Mathematical Induction
Discrete Mathematics

The sum of first odd integers:

$n = 1, \quad 1 = 1$

$n = 2, \quad 1 + 3 = 4$

$n = 3, \quad 1 + 3 + 5 = 9$

$n = 4, \quad 1 + 3 + 5 + 7 = 16$

$n = 5, \quad 1 + 3 + 5 + 7 + 9 = 25$

$n = 6, \quad 1 + 3 + 5 + 7 + 9 + 11 = 36$

$\vdots$

$n = k, \quad 1 + 3 + 5 + \cdots + (2k - 1) =?$

Statement of Problem

Let $P(n)$, a propositional function on a well-ordered set S. The problem is to prove that

$$\forall n \in S, P(n)$$

is true.

Definition: Mathematical Induction

Let $P(n)$, a propositional function on a well-ordered set S. If 1 is the minimum element of the set S, then, the rule of inference:

$$\begin{array}{rl} & P(1) \\ & \forall k (P(k) \rightarrow P(k+1)) \\ \hline \therefore & \forall n\, P(n) \end{array}$$

is known as the **principle of induction** (or the **first principle of induction**).

The first hypothesis $P(1)$ is called the **basis step**.
The second hypothesis $P(k) \rightarrow P(k+1)$ for any k is called the **inductive step**. The assumption that $P(k)$ is true is called the **inductive hypothesis**.

PRELIMINARY STEP: Let $P(n)$ be the propositional function: $\sum_{i=1}^{n}(2i-1) = n^2$.

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

We conclude that $P(n)$ is true for all positive integers n.

Example: The Sum of the First n Integers

PRELIMINARY STEP: Let $P(n)$ be the propositional function: $\sum_{i=1}^{n} i = n(n+1)/2$.

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

We conclude that $P(n)$ is true for all positive integers n.

Why Mathematical Induction Is Valid?

Suppose that the well-ordered set S is $\mathbb{N}$, that $P(1)$ is true and that the proposition $P(k) \rightarrow P(k+1)$ is true for any k. We use a proof by contradiction to prove that $P(n)$ must be true for all $n \in \mathbb{N}$. Suppose there exists at least one positive integer such that $P(n)$ is false. Let the set $F \subset \mathbb{N}$ be the set of integers such that $P(n)$ is false. By assumption, the set F is non empty. So, according to the well-ordering property, F has a least element, which will be designated by m. We know that m cannot be 1 because $P(1)$ is true. Therefore m is an integer greater than 1 and $m-1 \in \mathbb{N}$. Moreover, $m-1$ is less than m and does not belong to F, so $P(m-1)$ must be true. Since the conditional statement $P(k) \rightarrow P(k+1)$ is true for any k, then $P(m-1) \rightarrow P(m)$ and therefore $P(m)$ is true. This contradicts the choice of m.