - 2xy + y^2 + 2x - 4y + 3 = 0$ when the origin is transferred to the point $(-2,1)$ and the axes turned through an angle 60^0.
3. Find transform equation of $4x^2 + xy - y^2 - 8x + 2y + 5 = 0$ when the origin is transferred to the point $(-1, -2)$ and the axes turned through an angle 55^0.

EQUATION & ITS GEOMETRY

◙ Equation in 2 variables:

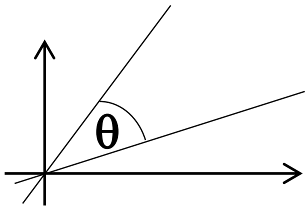
- **1^{st} degree General equation:**

$$ax + by + c = 0$$

It always represents a straight line in plane, provided at least $a \neq 0$ or, $b \neq 0$

❑ Homogeneous equation:

An equation in which degree of each term in it is equal is called Homogeneous equation. Such as $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of degree or order 2 because degree of its each term is two. It is noted that homogeneous equation always represents straight lines passing through the origin.



- **2^{nd} degree homogenous equation:**

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{----------------(*)}$$

In plane, it always represents two straight lines passing through the origin (0, 0).

- ❖ Lines be perpendicular if $a + b = 0$
- ❖ Lines be parallel/coincident if $h^2 = ab$
  Since two lines passes through the point $(0,0)$, so lines must be coincident.
- ❖ Lines be real and different if $h^2 > ab$.
- ❖ Lines be imaginary if $h^2 < ab$. But passes through the point $(0,0)$.

❑ Non-homogeneous equation:

An equation in which degree of each term in it is not equal is called Non-homogeneous equation. Such as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a non- homogeneous equation of degree or order 2.

- **2^{nd} degree General equation :**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{-----------------(**)}$$

Descartes found that the graphs of second-degree equations in two variables in plane always fall into one of seven categories: [1] single point, [2] pair of straight lines, [3] circle, [4] parabola, [5] ellipse, [6] hyperbola, and [7] no graph at all.

Note:

- $ax^2+2hxy+by^2+2gx+2fy+c=0$, represents pair of straight lines if

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2=0$$

(1) Two parallel lines if $\Delta=0, h^2=\text{ab}$.
(2) Two perpendicular lines if $\Delta=0, \text{a}+\text{b}=0$.

- Otherwise, (**) will represent:
  1. A circle if $\Delta\neq 0, a=b, h=0$.
  2. A parabola if $\Delta\neq 0, h^2=\text{ab}$
  3. An ellipse if $\Delta\neq 0, h^2-\text{ab}<0$.
  4. A hyperbola if $\Delta\neq 0, h^2-\text{ab}>0$.
  5. A rectangular hyperbola if $a+b=0, h^2-ab>0, \Delta\neq 0$.

If $\Delta=0$ then (**) will represent pair of straight line and if $\Delta\neq 0$ then (**) will represent curve.

If none of the above conditions are satisfied then (**) represents [1] or [7].

Also,

- ✓ if $c=0$, then (**) always passes through the origin(0,0).
- ✓ term containing xy can be transformed by a suitably chosen rotation into a form which does not contain xy term. The standard equation is easily obtained.
- ✓ the rotation angle 'θ' that will eliminate the 'xy' term is given by $\cot 2\theta=\frac{a-b}{2h}$ or $\tan 2\theta=\frac{2h}{a-b}$

❖ **Determination of intersecting point (α,β) of the straight lines represented by the equation** $ax^2+2hxy+by^2+2gx+2fy+c=0$:

Let $f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c$

Then set
$$\frac{\partial f}{\partial x}=2ax+2hy+2g=0\Rightarrow ax+hy+g=0 \quad \cdots\cdots\cdots(i)$$
$$\frac{\partial f}{\partial y}=2hx+2by+2f=0\Rightarrow hx+by+f=0 \quad \cdots\cdots\cdots(ii)$$

Now calculate the intersecting point of the equation (i) and (ii) we get the intersecting point (α,β) of the straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Angle 'Θ' between 2 (real) straight lines represented by (*) or (**):

We know that 2 straight lines always cut at an angle (real or imaginary). If 'Θ' be that angle, we have to use following formula to find 'Θ'.

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

> Case 1 : If $h^2 < ab$, then angle 'Θ' is **imaginary** and we can't view it.
>
> Case 2 : When $h^2 \nless ab$, then angle 'Θ' is not **imaginary**.

Some condition for angle 'Θ':-

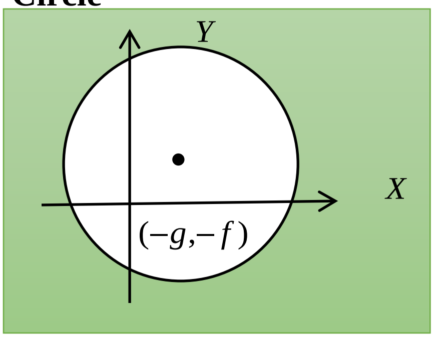
- If $\Theta = 0^0$, we say, the straight lines are either parallel or, coincident (*i.e.* same).
- If a + b = 0, then $\Theta = 90^0$, and we say, the straight lines are perpendicular.
- If Θ is +ve, we accept it & say that the angle is acute.
- If Θ is -ve, we add 180^0 , and then get an obtuse angle.

Separation of equation of each line from (*) or (**):

To determine the equation of each line separately, we need to solve (*) or (**).*i.e.* we write the equation as $\mathbf{Ax^2 + B\,x + C = 0}$ & then solve it.

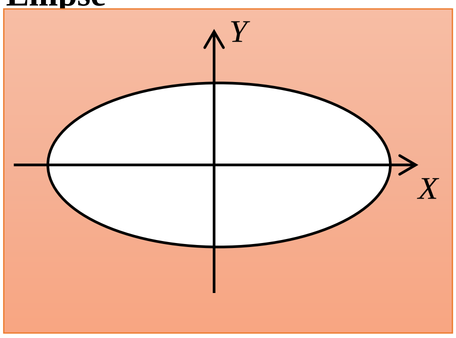
Some figure and standard form:-

Circle

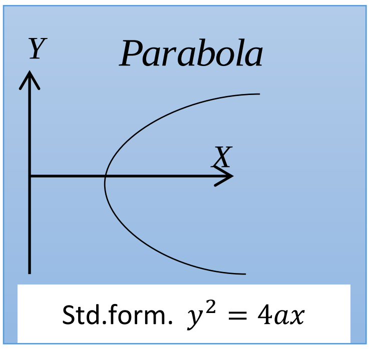


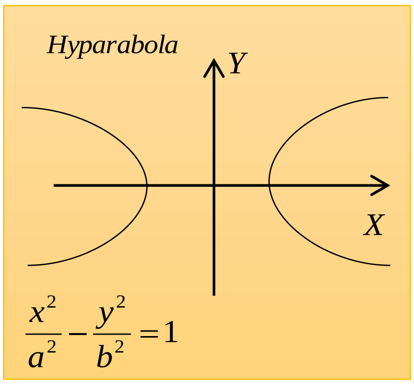
Std.form. $x^2 + y^2 + 2gx + 2fy + c = 0$

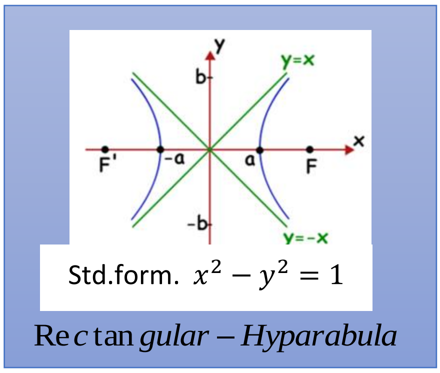
Ellipse



Std.form. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$







Invariant:

By the rotation of the rectangular coordinate axes about the origin the following equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ changes to the equation $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ in which the following expression remain unchanged

1. $a'+b'=a+b$
2. $h'^2-a'b'=h^2-ab$
3. $\Delta'=\Delta$ where $\Delta'=\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$ and $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

The unchanged quantities $a+b$, h^2-ab and Δ are called invariants in coordinate transformation.

Mathematical problem Solution

Problem 01:- If the equation is $3x^2-16xy+5y^2=0$, then find, **(a)** angle between the lines **(b)** equation of line.

Solution:

(a)

Given homogeneous equation is as follows

$$3x^2-16xy+5y^2=0$$

Comparing the given equation with the general homogeneous equation $ax^2+2hxy+by^2=0$ we have $a=3, h=-8 \ \ and\ b=5$.

Let an angle between the lines is θ.

Then we have $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

$$\text{Or, } \tan\theta=\frac{2\sqrt{(-8)^2-3.5}}{3+5}$$

$$\text{Or, } \tan\theta=\frac{2\sqrt{64-15}}{8}$$

$$\text{Or, } \tan\theta=\frac{2\sqrt{49}}{8}=\frac{2.7}{8}=\frac{14}{8}$$

$$\therefore\ \theta=\tan^{-1}\left(\frac{14}{8}\right)=60.26^0$$

Therefore the angle between the lines is 60.26^0.

(b)

Given homogeneous equation is as follows

$$3x^2-16xy+5y^2=0$$

We expressed the given equation as

$$3x^2-16xy+5y^2=0$$

$$\text{Or, } 3x^2-16y.x+5y^2=0$$

$$\text{Or, } x=\frac{16y\pm\sqrt{(-16y)^2-4.3.5y^2}}{2.3}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{Or, } x=\frac{16y\pm\sqrt{256y^2-60y^2}}{6}$$

$$\text{Or, } x=\frac{16y\pm\sqrt{196y^2}}{6}$$

$$\text{Or, } x=\frac{16y\pm 14y}{6}$$

Taking positive sign we get $x=\frac{16y+14y}{6}=\frac{30y}{6}=5y$

Therefore $x=5y\Rightarrow x-5y=0$

And taking negative sign we get $x=\frac{16y-14y}{6}=\frac{2y}{6}=\frac{y}{3}$

Therefore, $x=\frac{y}{3}\Rightarrow 3x=y\ \therefore 3x-y=0$

Therefore $x-5y=0$ and $3x-y=0$

These are the straight lines passing through the origin.

Problem 02:- Show that $6x^2-5xy-6y^2+14x+5y+4=0$ represents pair of straight lines.

Solution:

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0\ \cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=6\ ,h=-\frac{5}{2},b=-6,g=7,\ f=\frac{5}{2}\ \&c=4$$

Now,

$$\Delta=\begin{vmatrix}6 & -\frac{5}{2} & 7\\ -\frac{5}{2} & -6 & \frac{5}{2}\\ 7 & \frac{5}{2} & 4\end{vmatrix}=6(-24-\frac{25}{4})-\left(-\frac{5}{2}\right)\left(-10-\frac{35}{2}\right)+7\left(-\frac{25}{4}+42\right)$$

$$=6(-24-\frac{25}{4})-\left(-\frac{5}{2}\right)\left(-10-\frac{35}{2}\right)+7\left(-\frac{25}{4}+42\right)$$

$$=6(-24-\frac{25}{4})+\frac{5}{2}\left(-10-\frac{35}{2}\right)+7\left(-\frac{25}{4}+42\right)$$

$$=(-144-\frac{150}{4})+\left(-25-\frac{175}{4}\right)+\left(-\frac{175}{4}+294\right)$$

$$=\frac{-576-150}{4}+\left(\frac{-100-175}{4}\right)+\left(\frac{-175+1176}{4}\right)$$

$$=\frac{-726}{4}+\left(\frac{-275}{4}\right)+\left(\frac{1001}{4}\right)$$

$$=-\frac{1001}{4}+\frac{1001}{4}=0$$

Since $\Delta=0$ so the given equation represents a pair of straight lines.

Another process,

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0 \cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=6\ ,h=-\frac{5}{2},b=-6,g=7,\ f=\frac{5}{2}\ \&c=4$$

We know that, $\Delta= abc+2fgh-af^2-bg^2-ch^2$

$$=6*(-6)*4+2*\left(\frac{5}{2}\right)*7*\left(-\frac{5}{2}\right)-6*\left(\frac{5}{2}\right)^2-(-6)*(7)^2-4*\left(-\frac{5}{2}\right)^2$$

$$=0$$

Since $\Delta=0$ so the given equation represents a pair of straight lines.

Problem 03:- Find the angle between the straight lines represented by the equation

$$6x^2-5xy-6y^2+14x+5y+4=0.$$

Solution:

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=6\ ,h=-\frac{5}{2},b=-6,g=7,\ f=\frac{5}{2}\ \&c=4$$

Assume that θ be the angle between the straight lines then we have the followings

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow\quad \tan\theta=\frac{2\sqrt{\frac{25}{4}+36}}{6-6}$$

$$\Rightarrow\quad \tan\theta=\infty$$

$$\Rightarrow\quad \theta=\tan^{-1}(\infty)\qquad \therefore\ \theta=\frac{\pi}{2}$$

Problem 04:- Find the equation of the straight lines represented by the equation

$$x^2+6xy+9y^2+4x+12y-5=0.$$

Solution:

Given equation is,

$$x^2+6xy+9y^2+4x+12y-5=0$$

Arrange the above equation as a quadratic equation in x we get

$$x^2+6xy+9y^2+4x+12y-5=0$$

$$x^2+(6y+4)x+9y^2+12y-5=0$$

$$\therefore\ x=\frac{-(6y+4)\pm\sqrt{(6y+4)^2-4.1.(9y^2+12y-5)}}{2.1}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(6y+4)\pm\sqrt{(6y+4)^2-4(9y^2+12y-5)}}{2}$$

$$\Rightarrow \quad x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-\left(36y^2+48y-20\right)}}{2}$$

$$\Rightarrow \quad x=\frac{-(6y+4)\pm\sqrt{36y^2+48y+16-36y^2-48y+20}}{2}$$

$$\Rightarrow \quad x=\frac{-(6y+4)\pm\sqrt{16+20}}{2}$$

$$\Rightarrow \quad x=\frac{-(6y+4)\pm\sqrt{36}}{2}$$

$$\Rightarrow \quad x=\frac{-(6y+4)\pm 6}{2}$$

Taking positive we get $x=\dfrac{-(6y+4)+6}{2}$

$$\Rightarrow \quad x=\frac{-6y-4+6}{2}$$

$$\Rightarrow \quad x=\frac{-6y+2}{2}$$

$$\Rightarrow \quad 2x=-6y+2$$

$$\Rightarrow \quad x=-3y+1$$

$$\Rightarrow \quad x+3y-1=0$$

Taking negative we get $x=\dfrac{-(6y+4)-6}{2}$

$$\Rightarrow \quad x=\frac{-6y-4-6}{2}$$

$$\Rightarrow \quad x=\frac{-6y-10}{2}$$

$$\Rightarrow \quad 2x=-6y-10$$

$$\Rightarrow \quad x=-3y-5$$

$$\Rightarrow \quad x+3y+5=0$$

Therefore, required equations of the straight lines $x+3y-1=0$ and $x+3y+5=0$.

Problem 05:- Find the point of intersection of the straight lines represented by the equation

$$6x^2-5xy-6y^2+14x+5y+4=0.$$

Solution:

Given equation is,

$$6x^2-5xy-6y^2+14x+5y+4=0$$

Suppose $f(x,y)=6x^2-5xy-6y^2+14x+5y+4$

Now, Differentiating the function $f(x,y)$ with respect to x and y partially and equating with zero, we get

$$\frac{\partial f}{\partial x}=12x-5y+14$$

$$\Rightarrow 12x-5y+14=0 \cdots\cdots\cdots\cdots\cdots (i)$$

And

$$\frac{\partial f}{\partial x} = -5x - 12y + 5$$
$$\Rightarrow -5x - 12y + 5 = 0$$
$$\Rightarrow \; 5x + 12y - 5 = 0 \cdots\cdots\cdots\cdots\cdots (ii)$$

Solving equation (i) and (ii) we get the point of intersection of lines represented by the given equation.
Using cross multiplication method on equation (i) and (ii)

$$\frac{x}{25-168} = \frac{y}{70+60} = \frac{1}{144+25}$$
$$\frac{x}{-143} = \frac{y}{130} = \frac{1}{169}$$
$$x = -\frac{143}{169} = -\frac{11}{13} \;\; \& \;\; y = \frac{130}{169} = \frac{10}{13}$$

Therefore, the coordinates of point of intersection is $(x, y) = \left(-\frac{11}{13}, \frac{10}{13}\right)$.

Problem-06: Test the nature of the equation $3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$ and also find its center.

Solution:

1st part: Given that,

$$3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also the general equation of second degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Comparing (i) and (ii) we have,

$$a = 3, h = -4, b = -3, g = 5, f = -\frac{13}{2}, c = 8.$$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 3 \times (-3) \times 8 + 2 \times \left(-\frac{13}{2}\right) \times 5 \times (-4) - 3 \times (-\frac{13}{2})^2 - (-3) \times 25 - 8 \times 16$$
$$= -72 + 260 - \frac{507}{4} + 75 - 128$$
$$= \frac{33}{4}$$

Since, $\Delta = \frac{33}{4} \neq 0$ so the given equation represents a conic.

Again, $h^2 - \text{ab} = 16 + 9 = 25 > 0$

And, $a + b = 3 - 3 = 0$

Since, $a + b = 0, h^2 - ab > 0, \Delta = 0.$ so the given equation represents a rectangular hyperbola.

2nd part: Let, $f(x, y) = 3x^2 - 8xy - 3y^2 + 10x - 13y + 8 = 0$

$$\therefore \frac{\partial f}{\partial x} = 6x - 8y + 10 = 0$$
$$\text{And } \frac{\partial f}{\partial y} = 8x + 6y + 13 = 0$$

The center of the conic is the intersection of two lines,

$$6x - 8y + 10 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$
$$8x + 6y + 13 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

Solving (iii) and (iv) we have,

$$x = -\frac{41}{25}, y = \frac{1}{50}$$

Hence the center is at $\left(-\frac{41}{25}, \frac{1}{50}\right)$.

Problem-07: Test the nature of the equation $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$.

Solution: Given that,

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also the general equation of second degree is,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Comparing (i) and (ii) we have,

$$a = 9, h = -12, b = 16, g = -9, f = -\frac{101}{2}, c = 19.$$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 9 \times (-12) \times 19 + 2 \times \left(-\frac{101}{2}\right) \times (-9) \times (-12) - 9 \times \left(-\frac{101}{2}\right)^2 - 16 \times (-9)^2 - 19 \times (-12)^2$

$= -2052 - 10908 - \frac{91809}{4} - 1296 - 2736$

$= -\frac{159777}{4}$

Since, $\Delta = -\frac{159777}{4} \neq 0$ so the given equation represents a conic.

Again, $h^2 - \text{ab} = (-12)^2 - 9 \times 16 = 144 - 144 = 0$

Since, $h^2 - ab = 0, \Delta \neq 0$. so the given equation represents a hyperbola.

Problem-08: Reduce the equation $8x^2 + 4xy + 5y^2 - 24x - 24y = 0$ to the standard form.

Solution:

Given general equation of second degree is

$$8x^2 + 4xy + 5y^2 - 24x - 24y = 0 \cdots\cdots\cdots\cdots\cdots\cdots (i)$$

Comparing this above equation with the standard equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$$a = 8, h = 2, b = 5, g = -12, f = -12 \ \& c = 0$$

Now,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 8 & 2 & -12 \\ 2 & 5 & -12 \\ -12 & -12 & 0 \end{vmatrix} = 8(0-144) - 2(0-144) - 12(-24+60)$$

$= 8(-144) - 2(-144) - 12(-24+60)$

$= -1152 + 288 - 432$

$= -1296 \neq 0$

And

$$h^2 - ab = 2^2 - 40 = 4 - 40 = -36 < 0$$

Since $\Delta \neq 0$ and $h^2 - ab < 0$. So the equation represents an ellipse.

Suppose $f(x, y) = 8x^2 + 4xy + 5y^2 - 24x - 24y$

Now, Differentiating the function $f(x, y)$ with respect to x and y partially and equating with zero, we get

$\frac{\partial f}{\partial x} = 16x + 4y - 24$

$\Rightarrow 16x + 4y - 24 = 0$

$\Rightarrow 4x + y - 6 = 0 \cdots\cdots\cdots\cdots\cdots (ii)$

And

$\frac{\partial f}{\partial x} = 4x + 10y - 24$

$\Rightarrow 4x + 10y - 24 = 0$

$\Rightarrow 2x + 5y - 12 = 0 \cdots\cdots\cdots\cdots\cdots (iii)$

Solving equation (i) and (ii) we get center of the conic represented by the given equation.
Using cross multiplication method on equation (i) and (ii)

$$\frac{x}{-12+30}=\frac{y}{-12+48}=\frac{1}{20-2}$$

Or, $\frac{x}{18}=\frac{y}{36}=\frac{1}{18}$

$$x=\frac{18}{18}=1 \ \& \ y=\frac{36}{18}=2$$

Therefore, the coordinates of center is $(\alpha,\beta)=(x,y)=(1,2)$.

Therefore, the equation of the conic referred to center as origin is

$$8x^2+4xy+5y^2+c_1=0 \cdots\cdots\cdots\cdots(iv)$$

Where $c_1=g\alpha+f\beta+c=-12-24+0=-36$

So the equation (iv) becomes $8x^2+4xy+5y^2-36=0 \cdots\cdots\cdots\cdots(v)$

When the xy term is removed by the rotation of axes then the reduced equation is

$$a_1x^2+b_1y^2=36 \cdots\cdots\cdots\cdots(vi)$$

Then by invariants we have

$$a_1+b_1=a+b=8+5=13 \cdots\cdots\cdots\cdots\cdots\cdots(vii)$$

$$\Rightarrow \ a_1b_1-h_1^2=ab-h^2$$

$$\Rightarrow \ a_1b_1-0=40-4=36$$

$$\Rightarrow \ a_1b_1=36$$

We know,

$$(a_1-b_1)^2=(a_1+b_1)^2-4a_1b_1$$

$$\Rightarrow \ (a_1-b_1)^2=13^2-4\times 36$$

$$\Rightarrow \ (a_1-b_1)^2=169-144=25$$

$$\Rightarrow \ (a_1-b_1)^2=25$$

$$a_1-b_1=5 \cdots\cdots\cdots\cdots\cdots\cdots(viii)$$

Solving equations (vii) and $(viii)$ we have $a_1=9$ and $b_1=4$

The equation (vi) becomes $9x^2+4y^2=36$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Which is required equations.

Problem-09: Reduce the equation $32x^2+52xy-7y^2-64x-52y-148=0$ to the standard form.

Solution:
Given general equation of second degree is

$$32x^2+52xy-7y^2-64x-52y-148=0 \cdots\cdots\cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=32\ ,h=26,b=-7,g=-32,\ f=-26\ \&c=-148$$

Now,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 32 & 26 & -32 \\ 26 & -7 & -26 \\ -32 & -26 & -148 \end{vmatrix} = 162000 \neq 0$$

And

$$h^2 - ab = 26^2 + 224 = 900 > 0$$

Since $\Delta \neq 0$ and $h^2 - ab > 0$. So the equation represents hyperbola.

Suppose $f(x, y) = 32x^2 + 52xy - 7y^2 - 64x - 52y - 148$

Now, Differentiating the function $f(x, y)$ with respect to x and y partially and equating with zero, we get

$$\frac{\partial f}{\partial x} = 64x + 52y - 64$$

$$\Rightarrow 64x + 52y - 64 = 0$$

$$\Rightarrow 16x + 13y - 16 = 0 \cdots\cdots\cdots\cdots\cdots (ii)$$

And

$$\frac{\partial f}{\partial x} = 52x - 14y - 52$$

$$\Rightarrow 52x - 14y - 52 = 0$$

$$\Rightarrow 26x - 7y - 26 = 0 \cdots\cdots\cdots\cdots\cdots (iii)$$

Solving equation (ii) and (iii) we get center of the conic represented by the given equation.
Using cross multiplication method on equation (ii) and (iii)

$$\frac{x}{-112 - 338} = \frac{y}{-416 + 416} = \frac{1}{-112 - 338}$$

$$\frac{x}{-450} = \frac{y}{0} = \frac{1}{-450}$$

$$x = \frac{-450}{-450} = 1 \;\&\; y = \frac{0}{-450} = 0$$

Therefore, the coordinates of center is $(\alpha, \beta) = (x, y) = (1, 0)$.

Therefore, the equation of the conic referred to center as origin is

$$32x^2 + 52xy - 7y^2 + c_1 = 0 \cdots\cdots\cdots\cdots (iv)$$

Where $c_1 = g\alpha + f\beta + c = -32 + 0 - 148 = -180$

So the equation (iv) becomes $32x^2 + 52xy - 7y^2 - 180 = 0 \cdots\cdots\cdots\cdots (v)$

When the xy term is removed by the rotation of axes then the reduced equation is

$$a_1x^2 + b_1y^2 = 180 \cdots\cdots\cdots\cdots (vi)$$

Then by invariants we have

$$a_1 + b_1 = a + b = 32 - 7 = 25 \cdots\cdots\cdots\cdots\cdots\cdots (vii)$$

$$\Rightarrow \quad a_1b_1 - h_1^2 = ab - h^2$$

$$\Rightarrow \quad a_1b_1 - 0 = -224 - 676 = -900$$

$$a_1b_1 = -900$$

We know,

$$(a_1 - b_1)^2 = (a_1 + b_1)^2 - 4a_1b_1$$

$$\Rightarrow \quad (a_1 - b_1)^2 = 25^2 + 4 \times 900$$

$$\Rightarrow \quad (a_1 - b_1)^2 = 4225$$

$$a_1 - b_1 = 65 \cdots\cdots\cdots\cdots\cdots\cdots (viii)$$

Solving equations (vii) and $(viii)$ we have $a_1 = 45$ and $b_1 = -20$

The equation (vi) becomes $45x^2 - 20y^2 = 180$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Which is required equations.

Problem 10:- What are represented by the following equations?

(1) $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$

(2) $x^2 + 2xy + y^2 + 2x - 1 = 0$

(A) If it will be a straight line then find,

(i) The angle in between the lines; (ii) find the equation of each line.

(B) If it will be a curve then find,

(i) Find a rotation angle by which the xy-term will be eliminated; (ii) find the standard form.

Solution:- **(1)**

Given equation is,

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

Comparing this above equation with the standard equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$$a = 12, \quad b = -10, \quad c = -35, \quad g = \frac{13}{2}, \quad f = \frac{45}{2}, \quad h = \frac{7}{2}$$

Now we have, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 12 * (-10) * (-35) + 2 * \left(\frac{45}{2}\right) * \left(\frac{13}{2}\right) * \left(\frac{7}{2}\right) - 12 * \left(\frac{45}{2}\right)^2 - (-10) * \left(\frac{13}{2}\right)^2 - (-35) * \left(\frac{7}{2}\right)^2$$

$$= 0$$

Since $\Delta = 0$ so the given equation represents a pair of straight lines.

The angle between two lines:-

If θ be the angle between the straight lines then we know that

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\Rightarrow \quad \tan\theta = \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 12 * (-10)}}{12 + (-10)}$$

$$\Rightarrow \quad \tan\theta = 11.5$$

$$\Rightarrow \quad \theta = \tan^{-1}(11.5)$$

$$\Rightarrow \quad \theta = 85^0 1' 48.93''$$

Separation of lines:-

Given equation is,

$$12x^2+7xy-10y^2+13x+45y-35=0$$

$$\Rightarrow\ 12x^2+(7y+13)x+(-10y^2+45y-35)=0$$

$$\Rightarrow\ x=\frac{-(7y+13)\pm\sqrt{(7y+13)^2-4\cdot(-10y^2+45y-35)\cdot 12}}{2\cdot 12}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\ 24x=-(7y+13)\pm\sqrt{(23y-43)^2}$$

$$\Rightarrow\ 24x+7y+13=\pm(23y-43)$$

Taking positive,

$$24x+7y+13=(23y-43)$$

$$\Rightarrow\ 3x-2y+7=0$$

Taking negative,

$$24x+7y+13=-(23y-43)$$

$$4x+5y-5=0$$

Hence, $3x-2y+7=0$ and $4x+5y-5=0$ are the required equations of two straight lines represented by $12x^2+7xy-10y^2+13x+45y-35=0$.

Solution:- (2)

Given general equation of second degree is

$$x^2+2xy+y^2+2x-1=0\ \cdots\cdots\cdots\cdots\cdots\cdots(i)$$

Comparing this above equation with the standard equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ we get

$$a=1\ ,h=1,b=1,g=1,\ f=0\ \&\ c=-1$$

Now,

$$\Delta=\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & -1\end{vmatrix}=1\neq 0$$

And

$$h^2-ab=1-1=0$$

Since $\Delta\neq 0$ and $h^2-ab=0$. So the equation represents parabola.

To remove 'xy' term we need a rotation of 'θ' where

$$\cot 2\theta=\frac{a-b}{2h}$$

$$\text{Or, }\cot 2\theta=\frac{1-1}{2\cdot 1}=0$$

$$\text{Or, }2\theta=90^0$$

$$\theta=45^0$$

This is the required angle.

Now we know that,

$$x=x'\cos\theta-y'\sin\theta=x'\cos 45^\circ-y'\sin 45^\circ=\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'=\frac{1}{\sqrt{2}}(x'-y')$$

and

$$y=x'\sin\theta+y'\cos\theta=x'\sin 45^\circ+y'\cos 45^\circ=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'=\frac{1}{\sqrt{2}}(x'+y')$$

Putting this in (i) we get,

$$x^2+2xy+y^2+2x-1=0$$

Or, $(x+y)^2+2x-1=0$

Or, $\left\{\frac{1}{\sqrt{2}}(x'-y')+\frac{1}{\sqrt{2}}(x'+y')\right\}^2+2\cdot\frac{1}{\sqrt{2}}(x'-y')-1=0$

Or, $\left\{\frac{1}{\sqrt{2}}(x'-y'+x'+y')\right\}^2+2\cdot\frac{1}{\sqrt{2}}(x'-y')-1=0$

Or, $2(x')^2+\sqrt{2}x'-\sqrt{2}y'-1=0$

Or, $2(x')^2+\sqrt{2}x'=\sqrt{2}y'+1$ This can be written as $2(x)^2+\sqrt{2}x=\sqrt{2}y+1$

Standard form,

$$2(x)^2+\sqrt{2}x=\sqrt{2}y+1$$

Or, $(x)^2+\frac{1}{\sqrt{2}}x=\frac{1}{\sqrt{2}}y+\frac{1}{2}$ (Both side multiply by 2)

Or, $(x)^2+2\cdot x\frac{1}{2\sqrt{2}}+(\frac{1}{2\sqrt{2}})^2-\frac{1}{8}=\frac{1}{\sqrt{2}}y+\frac{1}{2}$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}y+\frac{5}{8}$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}(y+\sqrt{2}\cdot\frac{5}{8})$

Or, $(x+\frac{1}{2\sqrt{2}})^2=\frac{1}{\sqrt{2}}(y+\cdot\frac{5}{4\sqrt{2}})$

Or, $(x+\frac{1}{2\sqrt{2}})^2=4\frac{1}{4\sqrt{2}}(y+\cdot\frac{5}{4\sqrt{2}})$

$X^2=4AY$ where $X=(x+\frac{1}{2\sqrt{2}}), A=\frac{1}{4\sqrt{2}}\ \&\ Y=(y+\cdot\frac{5}{4\sqrt{2}})$

That is the standard form of Parabola.

H.W

What are represented by the following equations?

(1) $x^2+xy+y^2+x+y=0$. (2) $6x^2-5xy-6y^2=0$; (3) $6x^2-5xy-6y^2=0$

(4) $4x^2-15xy-4y^2-46x+14y+60=0$; (5) $x^2-5xy+y^2+8x-20y+15=0$;

(A) If it will be a straight line then find,

(i) The angle in between the lines ; (ii) find the equation of each line.

(B) If it will be a curve then find,

(i) Find a rotation angle by which the xy-term will be eliminated; (ii) find the standard form.

◙ Equation in 3 variables:

- **1st degree general equation :** $ax + by + cz + d = 0$

  It always represents a plane.

- **The 2nd degree general equation :**

  It is given by: $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0.$

  It represents a quadratic surface (in brief, quadrics).

  Some Surfaces in standard forms:

Equations	**Represents**
(a second order polynomial containing quadratic terms in x, y, and z) = constant	
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	an elliptic cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	an ellipsoid.
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	a hyperboloid of one sheet
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	a hyperboloid of two sheets.
(Equations containing two quadratic terms and one linear term)	
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$	an elliptic paraboloid.
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$	a hyperbolic paraboloid.
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	an elliptic cylinder
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$	a hyperbolic cylinder
$x^2+2kz=0$	a parabolic cylinder

COORDINATES TRANSFORMATION

■ **TWO DIMENSIONAL (2D):**

- Rectangular/Cartesian (x,y) system
- Polar (r, θ) system.

■ **THREE DIMENSIONAL (3D):**

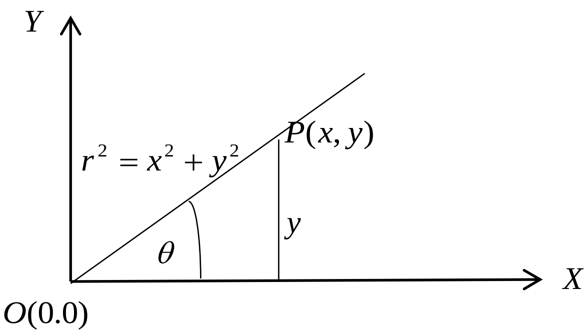
- Rectangular/Cartesian (x,y,z) system
- Cylindrical (r,θ,h) system.
- Spherical (ρ, φ, θ) system. Θ is called the azimuthal angle, φ the zenith angle.

■ TWO DIMENSIONAL SYSTEM (2D):

Cartesian /Rectangular coordinate System:-

In the Cartesian coordinate system in 2D, the point in a space or in three dimensional systems be represented by the symbol (x, y) where x is the distance on X axis and y is the distance on Y axis of the point (x, y).

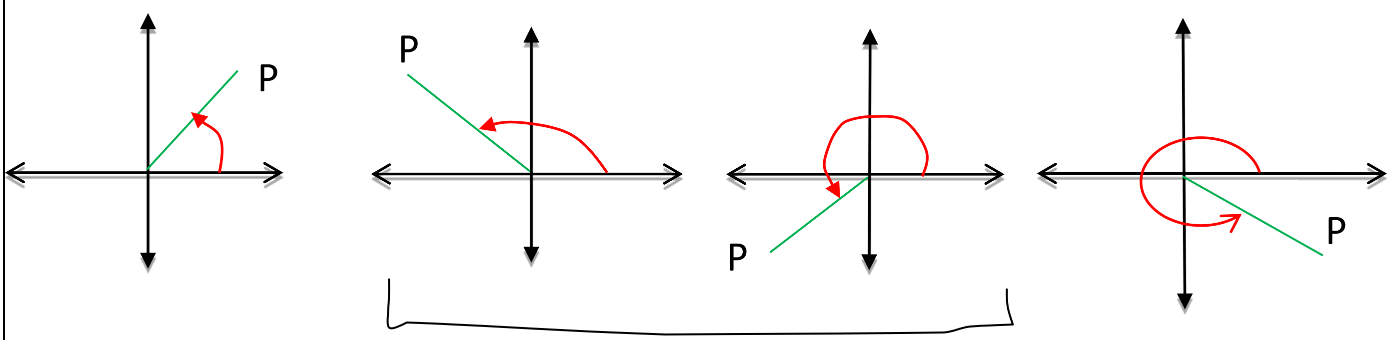
Figure:



■ Mutual relation (2D Cases)

$x = r \cos\theta$, $y = r \sin\theta$ **;** $r^2 = x^2 + y^2$,

There are 3 formulas to find θ for some given point P(x,y). These are:



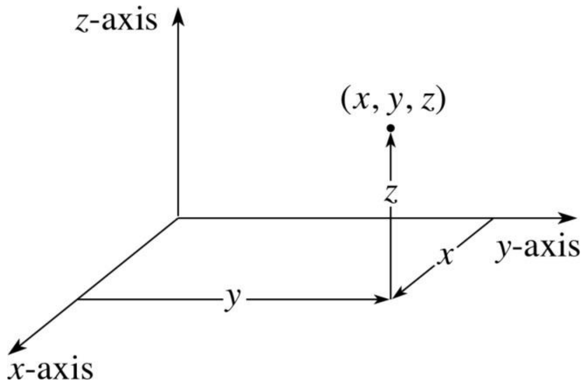
$\theta = \tan^{-1}(y/x)$ $\theta = \tan^{-1}(y/x) + 180^0$, $\theta = \tan^{-1}(y/x) + 360^0$

■ THREE DIMENSIONAL SYSTEM (3D):

❖ Cartesian /Rectangular coordinate System (RS):

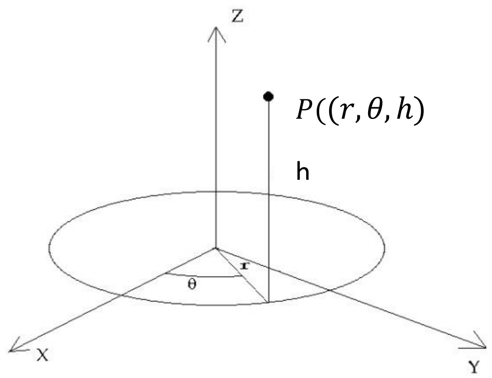
In the Cartesian coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (x, y, z) where x is the distance on x axis, y is the distance on y axis and z is the distance on z axis of the point (x, y, z).

Figure:



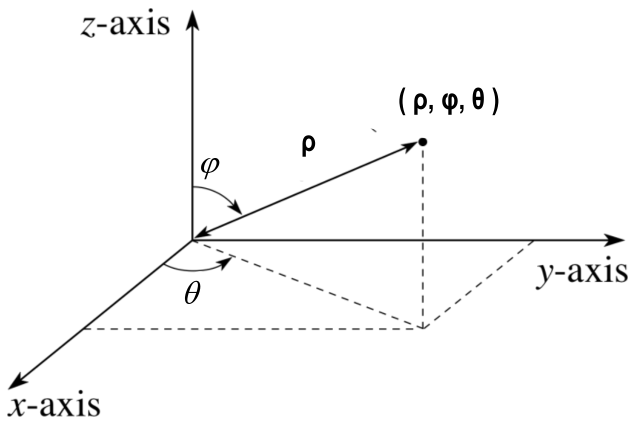
❖ Cylindrical coordinate System (CS):

In the Cylindrical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (r, θ, h) where r is the distance of the point from origin or length of radial line, θ is the angle between radial line and x axis and h is the distance of the point (r, θ, h) from the xy plane.



❖ Spherical coordinate system (SS):

In the Spherical coordinate system in 3D, the point in a space or in three dimensional systems be represented by the symbol (ρ, φ, θ) where ρ is the distance of the point from origin or length of radial line, φ is the angle between radial line and z axis and θ the angle between radial line (joining with the foot point of the perpendicular from the given point on the xy plane) and the x axis.



□□ Relation between Cartesian/Rectangular and Cylindrical System:

$CS \to RS$	$RS \to CS$
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\theta = \tan^{-1}(\frac{y}{x})$
$z = h$	$h = z$

□□ Relation between Cartesian/Rectangular and Spherical System:

$SS \to RS$	$RS \to SS$
$x = \rho\sin\varphi\cos\theta$	$\rho = \sqrt{x^2 + y^2 + z^2}$
$y = \rho\sin\varphi\sin\theta$	$\varphi = \cos^{-1}(\frac{z}{\sqrt{x^2 + y^2 + z^2}}) = \cos^{-1}(\frac{z}{\varphi})$
$z = \rho\cos\varphi$	$\theta = \tan^{-1}(\frac{y}{x})$

□□ Relation between Cylindrical and Spherical System:

$CS \to SS$	$SS \to CS$
$r = \rho\sin\varphi$	$\rho = \sqrt{r^2 + h^2}$
$\theta = \theta$	$\varphi = \tan^{-1}(\frac{r}{h})$
$h = \rho\cos\varphi$	$\theta = \theta$

Restriction:

$(x, y, z, h) \in (-\infty, \infty)$; $(\rho, r) \in [0, \infty)$;

$\theta \in [0, 360^0)$; $\varphi \in [0, 180^0]$;

Mathematical Problems

Problem no: 01

Convert $\left(3,\frac{\pi}{3},-4\right)$ from Cylindrical to Cartesian Coordinates.

Solution:

Given that,

Cylindrical coordinates of a point is $(r,\theta,h)=\left(3,\frac{\pi}{3},-4\right)$

We know that,

$$x=r\cos\theta, y=r\sin\theta, z=h$$

Now, $x=r\cos\theta=3\cos\frac{\pi}{3}=3\times\frac{1}{2}=\frac{3}{2}$

$$y=3\sin\frac{\pi}{3}=3\times\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$$

$$z=h=-4$$

Therefore the Cartesian coordinates of the given point is $(x\ ,\ y,\ z)=\left(\frac{3}{2},\frac{3\sqrt{3}}{3},-4\right)$.

H.W:

Convert the followings cylindrical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3},\frac{\pi}{4},-4\right)$ **2.** $\left(4\sqrt{3},0,\ 5\right)$ **3.** $(\sqrt{5},55^0,-3)$ 4. $(3,70^0,2)$

Problem no: 02

Convert $(-2,2,3)$ from Cartesian to Cylindrical Coordinates.

Solution:

Given that,

Cartesian coordinates of a point is $(x\ ,\ y,\ z)=(-2,2,3)$

We know that,

$$r=\sqrt{x^2+y^2}$$

$$\theta=\tan^{-1}(\frac{y}{x})$$

$$h=z$$

Now,

$$r=\sqrt{x^2+y^2}=\sqrt{(-2)^2+2^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)\Rightarrow\tan\theta=\frac{y}{x}\Rightarrow\tan\theta=\frac{2}{-2}=-1$$

Here

$$\tan\theta = -1$$

$$\tan\theta = -\tan\frac{\pi}{4}$$

$$\tan\theta = \tan(\pi - \frac{\pi}{4})$$

$$\tan\theta = \tan(\frac{3\pi}{4})$$

$$\theta = \frac{3\pi}{4}$$

And $h = z = 3$

Therefore the Cylindrical coordinates of the given point is $(r\ ,\ \theta,\ z) = \left(2\sqrt{2}, \frac{3\pi}{4}, 3\right)$.

H.W:

Convert the followings Cartesian coordinates to the Cylindrical Coordinates system:

1. $\left(4\sqrt{3}, 4, -4\right)$ 2. $\left(-\sqrt{3}, -4, 4\right)$ 3. $\left(-\sqrt{3}, 4, 2\right)$ 4. $\left(4\sqrt{2}, -1, -4\right)$
5. $\left(4\sqrt{3}, 0, 5\right)$ 6. $\left(0, 4, 9\right)$

Problem no: 03

Convert $\left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$ from Spherical to Cartesian Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho, \varphi, \theta) = \left(8, \frac{\pi}{4}, \frac{\pi}{6}\right)$

We know that,

$$x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$$

Now,

$$x = \rho\sin\varphi\cos\theta = 8\sin\frac{\pi}{4}\cos\frac{\pi}{6} = 8\times\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}$$

$$y = \rho\sin\varphi\sin\theta = 8\sin\frac{\pi}{4}\sin\frac{\pi}{6} = 8\times\frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

And

$$z = \rho\cos\varphi = 8\cos\frac{\pi}{4} = 8\times\frac{1}{\sqrt{2}} = 4\sqrt{2}$$

Therefore the Cartesian coordinates of the given point is $(x\ ,\ y,\ z) = \left(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2}\right)$

H.W:

Convert the followings Spherical coordinates to the Cartesian Coordinates system:

1. $\left(4\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ 2. $\left(-\sqrt{3}, 124^o, 75^o\right)$ 3. $\left(\sqrt{3}, 140^o, 140^o\right)$

Problem no: 04

Convert $\left(2\sqrt{3},6,-4\right)$ from Cartesian to Spherical Coordinates.

Solution:

Given that,

Cartesian coordinates of a point is $(x\ ,\ y,\ z)=\left(2\sqrt{3},6,-4\right)$

We know that,

$$\rho=\sqrt{x^2+y^2+z^2}$$

$$\varphi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\cos^{-1}\left(\frac{z}{\rho}\right)\qquad Or,\varphi=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

Now ,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{(2\sqrt{3})^2+6^2+(-4)^2}=\sqrt{12+36+16}=\sqrt{64}=8$$

$$\varphi=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\Rightarrow\tan\varphi=\frac{\sqrt{x^2+y^2}}{z}\qquad=\frac{\sqrt{(2\sqrt{3})^2+6^2}}{-4}$$

$$=\frac{\sqrt{12+36}}{-4}=\frac{\sqrt{48}}{-4}=\frac{4\sqrt{3}}{-4}=-\sqrt{3}$$

Here

$$\Rightarrow\ \tan\varphi=-\sqrt{3}$$

$$\tan\varphi=-\tan\frac{\pi}{3}$$

$$\tan\varphi=\tan\left(\pi-\frac{\pi}{3}\right)$$

$$\tan\varphi=\tan\left(\frac{2\pi}{3}\right)$$

$$\varphi=\frac{2\pi}{3}$$

And $\qquad\theta=\tan^{-1}\left(\frac{y}{x}\right)\Rightarrow\ tan\theta=\frac{y}{x}\qquad=\frac{6}{2\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}$

Here

$$tan\theta=\sqrt{3}$$

$$tan\theta=\tan\frac{\pi}{3}$$

$$\theta=\frac{\pi}{3}$$

Therefore the Spherical coordinates of the given point is $(\rho,\varphi,\theta)=\left(8,\frac{2\pi}{3},\frac{\pi}{3}\right)$.

H.W:

Convert the followings Cartesian coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3},4,-4\right)$ 2. $\left(-\sqrt{3},-4,4\right)$ 3. $\left(-\sqrt{3},4,2\right)$ 4. $\left(4\sqrt{2},-1,-4\right)$
5. $\left(\sqrt{3},0,0\right)$ 6. $\left(0,4,9\right)$ 7. $\left(4\sqrt{3},0,5\right)$

Problem no: 05

Convert $\left(1\,,\,\frac{\pi}{2}\,,1\right)$ from Cylindrical to Spherical Coordinates.

Solution:

Given that,

Cylindrical coordinates of a point is $(r,\theta,h)=\left(1\,,\,\frac{\pi}{2}\,,1\right)$

We know that,

$$\rho=\sqrt{r^2+h^2}$$

$$\varphi=\tan^{-1}(\frac{r}{h})$$

$$\theta=\theta$$

Now,

$$\rho=\sqrt{r^2+h^2}\quad=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$$

$$\varphi=\tan^{-1}(\frac{r}{h})\quad\Rightarrow\tan\varphi\quad=\frac{r}{z}=\frac{1}{1}=1$$

Here

$$\tan\varphi\quad=1$$

$$\tan\varphi\quad=\tan\frac{\pi}{4}$$

$$\varphi\quad=\frac{\pi}{4}$$

And

$$\theta=\theta=\frac{\pi}{2}$$

Therefore the Spherical coordinates of the given point is $(\rho,\varphi,\theta)=(\sqrt{2},\frac{\pi}{4},\frac{\pi}{2})$.

H.W:

Convert the followings Cylindrical coordinates to the Spherical Coordinates system:

1. $\left(4\sqrt{3},42^o,-4\right)$ 2. $\left(4\sqrt{3},0^o,5\right)$ 3. $\left(-\sqrt{3},134^o,-4\right)$

Problem no: 06

Convert $\left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$ from Spherical to Cylindrical Coordinates.

Solution:

Given that,

Spherical coordinates of a point is $(\rho,\varphi,\theta)=\left(4\sqrt{3}\ ,\ \frac{\pi}{4}\ ,\ \frac{\pi}{4}\right)$

We know that,

$$r=\rho\sin\varphi$$
$$\theta=\theta$$
$$h=\rho\cos\varphi$$

Now,

$$r=\rho\sin\varphi \quad =4\sqrt{3}\sin\frac{\pi}{4}=4\sqrt{3}\times\frac{1}{\sqrt{2}}=2\sqrt{6}$$

$$\theta=\theta \quad =\frac{\pi}{4}$$

And $$h=\rho\cos\varphi=4\sqrt{3}\times\cos\frac{\pi}{4}=4\sqrt{3}\times\frac{1}{\sqrt{2}}=2\sqrt{6}$$

Therefore the Cylindrical coordinates of the given point is $(r,\theta,h)=\left(2\sqrt{6},\frac{\pi}{4},2\sqrt{6}\right)$.

H.W:

Convert the followings Spherical coordinates to the Cylindrical Coordinates system:

1. $(\sqrt{5},\frac{\pi}{4},\frac{\pi}{3})$ 2. $(-\sqrt{5},140^0,75^0)$ 3. $(-3,40^0,15^0)$

Example:- Convert each of (i) $(8,-3,-7)$ (ii) $(5,120^0,330^0)$ to the other two system.

Solution:-

(i)

Hear the given coordinate system is $(8,-3,-7)$ which is RS, so we need to convert it to CS and SS.

In CS:

$$r=\sqrt{x^2+y^2}=\sqrt{8^2+(-3)^2}=8.54$$
$$\theta=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{-3}{8})=-\tan^{-1}(\frac{3}{8})=360^0-\tan^{-1}(\frac{3}{8})=339^0 26' 38''$$
$$h=z=-7$$

Hence the point is, $(r,\theta,h)=(8.54,339^0 26' 38'',-7)$

In SS:

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{8^2+(-3)^2+(-7)^2}=11.045$$

$$\varphi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\cos^{-1}\left(\frac{z}{\rho}\right)=\cos^{-1}\left(\frac{-7}{11.045}\right)=129^0 18' 27''$$

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-3}{8}\right)=-\tan^{-1}\left(\frac{3}{8}\right)=360^0-\tan^{-1}\left(\frac{3}{8}\right)=339^0 26' 38''$$

Hence the point is, (ρ,φ,θ) =$(11.045, 129^0 18' 27'', 339^0 26' 38'')$.

(ii)

Hear the given coordinate system is $(5, 120^0, 330^0)$ which is SS, so we need to convert it to CS and RS.

In RS:

$$x=\rho\sin\varphi\cos\theta=5\sin 120^0\cos 330^0=3.75$$

$$y=\rho\sin\varphi\sin\theta=5\sin 120^0\sin 330^0=-2.17$$

$$z=\rho\cos\varphi=5\cos 120^0=-2.5$$

Hence the point is, (x,y,z) =$(3.75, -2.17, -2.5)$.

In CS:

$$r=\rho\sin\varphi=5\sin 120^0=4.43$$

$$\theta=\theta=330^0$$

$$h=z=-2.5$$

Hence the point is, (r,θ,h) = $(4.43, 330^0, -2.5)$.

H.W:

Convert each of from below to other 2 systems.

(i) (4√3, 4, -4) ; (ii) (-√3, 4, 2) ; (iii) (-√3, -4, 4);

(iv) $(5, 120^0, 330^0)$; (v) (4√2, -1, -4); (vi)(0, 4, 9) ;

(vii) (√3, 0, 0) ; (viii) (4√3, 42^0, -4) ; (ix) (4√3, 0^0, 5) ;

(x) (-√3, 134^0, -4); (xi) (4√3, 45^0,45^0) ; (xii)(-√3, 124^0,75^0) ;

(xiii) (√3, 140^0,140^0) ;

Transformation of Equations

Mathematical problems

Problem 01:- Express Cartesian Equation $x^2 - y^2 = 25$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 - y^2 = 25$

We have

$$x = r\cos\theta \ , \ y = r\sin\theta \ and \ z = h$$

Replacing x and y from the given equation we get desired cylindrical equation as follows,

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 25$$

$$\Rightarrow \quad r^2\cos^2\theta - r^2\sin^2\theta = 25$$

$$\Rightarrow \quad r^2\left(\cos^2\theta - \sin^2\theta\right) = 25$$

$$\Rightarrow \quad r^2\cos(2\theta) = 25$$

$$r^2 = 25Sec(2\theta)$$

This is the required cylindrical equation.

Problem 02:- Express Cartesian Equation $x^2 + y^2 + z^2 = 0$ in Cylindrical Equation.

Solution:

Given Cartesian Equation is $x^2 + y^2 + z^2 = 0$

We have,

$$x = r\cos\theta \ , \ y = r\sin\theta \ and \ z = h$$

Replacing x, y and z from the given equation we get desired Cylindrical equation as follows,

$$(r\cos\theta)^2 + (r\sin\theta)^2 + z^2 = 0$$

$$\Rightarrow \quad r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 0$$

$$\Rightarrow \quad r^2\left(\cos^2\theta + \sin^2\theta\right) + z^2 = 0$$

$$r^2 + z^2 = 0$$

This is the required cylindrical equation.

H.W:

Transform the following Cartesian equations into the Cylindrical Equations:

1. $x^2 - y^2 + 2z^2 = 3x$
2. $x^2 + y^2 + z^2 = 2z$
3. $z^2 = y^2 - x^2$
4. $x + y + z = 1$

Problem 03:- Transform Cartesian Equation $x^2 + y^2 - z^2 = 1$ to Spherical Equation.

Solution:

Given Cartesian Equation is $x^2 + y^2 - z^2 = 1$

We have,

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

Replacing x, y and z from the given equation we get desired Cylindrical equation as follows,

$$(\rho \sin\varphi \cos\theta)^2 + (\rho \sin\varphi \sin\theta)^2 - (\rho \cos\varphi)^2 = 1$$
$$\Rightarrow \quad \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta - \rho^2 \cos^2\varphi = 1$$
$$\Rightarrow \quad \rho^2 \sin^2\varphi(\cos^2\theta + \sin^2\theta) - \rho^2 \cos^2\varphi = 1$$
$$\Rightarrow \quad \rho^2 \sin^2\varphi - \rho^2 \cos^2\varphi = 1$$
$$\Rightarrow \quad \rho^2 (\sin^2\varphi - \cos^2\varphi) = 1$$
$$\Rightarrow \quad -\rho^2 (\cos^2\varphi - \sin^2\varphi) = 1$$
$$\Rightarrow \quad -\rho^2 \cos(2\varphi) = 1$$
$$\rho^2 = -\sec(2\varphi)$$

This is the required Spherical Equation.

H.W:

Transform the following Cartesian equations into the Spherical Equations:

1. $x^2 - y^2 + 2z^2 = 3x$ 2. $x^2 + y^2 + z^2 = 2z$ 3. $z^2 = y^2 - x^2$ 4. $x + y + z = 1$

Problem 04:- Transform Spherical Equation $\rho = 2\cos\varphi$ to Cylindrical Equation.

Solution:

Given Spherical Equation is $\rho = 2\cos\varphi$

We have,

$$\rho = \sqrt{r^2 + h^2}, \varphi = \tan^{-1}(\frac{r}{h}), \theta = \theta$$

Replacing ρ and φ from the given equation we get desired Cylindrical equation as follows,

$$\sqrt{r^2 + h^2} = 2\cos\varphi$$
$$\Rightarrow \quad \sqrt{r^2 + h^2} = 2 \times \frac{h}{\rho} \qquad [\because h = \rho\cos\varphi]$$
$$\Rightarrow \quad \sqrt{r^2 + h^2} = 2 \times \frac{h}{\sqrt{r^2 + h^2}}$$
$$\therefore r^2 + h^2 = 2h$$

This is the required Cylindrical equation.

H.W:

Transform the following Spherical Equations into the Cylindrical Equations:

1. $\varphi = \frac{\pi}{4}$ 2. $\rho = 2\sec\theta$ 3. $\rho = \cos ec\theta$

Problem 05:- Transform Cylindrical Equation $r^2 \cos 2\theta = h$ to Cartesian/Rectangular Equation.

Solution:

Given Cylindrical Equation is $r^2 \cos 2\theta = h$

We have,

$$x = r\cos\theta \text{ , } y = r\sin\theta \text{ and } z = h$$

Given equation is

$$r^2 \cos 2\theta = h$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = z$$

$$\Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = z$$

$$\Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = z$$

$$\therefore (x)^2 - (y)^2 = z \quad \text{[Putting values]}$$

This is the required Cartesian/Rectangular Equation.

H.W:

Transform the following Cylindrical Equations into the Cartesian/Rectangular Equations:

1. $r = 2\sin\theta$ **2.** $z = 5\sin\theta$

Problem 05:- Transform Spherical Equation $\rho\sin\varphi = 1$ to Cartesian/Rectangular Equation.

Solution:

Given Spherical Equation is $\rho\sin\varphi = 1$

We have, $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$ and $\rho = \sqrt{x^2 + y^2 + z^2}$

Now,

$$\rho\sin\varphi = 1$$

$$\Rightarrow \rho^2\sin^2\varphi = 1$$

$$\Rightarrow \rho^2(1 - \cos^2\varphi) = 1$$

$$\Rightarrow \rho^2 - (\rho\cos\varphi)^2 = 1$$

$$\Rightarrow x^2 + y^2 + z^2 - z^2 = 1$$

$$\therefore x^2 + y^2 = 1$$

H.W:

Transform the following Spherical Equations into the Cartesian/Rectangular Equations:

1. $\rho\sin\varphi = 1$ 2. $\rho = 2\sec\varphi$ 3. $\rho = \csc\varphi$ 4. $\rho\sin\varphi = 2\cos\theta$