Predicates and Quantifiers
Discrete Mathematics — CSE 131

Definition: Variable and Predicate

Let the declarative statement:

"x is greater than 3".

- This declarative statement is neither true nor false because the value of x is not specified. Therefore this declarative statement is not a proposition.
- In this declarative statement,
    - the **variable** x is the subject of the statement,
    - "is greater than 3" is the **predicate** that refers to a property that the subject of the statement can have.

Definition: Propositional Function

Let again the declarative statement:

"*x is greater than* 3".

We denote this declarative statement by $P(x)$ where

- x is the variable,
- P is the predicate "is greater than 3".

The declarative statement $P(x)$ is said to be the value of the **propositional function** P at x.

Once a value has been assigned to the variable x, the declarative statement $P(x)$ becomes a proposition and has a truth value, true or false.

Examples of Propositional Functions

Let the set of integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.

Let $P(x)$, the propositional function "$x > 3$".

- If no value has been assigned to x, this propositional function has no truth value.
- $P(-3)$ is "$-3 > 3$" which is a false proposition.
- $P(5)$ is "$5 > 3$" which is a true proposition.
- $P(y) \wedge \neg P(0)$ is not a proposition because the variable y has no value yet.
- $P(5) \wedge \neg P(0)$ is "$(5 > 3) \wedge \neg(0 > 3)$" which is a true proposition.

Definition: Quantification

Assigning values to a variable is one method to transform a propositional function into a proposition.

Quantification is another method to transform a propositional function into a proposition. Quantification expresses the extend to which a predicate is true over a range of elements. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

We will study two types of quantification:

- the universal quantification,
- the existential quantification.

Definition: Universe of Discourse

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **universe of discourse** (or **domain of discourse**).

In a propositional function $P(x)$, the universe of discourse specifies the possible values of the variable x and must always be provided when a universal quantifier is used.

Definition: Universal Quantification

Definition

The **universal quantification** of $P(x)$ is the proposition:

"$P(x)$ for all values of x in the universe of discourse."

The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$. The symbol $\forall$ is the **universal quantifier**. We read the proposition $\forall x\, P(x)$ as "for all x, $P(x)$" or "for every x, $P(x)$."

Remarks:
An element in the universe of discourse for which $P(x)$ is false is called a **counterexample** of $\forall x\, P(x)$.
If the universe of discourse is empty, then $\forall x\, P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the empty universe of discourse for which $P(x)$ is false.

Universe of Discourse of Finite Dimension

When all the elements in the universe of discourse can be listed — say x_1, x_2, ..., x_n — it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, ..., $P(x_n)$ are all true.

Example: Let the universe of discourse be $U = \{1, 2, 3\}$. Then

$$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3).$$

Definition: Existential Quantification

Definition

The **existential quantification** of $P(x)$ is the proposition:

> *"there exists an element x in the universe of discourse such that $P(x)$."*

The notation $\exists x\, P(x)$ denotes the existential quantification of $P(x)$. The symbol $\exists$ is the **existential quantifier**. We read the proposition $\exists x\, P(x)$ as "there exists an x such that $P(x)$" or "there is an x such that $P(x)$" or "there is at least one x such that $P(x)$" or "for some x, $P(x)$."

If the universe of discourse is empty, then $\exists x\, P(x)$ is false for any propositional function $P(x)$ because there are no elements x in the empty universe of discourse for which $P(x)$ is true.

Universe of Discourse of Finite Dimension

When all the elements in the universe of discourse can be listed — say x_1, x_2, ..., x_n — it follows that the existential quantification $\exists x\, P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

because this disjunction is true if and only if at least one of $P(x_1)$, $P(x_2)$, ..., $P(x_n)$ is true.

Example: Let the universe of discourse be $U = \{1, 2, 3\}$. Then

$$\exists x\, P(x) \equiv P(1) \vee P(2) \vee P(3).$$

Definition: Uniqueness Quantification

Definition

The **uniqueness quantification** of $P(x)$ is the proposition:

> *"there exists a unique x in the universe of discourse such that $P(x)$."*

The notation $\exists! x\, P(x)$ denotes the uniqueness quantification of $P(x)$. The symbol $\exists!$ is the **uniqueness quantifier**. We read the proposition $\exists! x\, P(x)$ as "there exists a unique x such that $P(x)$" or "there is exactly one x such that $P(x)$" or "there is one and only one x such that $P(x)$."

We can avoid the use of uniqueness quantification with the following logical equivalences:

$$\begin{aligned} \exists! x\, P(x) &\equiv \exists x(P(x) \wedge \forall y(P(y) \rightarrow (y = x))) \\ &\equiv \exists x \forall y(P(y) \leftrightarrow (y = x)) \end{aligned}$$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

Operator	Precedence
()	-1
$\forall$, $\exists$	0
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$, not $\forall x (P(x) \vee Q(x))$.

Don't make the assumption that precedence of quantifiers and logical operators are well known. Put parentheses instead to make it clear.

Definitions: Bound and Free Variables

- If a quantifier is used on the variable x, then this variable is **bound**.
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- When all the variables that occur in a propositional function are bound or set to a particular value, then the propositional function is a proposition.

Examples of Bound and Free Variables

- Let $P(x, y)$, the propositional function "$x + y = 0$".
- The logical variables x and y are free and we cannot evaluate the truth value of $P(x, y)$.
- If the value 3 is set to x, then x is no longer a free variable, but $P(3, y)$ is still a propositional function because y is still a free variable.
- If we apply the universal quantification to the variable y, the propositional function $\forall y\, P(3, y)$ is now a proposition. Both variables x and y are no longer free and the truth value of the proposition is false.

Negating Quantified Expressions

The negation of a quantifier changes the universal quantifier into the existential quantifier and vice versa.

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x),$$
$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x).$$

These rules for negations for quantifiers are called **De Morgan's laws for quantifiers**.