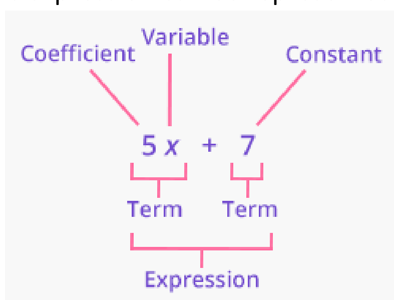
06 Polynomial Equations

Expression:
An *expression* is a finite combination of mathematical symbols that is well-formed according to the rules that depend on the context.
For example: An algebraic expression can be represented as:



Fun Facts:

- An expression does not contain equal to sign or any inequalities signs.
- When we add inequality or equality sign to an expression, it becomes an equation.
- Both sides of an equation are an expression.
- In expression power of the variable is any number.

Polynomial:
A polynomial is an expression consisting of variables (also called indeterminates) and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents of variables. For example: A polynomial of a single indeterminate, x, is $x^2 - 4x + 7$.
Zeros:
Zeros are the values of the variables that vanishes the expression or polynomial.
For example: 1 & 3 are the zeroes of the polynomial $x^2 - 4x + 3$.
Equation & Identity:
Equation is a mathematical statement that the values of two expressions are equal and indicated by the sign =. Identity is also an equation but it number of roots are more than its degree. For example, the equality of two expression $x^2 = 4x - 3$ is called an equation. On the other hand, the equality of two expression $x^2 - x = x(x-1)$ is called Identity due to it has more roots from its degree.
Roots/solutions of an equation:
The roots /solutions of an equation are the values of the variables that satisfies the equation or Identities. For example, the equation $x^2 - 4x + 3 = 0$ has two roots as 1 and 3. But the identity $x^2 - x = x(x-1)$ has infinitely many roots.
Remainder theorem:
It states that the remainder of the division of a polynomial $f(x)$ by a linear polynomial $x - r$ is equal to $f(r)$. For example: For the polynomial $f(x) = x^2 + 5x - 6$, the division of the polynomial $f(x)$ by $(x-3)$ yields 18 ,so $f(3) = 18$ (Remainder).

Factor theorem:
The factor theorem states that a polynomial $f(x)$ has a factor $(x-k)$ if and only if $f(k)=0$ where k is the root of the polynomial. For example, the polynomial x^2-4x+3 has a factor $(x-1)$ for account of $f(1)=0$ if we say $f(x)=x^2-4x+3$.

Quadratic Equations:
An equation of the form $ax^2+bx+c=0, a\neq 0$ is called quadratic equation because quadratic comes from Latin *quadratus* which mean "square" . The constants a, b & c are called the coefficients of the equation and may be distinguished by calling them, respectively, the quadratic coefficient, the linear coefficient and the constant or free term.

Solution of the quadratic Equation:
General quadratic equation is $ax^2+bx+c=0, a\neq 0$
Multiplying the above equation by $4a$ we get,

$$4a.ax^2+4a.bx+4a.c=4a.0$$
$$4a^2x^2+4abx+4ac=0$$
$$(2ax)^2+2.(2ax)b+b^2-b^2+4ac=0$$
$$(2ax)^2+2.(2ax)b+b^2=b^2-4ac$$
$$(2ax+b)^2=b^2-4ac$$
$$(2ax+b)=\pm\sqrt{b^2-4ac}$$
$$2ax=-b\pm\sqrt{b^2-4ac}$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ (Here a must be positive)

Discriminant: Discriminant is a function of the coefficients of a polynomial equation whose value gives information about the roots of the polynomial.
Here discriminant, $D=b^2-4ac$
Nature of the roots are:

Cubic Equation:
Rene de cartes sign rules

Mathematical problems

1. Solve the equation $x^2+5x+6=0$ by using factoring Method
   Solution:
   Factorization Method:
   We have, $x^2+5x+6=0$
   $$x^2+3x+2x+6=0$$
   $$x(x+3)+2(x+3)=0$$
   $$(x+2)(x+3)=0$$
   Therefore $x+2=0$ or $x+3=0$
   So $x=-2$ or $x=-3$
   2nd Method:
   We have, $x^2+5x+6=0$

$$x=\frac{-5\pm\sqrt{25-4.1.6}}{2.1}=\frac{-5\pm\sqrt{25-24}}{2}=\frac{-5\pm\sqrt{1}}{2}=\frac{-5\pm1}{2}$$

Taking(+ve) we get $x=-2$ and taking (-ve) $x=-3$. (Ans)

2. Solve the equation $x^3-3x^2+3x-1=0$ using Remainder theorem.

Solution: Given equation is $x^3-3x^2+3x-1=0$.

Let $f(x)=x^3-3x^2+3x-1$. For $x=1$, $f(1)=1^3-3.1^2+3.1-1=0$, so one factor of f(x) is (x-1).

Now,

$$x^3-3x^2+3x-1=0$$
$$x^2(x-1)-2x(x-1)+1(x-1)=0$$
$$(x-1)\left(x^2-2x+1\right)=0$$
$$(x-1)(x-1)^2=0$$
$$(x-1)(x-1)(x-1)=0$$

Therefore, $x-1=0 \quad or \quad x-1=0 \quad or \quad x-1=0$

$x=1 \quad or \quad x=1 \quad or \quad x=1$ (Ans)

3. Solve the equation $4x^3-24x^2+23x+18=0$ having that the roots are in arithmetical progression.

Solution: We have, $4x^3-24x^2+23x+18=0$

In accordance with the question, assume that the roots are $\alpha-\beta, \alpha \;\&\; \alpha+\beta$.

Now,

$$\alpha-\beta+\alpha+\alpha+\beta=-\frac{-24}{4}\Rightarrow 3\alpha=6\Rightarrow\alpha=2.$$

And $\alpha(\alpha-\beta)(\alpha+\beta)=-\frac{18}{4}\Rightarrow\alpha\left(\alpha^2-\beta^2\right)=-\frac{9}{2}\Rightarrow 2\left(4-\beta^2\right)=-\frac{9}{2}\Rightarrow\left(4-\beta^2\right)=-\frac{9}{4}\Rightarrow\beta^2=4+\frac{9}{4}$

$$\Rightarrow\beta^2=4+\frac{9}{4}\Rightarrow\beta^2=\frac{16+9}{4}=\frac{25}{4}\Rightarrow\beta=\pm\frac{5}{2}$$

Therefore, the roots are $-\frac{1}{2}, 2 \;\&\; \frac{9}{2}$ (Ans)

4. Solve the equation $3x^3-26x^2+52x-24=0$ having that the roots are in geometrical progression.

Solution: We have, $3x^3-26x^2+52x-24=0$

In accordance with the question, assume that the roots are $\frac{\alpha}{r}, \alpha \;\&\; \alpha r$.

Now, $\frac{\alpha}{r}+\alpha+\alpha r=-\frac{-26}{3}\Rightarrow\frac{\alpha}{r}+\alpha+\alpha r=\frac{26}{3}\Rightarrow\alpha\left(\frac{1}{r}+1+r\right)=\frac{26}{3}$

And $\frac{\alpha}{r}.\alpha.\alpha r=-\frac{-24}{3}\Rightarrow\alpha.\alpha.\alpha=8\Rightarrow\alpha^3=8\Rightarrow\alpha=2.$

The value $\alpha=2$ implies $2\left(\frac{1}{r}+1+r\right)=\frac{26}{3}\Rightarrow\frac{1}{r}+1+r=\frac{13}{3}\Rightarrow\frac{1+r+r^2}{r}=\frac{13}{3}\Rightarrow 3+3r+3r^2=13r$

$$or \quad 3r^2-10r+3=0$$
$$or \quad 3r^2-9r-r+3=0$$
$$or \quad 3r(r-3)-1(r-3)=0$$
$$or \quad (r-3)(3r-1)=0$$
$$r-3=0 \quad or \quad 3r-1=0$$
$$r=3 \quad or \quad r=\frac{1}{3}$$

Therefore, the roots are $\frac{2}{3}, 2 \;\&\; 6$. (Ans)

5. Solve the equation $2x^3-x^2-22x-24=0$ having that the roots are in the ratio of 3:4.

Solution: Given that, $2x^3-x^2-22x-24=0$

In accordance with the question, assume that the roots are $3\alpha, 4\alpha \;\&\; \beta$.

Now, $3\alpha+4\alpha+\beta=-\frac{-1}{2} \Rightarrow 7\alpha+\beta=\frac{1}{2}$ ......(i)

And $3\alpha.4\alpha.\beta=-\frac{-24}{2} \Rightarrow 12\alpha^2\beta=12 \Rightarrow \alpha^2\beta=1 \Rightarrow \beta=\frac{1}{\alpha^2}$

From (i), we get $7\alpha+\frac{1}{\alpha^2}=\frac{1}{2} \Rightarrow \frac{7\alpha^3+1}{\alpha^2}=\frac{1}{2} \Rightarrow 14\alpha^3+2=\alpha^2 \Rightarrow 14\alpha^3-\alpha^2+2=0$

$\Rightarrow 14\alpha^3-\alpha^2+2=0$

Let $f(\alpha)=14\alpha^3-\alpha^2+2$,then $f\left(-\frac{1}{2}\right)=14\left(-\frac{1}{2}\right)^3-\left(-\frac{1}{2}\right)^2+2=-\frac{14}{8}-\frac{1}{4}+2=-\frac{7}{4}-\frac{1}{4}+2=-\left(\frac{7}{4}+\frac{1}{4}\right)+2=-2+2=0$.

So, $(2\alpha+1)$ is a one factor of $f(\alpha)$.

Therefore, $7\alpha^2(2\alpha+1)-4\alpha(2\alpha+1)+2(2\alpha+1)=0 \Rightarrow (2\alpha+1)\left(7\alpha^2-4\alpha+2\right)=0$

$\Rightarrow (2\alpha+1)\left(7\alpha^2-4\alpha+2\right)=0$

$\Rightarrow 2\alpha+1=0 \quad or \quad 7\alpha^2-4\alpha+2=0$

$\Rightarrow \alpha=-\frac{1}{2} \quad or \quad \alpha=\frac{-(-4)\pm\sqrt{(-4)^2-4.7.2}}{2.7}$

$\Rightarrow \alpha=-\frac{1}{2} \quad or \quad \alpha=\frac{4\pm\sqrt{16-84}}{14}==\frac{4\pm\sqrt{-68}}{14}(not\ real)$

From (i) $-\frac{7}{2}+\beta=\frac{1}{2} \Rightarrow \beta=\frac{1}{2}+\frac{7}{2}=4$

Therefore, the roots of the equation are $-\frac{3}{2}, -2 \,\&\, 4$.

6. Solve the equation $24x^3-14x^2-63x+45=0$ having that one root being double another.

Solution: Given equation is $24x^3-14x^2-63x+45=0$.

Let us consider the roots according to the question are $2\alpha, \alpha \,\&\, \beta$.

Now, $2\alpha+\alpha+\beta=-\frac{-14}{24} \Rightarrow 3\alpha+\beta=\frac{7}{12}$ ......(i)

$2\alpha^2+\alpha\beta+2\alpha\beta=\frac{-63}{24} \Rightarrow 2\alpha^2+3\alpha\beta=-\frac{21}{8}$......(ii)

And $2\alpha.\alpha.\beta=-\frac{45}{24} \Rightarrow \alpha^2\beta=-\frac{45}{48} \Rightarrow \beta=-\frac{15}{16\alpha^2}$ ....(iii)

From (i) & (ii), we get $2\alpha^2+3\alpha\left(\frac{7}{12}-3\alpha\right)=-\frac{21}{8} \Rightarrow 2\alpha^2+\left(\frac{7}{4}\alpha-9\alpha^2\right)=-\frac{21}{8}$

$\Rightarrow 2\alpha^2+\frac{7}{4}\alpha-9\alpha^2=-\frac{21}{8}$

$\Rightarrow \frac{7}{4}\alpha-7\alpha^2=-\frac{21}{8}$

$\Rightarrow 14\alpha-56\alpha^2=-21$

$\Rightarrow 2\alpha-8\alpha^2=-3$

$\Rightarrow 8\alpha^2-2\alpha-3=0$

$\therefore \alpha=\frac{-(-2)\pm\sqrt{(-2)^2-4.8.(-3)}}{2.8}=\frac{2\pm\sqrt{4+4.8.3}}{16}=\frac{2\pm\sqrt{4+96}}{16}=\frac{2\pm\sqrt{100}}{16}=\frac{2\pm10}{16}$

Taking (+ve), we get $\therefore \alpha=\frac{2+10}{16}=\frac{12}{16}=\frac{3}{4}$ and for (-ve) $\alpha=\frac{2-10}{16}=-\frac{8}{16}=-\frac{1}{2}$.

From equation (i), we get $\beta=\frac{7}{12}-3\alpha$

For $\alpha=\frac{3}{4}$, $\beta=\frac{7}{12}-3.\frac{3}{4}=\frac{7}{12}-\frac{9}{4}=\frac{7-27}{12}=-\frac{20}{12}=-\frac{5}{3}$.

For $\alpha=-\frac{1}{2}$, $\beta=\frac{7}{12}-3.\left(-\frac{1}{2}\right)=\frac{7}{12}+\frac{3}{2}=\frac{7+18}{12}=\frac{25}{12}$

But in the equation (iii) for $\alpha=\frac{3}{4}$, $\beta=-\frac{15}{16.\frac{9}{16}}=-\frac{15}{9}=-\frac{5}{3}$ and for $\alpha=-\frac{1}{2}$, $\beta=-\frac{15}{16.\frac{1}{4}}=-\frac{15}{4}$.

It is found that for $\alpha=-\frac{1}{2}$ the third equation is not satisfied, so the roots are $\frac{3}{4}$, $\frac{3}{2}$ & $-\frac{5}{3}$. (Ans.)

7. From an equation whose roots are 1, 2, 3 &4.

Solution: The roots of the equations are 1, 2, 3 &4.

Therefore, $(x-1)(x-2)(x-3)(x-4)=0$

$$\left(x^2-3x+2\right)\left(x^2-7x+12\right)=0$$

$$x^4-7x^3+12x^2-3x^3+21x^2-36x+2x^2-14x+24=0$$

$$x^4-10x^3+35x^2-50x+24=0 \quad \text{(Ans)}$$

8. Solve the equation $x^4-16x^3+86x^2-176x+105=0$ whose two roots being 1 & 7.

Solution: Given equation is $x^4-16x^3+86x^2-176x+105=0$.

Here two roots $x=1 \& x=7$.

So $(x-1)(x-7)=0 \Rightarrow x^2-8x+7=0$

Write the given equation with the help of $x^2-8x+7=0$,we get

$$x^2\left(x^2-8x+7\right)-8x\left(x^2-8x+7\right)+15\left(x^2-8x+7\right)=0$$

$$\left(x^2-8x+7\right)\left(x^2-8x+15\right)=0$$

There other two roots are in the quadratic equation $x^2-8x+15=0$

$$x^2-5x-3x+15=0$$

$$x(x-5)-3(x-5)=0$$

$$(x-5)(x-3)=0$$

Therefore $x-5=0 \quad or \quad x-3=0 \Rightarrow x=5 \quad or \quad x=3.$

Finally, the four roots of the given equation are 1,3,5 &7 (Ans.)

9. Solve the equation $6x^4-13x^3-35x^2-x+3=0$ whose one root being $2-\sqrt{3}$.

Solution: Given equation is $6x^4-13x^3-35x^2-x+3=0$.

According to the question $x=2-\sqrt{3}$.

Now, $x-2=-\sqrt{3}$

Squaring the above equation, we get $(x-2)^2=\left(-\sqrt{3}\right)^2 \Rightarrow x^2+4x+4=3 \Rightarrow x^2-4x+1=0.$

Write the given equation with the help of $x^2-4x+1=0$,we get

$$6x^2\left(x^2-4x+1\right)+11x\left(x^2-4x+1\right)+3\left(x^2-4x+1\right)=0$$

$$\left(x^2-4x+1\right)\left(6x^2+11x+3\right)=0$$

Therefore $x^2-4x+1=0 \quad or \quad 6x^2+11x+3=0$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4.1.1}}{2.1}=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm\sqrt{4.3}}{2}=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

$$or \quad x=\frac{-11\pm\sqrt{11^2-4.6.3}}{2.6}=\frac{-11\pm\sqrt{121-72}}{12}=\frac{-11\pm\sqrt{49}}{12}=\frac{-11\pm7}{12}$$

Taking(+ve) $x=\frac{-11+7}{12}=-\frac{4}{12}=-\frac{1}{3}$ and for (-ve) $x=\frac{-11-7}{12}=-\frac{18}{12}=-\frac{3}{2}$.

Finally, the four roots of the given equation are $2+\sqrt{3}$, $2+\sqrt{3}$, $-\frac{1}{3}$ & $\frac{3}{2}$.

10. How many real, positive, negative & imaginary roots of the equation $6x^4-13x^3-35x^2-x+3=0$ have.

Solution: Let $f(x)=6x^4-13x^3-35x^2-x+3$.

In the above function f(x) the sign of the terms are + - - - +. There are two change in the signs, so it has two positive roots of the given equation.

Replacing x by -x in f(x) , we get $f(-x)=6(-x)^4-13(-x)^3-35(-x)^2-(-x)+3=6(x)^4+13(x)^3-35(x)^2+(x)+3$.

In the above function f(-x) the sign of the terms are + + - + +. There are two change in the signs, so it has two negative roots of the given equation. It has no complex root since its degree is 4. (Ans.)

11. Solve the equation $x^2-6x+9=4\sqrt{x^2-6x+6}$.

Solution: Given that $x^2-6x+9=4\sqrt{x^2-6x+6}$.

Let $u=x^2-6x+9$ then the given equation reduces to $u=4\sqrt{u-3}$.

Squaring both sides, we get $u^2=16(u-3)=16u-48$

$$u^2-16u+48=0$$
$$u^2-12u-4u+48=0$$
$$u(u-12)-4(u-12)=0$$
$$(u-12)(u-4)=0$$

Therefore, $u-12=0 \quad or \quad u-4=0$

$x^2-6x+9-12=0 \quad or \quad x^2-6x+9-4=0$ [Putting value of u]

$$x^2-6x-3=0 \quad or \quad x^2-6x+5=0$$
$$x^2-6x-3=0 \quad or \quad x^2-6x+5=0$$

Therefore, $x=\frac{-(-6)\pm\sqrt{(-6)^2-4.1.(-3)}}{2.1}=\frac{6\pm\sqrt{36+12}}{2}=\frac{6\pm4\sqrt{3}}{2}=3\pm2\sqrt{3}$ and $x=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm\sqrt{16}}{2}=\frac{6\pm4}{2}$

Taking (+ve) & (-ve) we get $x=\frac{6+4}{2}=5$ & $x=\frac{6-4}{2}=1$. **(Accuracy Test is highly needed.)**

12. Solve the equation $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=3$

Solution: Given equation is $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}=3$.

$$\sqrt{1+x^2}+\sqrt{1-x^2}=3\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)$$
$$\sqrt{1+x^2}+\sqrt{1-x^2}=3\sqrt{1+x^2}-3\sqrt{1-x^2}$$
$$\sqrt{1-x^2}+3\sqrt{1-x^2}=3\sqrt{1+x^2}-\sqrt{1+x^2}$$
$$4\sqrt{1-x^2}=2\sqrt{1+x^2}$$
$$4\sqrt{1-x^2}=2\sqrt{1+x^2}$$
$$2\sqrt{1-x^2}=\sqrt{1+x^2}$$

Squaring both-sides, we get $4\left(1-x^2\right)=1+x^2 \Rightarrow 4-4x^2=1+x^2 \Rightarrow 3=5x^2 \Rightarrow x^2=\frac{3}{5} \Rightarrow x=\pm\sqrt{\frac{3}{5}}$.

(Accuracy Test is highly needed.)

13. If $\alpha\&\beta$ are the roots of the equation $2x^2-4x+1=0$ then form the equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$.

Solution: The given equation is $2x^2-4x+1=0$ whose roots are $\alpha\&\beta$.

So, $\alpha+\beta=-\frac{-4}{2}=2$ & $\alpha\beta=\frac{1}{2}$.

Therefore, the equation whose roots are $\alpha^2+\beta$ and $\beta^2+\alpha$ is

$$x^2-\left(\alpha^2+\beta+\beta^2+\alpha\right)x+\left(\alpha^2+\beta\right)\left(\beta^2+\alpha\right)=0$$

$$x^2-\left(\alpha^2+\beta^2+\alpha+\beta\right)x+\left(\alpha^2\beta^2+\alpha^3+\beta^3+\alpha\beta\right)=0$$

$$x^2-\left\{(\alpha+\beta)^2-2\alpha\beta+\alpha+\beta\right\}x+\left\{\alpha^2\beta^2+(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)+\alpha\beta\right\}=0$$

$$x^2-\{4-1+2\}x+\left\{\frac{1}{4}+8-3+\frac{1}{2}\right\}=0$$

$$x^2-5x+\left\{\frac{1}{4}+5+\frac{1}{2}\right\}=0$$

$$4x^2-20x+\{1+20+2\}=0$$

$$4x^2-20x+23=0 \qquad \text{(Ans.)}$$

14. The quadratic equation $x^2-4x-1=2k(x-5)$ where k is a constant, has two equal roots. Calculate the possible value of k.

Solution: Given equation is $x^2-4x-1=2k(x-5)$

$$x^2-4x-1=2kx-10k$$

$$x^2-4x-2kx+10k-1=0$$

$$x^2-(4+2k)x+(10k-1)=0$$

The two roots of the above equation, is equal if $b^2-4ac=0$.

$$\{-(4+2k)\}^2-4.1.(10k-1)=0$$

$$(4+2k)^2-40k+4=0$$

$$16+16k+4k^2-40k+4=0$$

$$4k^2-24k+20=0$$

$$k^2-6k+5=0$$

$$k^2-5k-k+5=0$$

$$k(k-5)-1(k-5)=0$$

$$(k-5)(k-1)=0$$

Therefore, $k-5=0 \quad or \quad k-1=0 \Rightarrow k=5 \quad or \quad k=1$ (Ans.)

15. Find the values of k for which the equation $2x^2+5x+3-k=0$ has two real distinct roots.

Solution: Given equation is $2x^2+5x+3-k=0$

The roots of the equation will be distinct if $b^2-4ac>0$

$$5^2-4.2.(3-k)>0$$

$$25-8(3-k)>0$$

$$25-24+8k>0$$

$$1+8k>0$$

$$8k>-1$$

Therefore, $k>-\frac{1}{8}$ (Ans.)