05 Partial Fractions

Rational Fraction: If $P(x)$ & $Q(x)$ are two polynomials in x and $Q(x) \neq 0$ then the quotient $\dfrac{P(x)}{Q(x)}$ is called a rational fraction.

Example: $\dfrac{x^2+1}{x^3-2x+3}$ is a rational fraction.

Proper Fraction: A fraction in which the degree of the numerator is less than the degree of denominator is called a proper fraction.

Example: $\dfrac{x^2+1}{x^3-2x+3}$ is a proper fraction.

Improper Fraction: A fraction in which the degree of the numerator is greater or equal to the degree of denominator is called an improper fraction.

Example: $\dfrac{x^2+1}{x^2-2x+3}$ & $\dfrac{x^3+1}{x^2-2x+3}$ are improper fractions.

Partial Fraction: A given fraction may be written as a sum of other fractions (called partial fractions) whose denominator is less than the denominator of the given fraction.

Fundamental theorem: Any fraction may be written as the sum of partial fractions according the following rules:

Case-1: When the fraction is **Proper fraction**:

a. When all factors are linear and different
   i.e.,

$$\frac{f(x)}{(x\pm a)(x\pm b)} = \frac{(?)}{x\pm a} + \frac{(?)}{x\pm a} \cdots\cdots (1)$$

where the coefficients of the blank spaces cannot be zero.

NOTE: Using the **Cover up method** we can find the values of the blank spaces of (1).

Cover up method: This method is applicable only for linear factors.

If $\dfrac{f(x)}{(x-a)(x-b)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}$ then

For A: Cover $(x-a)$ term in the denominator of the left-hand side and substitute $x=a$ in the remaining expression.

For B: Cover $(x-b)$ term in the denominator of the left-hand side and substitute $x=b$ in the remaining expression.

b. When all factors are linear and some are repeated
i.e.,

$$\frac{f(x)}{(x\pm a)(x\pm b)^n}=\frac{(?)}{(x\pm a)}+\frac{(?)}{(x\pm b)^n}+\frac{A}{(x\pm b)^{n-1}}+\cdots\cdots+\frac{B}{(x\pm b)}\quad\cdots\cdots(2)$$

NOTE: Find the coefficients of the blank spaces by using **Cover up method** and then to find A substitute any value for x except $x=\pm a$ & $x=\pm b$.

c. When all factors are quadratic and different
i.e.,

$$\frac{f(x)}{(x^2\pm a)(x^2\pm b)}=\frac{Ax+B}{x^2\pm a}+\frac{Cx+D}{x^2\pm b}\quad\cdots\cdots(3)$$

NOTE: To find the values of A, B, C & D multiplying both sides of (3) by $(x^2\pm a)(x^2\pm b)$ and then substitute the appropriate values for x.

d. When all factors are quadratic and some are repeated
i.e.,

$$\frac{f(x)}{(x^2\pm a)(x^2\pm b)^2}=\frac{Ax+B}{(x^2\pm a)}+\frac{Cx+D}{(x^2\pm b)^2}+\frac{C_1x+D_1}{(x^2\pm b)}\quad\cdots\cdots(4)$$

NOTE: To find the values of A, B, C, D, C_1 & D_1 multiplying both sides of (4) by $(x^2\pm a)(x^2\pm b)^2$ and then substitute the appropriate value for x.

Case-2: When the fraction is **improper fraction**: To split an improper fraction into a partial fraction, we will have to divide the numerator by denominator.

Example: if $\frac{3x^2-2x-2}{x^2-3x+2}$ then

$$x^2-3x+2\left|\begin{array}{l}3x^2-3x-2\\ \underline{3x^2-9x+6}\end{array}\right|3$$
$$6x-8$$

$$Since,\ Dividend = \left(Divisor \times Quotient\right) + \text{Re}\,mainder$$

Rewriting the given improper fraction we get

$$\frac{3x^2-2x-2}{x^2-3x+2} = 3+\frac{6x-8}{x^2-3x+2}$$

Now using the Cover up method anyone can solve the fraction.

Problem-1: Separate $\frac{5x-11}{2x^2+x-6}$ into partial fractions.

Solution: We have $\frac{5x-11}{2x^2+x-6}$

$$=\frac{5x-11}{2x^2+4x-3x-6}$$

$$=\frac{5x-11}{2x(x+2)-3(x+2)}$$

$$=\frac{5x-11}{(x+2)(2x-3)}$$

$$=\frac{3}{x+2}+\frac{-1}{2x-3}$$

Problem-2: Separate $\frac{3x^2+x-2}{(x-2)^2(1-2x)}$ into partial fractions.

Solution: We have $\frac{3x^2+x-2}{(x-2)^2(1-2x)}$

$$=\frac{-4}{(x-2)^2}+\frac{-\frac{1}{3}}{(1-2x)}+\frac{A}{(x-2)} \quad \cdots\cdots (1)$$

Putting $x=0$ in (1) we get,

$$\frac{3(0)^2+0-2}{(0-2)^2(1-2\times 0)}=\frac{-4}{(0-2)^2}+\frac{-\frac{1}{3}}{(1-2\times 0)}+\frac{A}{(0-2)}$$

$$or,\ \frac{-2}{4}=\frac{-4}{4}+\frac{-\frac{1}{3}}{1}+\frac{A}{-2}$$

$$or,\ -\frac{1}{2}=-1-\frac{1}{3}-\frac{A}{2}$$

$$or,\ \frac{A}{2}=-1-\frac{1}{3}+\frac{1}{2}$$

$$or,\ \frac{A}{2}=\frac{-6-2+3}{6}$$

$$or,\ A=-\frac{5}{3}$$

From (1) we get,

$$\frac{3x^2+x-2}{(x-2)^2(1-2x)}=\frac{-4}{(x-2)^2}+\frac{-\frac{1}{3}}{(1-2x)}+\frac{-\frac{5}{3}}{(x-2)}$$

$$=-\frac{4}{(x-2)^2}-\frac{1}{3}\cdot\frac{1}{(1-2x)}-\frac{5}{3}\cdot\frac{1}{(x-2)}$$

Problem-3: Separate $\frac{7+x}{(1+x)(1+x^2)}$ into partial fractions.

Solution: We have $\frac{7+x}{(1+x)(1+x^2)}$

$$=\frac{3}{(1+x)}+\frac{Ax+B}{(1+x^2)}\ \cdots\cdots\ (1)$$

Putting $x=0$ in (1) we get,

$$\frac{7+0}{(1+0)(1+0)}=\frac{3}{(1+0)}+\frac{A(0)+B}{(1+0)}$$

$$or,\ 7=3+B$$

$$or,\ B=4$$

Again putting $x=1$ in (1) we get,

$$\frac{7+1}{(1+1)(1+1)}=\frac{3}{(1+1)}+\frac{A(1)+B}{(1+1)}$$

$$or,\ \frac{8}{2\times2}=\frac{3}{2}+\frac{A+4}{2}$$

$$or,\ 2=\frac{3}{2}+\frac{A+4}{2}$$

$$or,\ \frac{A+4}{2}=2-\frac{3}{2}$$

$$or,\ \frac{A+4}{2}=\frac{1}{2}$$

$or,\ A+4=1$

$or,\ A=-3$

From (1) we get,

$$\frac{7+x}{(1+x)\left(1+x^2\right)}=\frac{3}{(1+x)}+\frac{-3x+4}{\left(1+x^2\right)}$$

$$=\frac{3}{(1+x)}-\frac{3x-4}{\left(1+x^2\right)}$$

Problem-4: Separate $\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}$ into partial fractions.

Solution: We have $\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}$

$$=\frac{Ax+B}{\left(x^2+5\right)}+\frac{Cx+D}{\left(x^2-3\right)} \cdots\cdots (1)$$

Multiplying both sides of (1) by $\left(x^2+5\right)\left(x^2-3\right)$ we get,

$x+1=(Ax+B)\left(x^2-3\right)+(Cx+D)\left(x^2+5\right)$

$or,\ x+1=Ax^3-3Ax+Bx^2-3B+Cx^3+5Cx+Dx^2+5D$

$or,\ x+1=(A+C)x^3+(B+D)x^2+(5C-3A)x-3B+5D$

Equating the coefficients of like term we get,

$A+C=0\ ;B+D=0\ ;\ 5C-3A=1\ ;\ -3B+5D=1$

$A=-C\ ;B=-D\ ;\ 5C-3A=1\ ;\ -3B+5D=1$

Since $A=-C$ so $5C-3A=1\Rightarrow 5C-3(-C)=1$

$or,\ 5C+3C=1$

$or,\ 8C=1$

$or,\ C=\frac{1}{8}$ and $A=-\frac{1}{8}$

Again $B=-D$ so $-3B+5D=1\Rightarrow -3(-D)+5D=1$

$or,\ 3D+5D=1$

$or,\ 8D=1$

$or,\ D=\frac{1}{8}$ and $B=-\frac{1}{8}$

From (1) we get,

$$\frac{x+1}{\left(x^2+5\right)\left(x^2-3\right)}=\frac{-\frac{1}{8}x-\frac{1}{8}}{\left(x^2+5\right)}+\frac{\frac{1}{8}x+\frac{1}{8}}{\left(x^2-3\right)}$$

$$=\frac{1}{8}\cdot\frac{x+1}{\left(x^2-3\right)}-\frac{1}{8}\cdot\frac{x+1}{\left(x^2+5\right)}$$

Problem-5: Separate $\frac{2x^2+x+1}{x^2+2x-3}$ into partial fractions.

Solution: We have $\frac{2x^2+x+1}{x^2+2x-3}$

$$=2+\frac{7-3x}{x^2+2x-3}$$

$$=2+\frac{7-3x}{x^2+3x-x-3}$$

$$=2+\frac{7-3x}{x(x+3)-1(x+3)}$$

$$=2+\frac{7-3x}{(x+3)(x-1)}$$

$$=2+\frac{1}{x-1}+\frac{-4}{x+3}$$

$$=2+\frac{1}{x-1}-\frac{4}{x+3}$$

Exercise:

1. Resolve $\frac{x+2}{(x-1)(x+3)}$ into partial fractions.
2. Resolve $\frac{1}{(x+2)(x+1)}$ into partial fractions.
3. Resolve $\frac{x}{(x-2)(x+1)^2}$ into partial fractions.
4. Resolve $\frac{42-19x}{\left(x^2+1\right)(x-4)}$ into partial fractions.
5. Find the decomposition of $\frac{1}{\left(x^2+5\right)\left(x^2-3\right)}$.
6. Resolve $\frac{x^2+5x-7}{x^2-x-2}$ into partial fractions.
7. Resolve $\frac{6x^3+5x^2-7}{3x^2-2x-1}$ into partial fractions.
8. Resolve $\frac{x^4+5x^3-7}{x^2+5x+6}$ into partial fractions.