Propositional Logic

Discrete Mathematics — CSE 131

Definition: Declarative Sentence

Definition

A **declarative sentence** is a sentence that declares a fact.

Examples of declarative sentences:

- Toronto Maple Leaf will not win the Stanley cup this year.
- $x + 1 = 3$.

Examples of sentences that are not declarative sentences:

- What time is it?
- Read this carefully.

Definition: Proposition

Definition

A **proposition** is a declarative sentence that is either true or false, but not both.

Examples of declarative sentences that are *not* propositions:

- Toronto Maple Leaf will not win the Stanley cup this year.
- $x + 1 = 3$.

Examples of declarative sentences that are propositions:

- Washington is the capital of Canada.
- $2 + 2 = 4$.

Notation

We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s, ... The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.

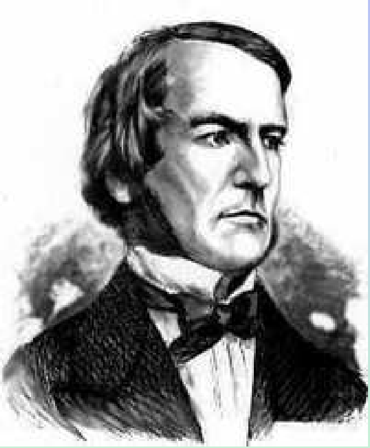
Definition

The **truth value** of a proposition is **true** (denoted T) if it is a true proposition; the **truth value** of a proposition is **false** (denoted F) otherwise.

Definition: Compound Proposition

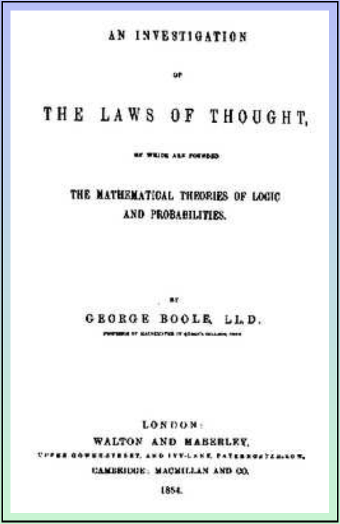
Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

George Boole



Born on November 2, 1815 in Lincoln, England.
Died on December 8, 1864 in Ballintemple, Ireland at 49 ans years old.

www-groups.dcs.st-and.ac.uk/
~history/Mathematicians/Boole.html


AN INVESTIGATION

THE LAWS OF THOUGHT,

THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

GEORGE BOOLE, LL.D.

LONDON:
WALTON AND MABERLEY,
CAMBRIDGE: MACMILLAN AND CO.
1854.

In 1854, George Boole established the rules of symbolic logic in his book The Laws of Thought.

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~history/Mathematicians/Boole.html

Definition: Negation

Definition

Let p be a proposition. The compound proposition

"it is not the case that p"

is an other proposition, called the **negation** of p, and denoted $\neg p$. The truth value of the negation of p is the opposite of the truth value of p. The proposition $\neg p$ is read "not p".

Truth Table of the Negation

A **truth table** presents the relations between the truth value of many propositions involved in a compound proposition. This table has a row for each possible truth value of the propositions.

Truth table for the negation $\neg p$ of the proposition p:

p	$\neg p$
T	F
F	T

Definition: Conjunction

Definition

Let p and q be propositions. The compound proposition

"p and q",

denoted $p \wedge q$, is true when both p and q are true and false otherwise. This compound proposition $p \wedge q$ is called the **conjunction** of p and q.

Truth table for the conjunction $p \wedge q$ of the propositions p and q:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition: Disjunction

Definition

Let p and q be propositions. The compound proposition

"*p or q*",

denoted $p \vee q$, is false when both p and q are false and true otherwise. This compound proposition $p \vee q$ is called the **disjunction** of p and q.

Truth table for the disjunction $p \vee q$ of the propositions p and q:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition: Exclusive Disjunction

Definition

Let p and q be propositions. The compound proposition

"p exclusive or q",

denoted $p \oplus q$, is true when exactly one of p and q is true and is false otherwise. This compound proposition $p \oplus q$ is called the **exclusive disjunction** of p and q.

Truth table for the exclusive disjunction $p \oplus q$ of the propositions p and q:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Definition: Implication

Definition

Let p and q be propositions. The compound proposition

> *"if p, then q"*,

denoted $p \rightarrow q$, is false when p is true and q is false, and is true otherwise. This compound proposition $p \rightarrow q$ is called the **implication** (or the **conditional statement**) of p and q.

In this implication, p is called the **hypothesis** (or **antecedent** or **premise**) and q is called the **conclusion** (or **consequence**).

Truth Table of the Implication

Truth table for the implication $p \longrightarrow q$ of the propositions p and q:

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Remarks:

- The implication $p \longrightarrow q$ is false only when p is true and q is false.
- The implication $p \longrightarrow q$ is true when p is false whatever the truth value of q.

The Implication

Variety of terminology is used to express the implication $p \rightarrow q$.

- "if p, then q";
- "p implies q";
- "q if p";
- "p only if q";
- "q when p";
- "p is sufficient for q";
- "a sufficient condition for q is p";
- "q follows from p";
- "q whenever p".

The Implication

In natural language, there is a relationship between the hypothesis and the conclusion of an implication. In mathematical reasoning, we consider conditional statements of a more general sort that we use in English. The implication

"If today is Friday, then $2+3=6$*"*

is true every day except Friday, even though $2+3=6$ is false.

The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. We only parallel English usage to make it easy to use and remember.

Definitions: Converse, Contrapositive and Inverse

We can form some new conditional statements starting with the implication $p \rightarrow q$. There are three related implications that occur so often that they have special names.

- The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$.
- The **inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.
- The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

Remember the contrapositive. The contrapositive $\neg q \rightarrow \neg p$ of the implication $p \rightarrow q$ always has the same truth value as $p \rightarrow q$.

Definition: Biconditional Statement

Definition

Let p and q be propositions. The compound proposition

"p if and only if q",

denoted $p \leftrightarrow q$, is true when p and q have the same truth value, and is false otherwise. This compound proposition $p \leftrightarrow q$ is called the **biconditional statement** (or the **bi-implication**) of p and q.

Note that the biconditional statement $p \leftrightarrow q$ is true when both implications $p \longrightarrow q$ and $q \longrightarrow p$ are true and is false otherwise. There are some other common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q";
- "if p then q, and conversely";
- "p iff q".

Truth Table of the Bi-Implication

Truth table for the bi-implication $p \leftrightarrow q$ of the propositions p and q:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table of Compound Propositions – Step 1

The truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

is given by

p	$\underbrace{q}_{a}$	$\underbrace{\neg q}_{\neg a}$
T	T	F
T	F	T
F	T	F
F	F	T

a	$\neg a$
T	F
F	T

Truth Table of Compound Propositions – Step 2

The truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

is given by

$\underbrace{p}_{a}$	q	$\underbrace{\neg q}_{b}$	$\underbrace{p}_{a} \vee \underbrace{\neg q}_{b}$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

a	b	$a \vee b$
T	T	T
T	F	T
F	T	T
F	F	F

The truth table of the compound proposition

$$(p \lor \neg q) \rightarrow (p \land q)$$

is given by

$\underbrace{p}_{a}$	$\underbrace{q}_{b}$	$\neg q$	$p \lor \neg q$	$\underbrace{p}_{a} \land \underbrace{q}_{b}$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	F
F	F	T	T	F

a	b	$a \land b$
T	T	T
T	F	F
F	T	F
F	F	F

The truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

is given by

p	q	$\neg q$	$\underbrace{p \vee \neg q}_{a}$	$\underbrace{p \wedge q}_{b}$	$\underbrace{(p \vee \neg q)}_{a} \rightarrow \underbrace{(p \wedge q)}_{b}$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

a	b	$a \rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Precedence of Logical Operators

Operator	Precedence
$()$	0
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Don't make the assumption that precedence of logical operators are well known. Put parentheses instead to make it clear.

Definition: Boolean Variable

A variable is called a **Boolean variable** if its value is either true or false.

Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one).

Consequently, a Boolean variable can be represented using a bit.

As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false.

Definition: Bit Operations

Computer **bit operations** correspond to the logical operators. By replacing true by 1 and false by 0 in the truth tables for the operators $\wedge$, $\vee$ and $\oplus$, we get the following tables:

x	y	$x \wedge y$
1	1	1
1	0	0
0	1	0
0	0	0

x	y	$x \vee y$
1	1	1
1	0	1
0	1	1
0	0	0

x	y	$x \oplus y$
1	1	0
1	0	1
0	1	1
0	0	0

We will also use the notation *AND*, *OR* and *XOR* for the operators $\wedge$, $\vee$ and $\oplus$, as is done in various programming languages.

Definition: Bit String

Definition

A **bit string** is a sequence of zero or more bits. The **length** of this bit string is the number of bits in the string.

Example: 1 1001 0111 is a bit string of length 9.

We define the bitwise *AND*, bitwise *OR* and bitwise *XOR* of two bit strings of the same length to be the strings that have as their bits the *AND*, *OR* or *XOR* of the corresponding bits in the two strings respectively. We use the symbols $\wedge$, $\vee$ and $\oplus$ to represent the bitwise *AND*, bitwise *OR* and bitwise *XOR* operations, respectively.

Example:
$$\begin{array}{cr} \wedge & 1\,0110 \\ & 1\,0101 \\ \hline & 1\,0100 \end{array} \qquad \begin{array}{cr} \vee & 1\,0110 \\ & 1\,0101 \\ \hline & 1\,0111 \end{array} \qquad \begin{array}{cr} \oplus & 1\,0110 \\ & 1\,0101 \\ \hline & 0\,0011 \end{array}$$