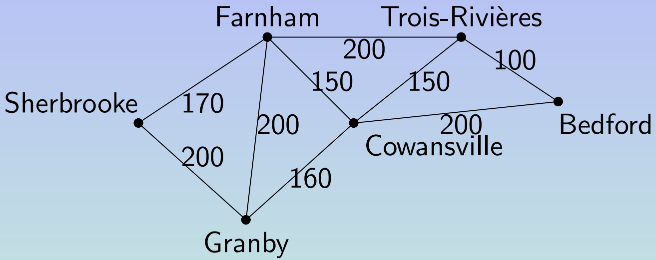
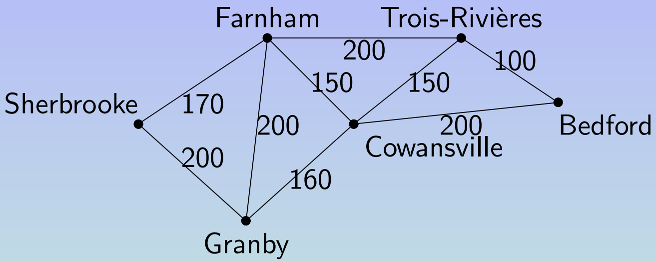
Minimum Spanning Trees
Discrete Mathematics

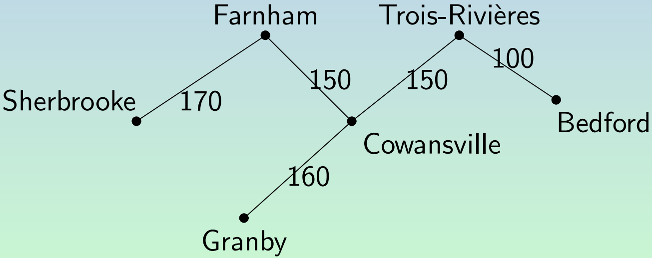
Example: Minimum Road System



In winter, the highway department wants to plow the fewest roads so that there will always be cleared roads connecting any two towns, but at the lowest possible cost.

Example: Minimum Road System





Minimum Spanning Tree

Definition

A **minimum spanning tree**, in a connected weighted graph, is a spanning tree that has the smallest possible sum of weights of its edges.

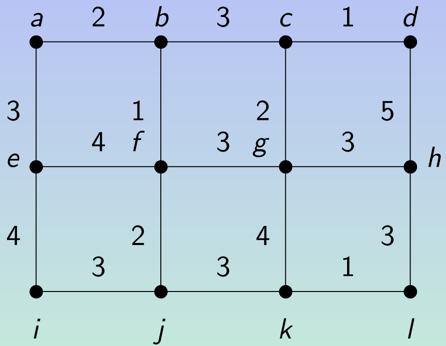
Prim's Algorithm

Robert C. Prim, *Shortest connection networks and some generalizations*, Bell System Techn. J. vol 36 (1957) pp. 1389–1401.

Prim's Algorithm

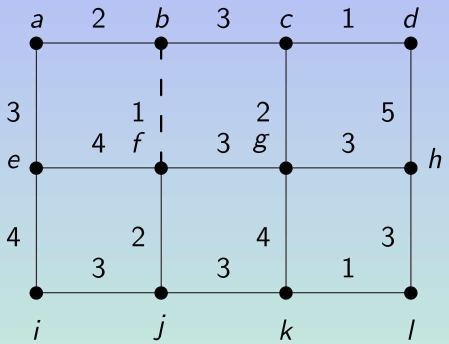
```
procedure Prim (G: weighted connected undirected graph
            with n vertices)
T := a minimum weight edge
for i := 1 to n − 2
begin
      e := an edge of minimum weight incident to a vertex
            in T and not forming a simple circuit in T
            if added to T.
      T := T with e added
end
{T is a minimum spanning tree of G}
```

Example of Prim's Algorithm, Step 1 of 5

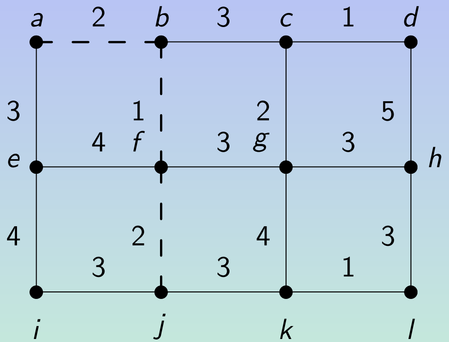


Find the edges $\{b, f\}$, $\{c, d\}$ and $\{k, l\}$ with minimum cost.

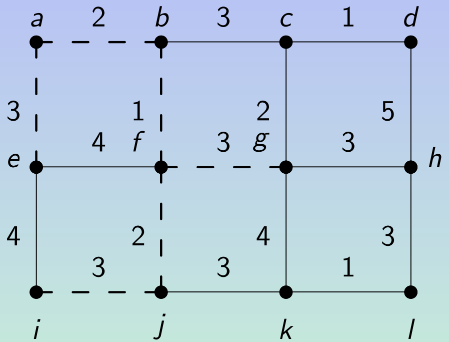
Example of Prim's Algorithm, Step 2 of 5



Initialise the minimum spanning tree T with the edge $\{b, f\}$, which is the first in lexicographic order amongst the minimum edges.

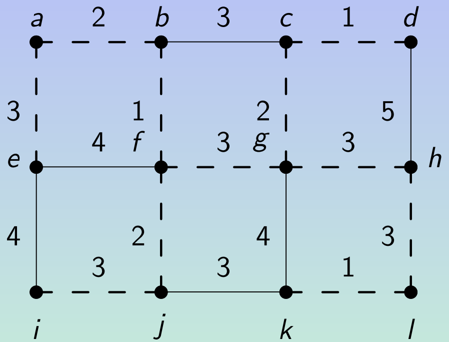


Add the edges $\{a, b\}$ and $\{f, j\}$ to T.



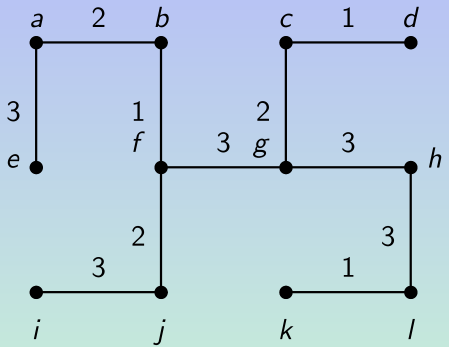
Add the edges $\{a, e\}$, $\{i, j\}$ and $\{f, g\}$ to T

Example of Prim's Algorithm, Step 5 of 5



Add the edges $\{c, g\}$, $\{c, d\}$, $\{g, h\}$, $\{h, l\}$ and $\{k, l\}$ to T.

Example of Prim's Algorithm



Final Solution. T is a minimum spanning tree of the graph G.

Joseph B. Kruskal

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On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem, Proc. Amer. Math. Soc., vol 7, no 1 (Feb. 1956), pp. 48–50.

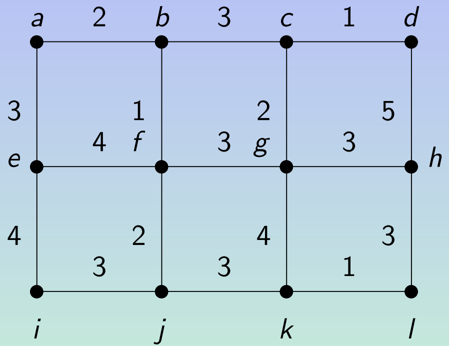
Kruskal's Algorithm

procedure *Kruskal* (G: weighted connected undirected graph
with n vertices)
$T :=$ empty tree
list $:=$ list of edges of G sorted by weight
while T has less than $n - 1$ edges
begin
$e :=$ next edge in the *list*
if e does not form a simple circuit
when added to T,
then $T := T$ with e added.
end
$\{T$ is a minimum spanning tree of $G\}$

Difference between Prim's and Kruskal's Algorithms

Note the difference between Prim's and Kruskal's algorithms. In Prim's algorithm, edges of minimum weight that are incident to a vertex already in the tree, and not forming a circuit, are chosen.

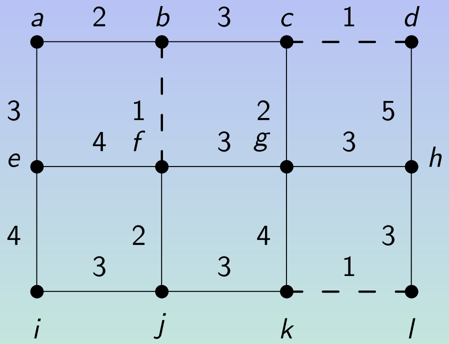
In Kruskal's algorithm, edges of minimum weight that are not necessarily incident to a vertex already in the tree, and that do not form a circuit, are chosen.



List of edges of G sorted by weight.

edge	\$
$\{b, f\}$	1
$\{c, d\}$	1
$\{k, l\}$	1
$\{a, b\}$	2
$\{c, g\}$	2
$\{f, j\}$	2
$\{a, e\}$	3
$\{b, c\}$	3
$\{f, g\}$	3

edge	\$
$\{g, h\}$	3
$\{h, l\}$	3
$\{i, j\}$	3
$\{j, k\}$	3
$\{e, f\}$	4
$\{e, i\}$	4
$\{g, k\}$	4
$\{d, h\}$	5

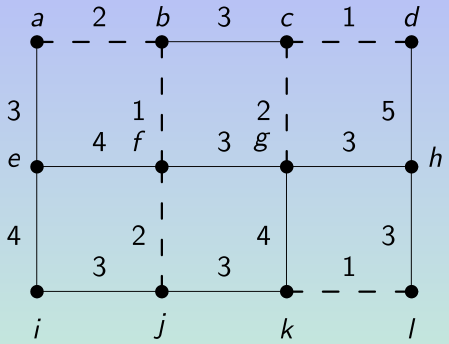


Add the edges $\{b, f\}$, $\{c, d\}$ and $\{k, l\}$.

edge	\$
$\{b, f\}$	**1**
$\{c, d\}$	**1**
$\{k, l\}$	**1**
$\{a, b\}$	2
$\{c, g\}$	2
$\{f, j\}$	2
$\{a, e\}$	3
$\{b, c\}$	3
$\{f, g\}$	3

edge	\$
$\{g, h\}$	3
$\{h, l\}$	3
$\{i, j\}$	3
$\{j, k\}$	3
$\{e, f\}$	4
$\{e, i\}$	4
$\{g, k\}$	4
$\{d, h\}$	5

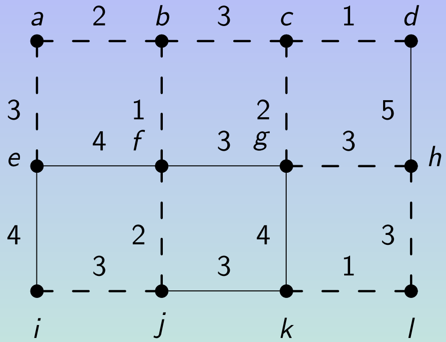
Example of Kruskal's Algorithm, Step 3 of 4



edge	\$
$\{b, f\}$	1
$\{c, d\}$	1
$\{k, l\}$	1
$\{a, b\}$	**2**
$\{c, g\}$	**2**
$\{f, j\}$	**2**
$\{a, e\}$	3
$\{b, c\}$	3
$\{f, g\}$	3

edge	\$
$\{g, h\}$	3
$\{h, l\}$	3
$\{i, j\}$	3
$\{j, k\}$	3
$\{e, f\}$	4
$\{e, i\}$	4
$\{g, k\}$	4
$\{d, h\}$	5

Add the edges $\{a, b\}$, $\{c, g\}$ and $\{f, j\}$.

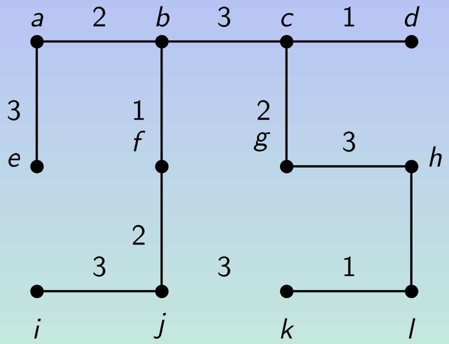


edge	\$
$\{b, f\}$	1
$\{c, d\}$	1
$\{k, l\}$	1
$\{a, b\}$	2
$\{c, g\}$	2
$\{f, j\}$	2
$\{a, e\}$	**3**
$\{b, c\}$	**3**
$\{f, g\}$	3

edge	\$
$\{g, h\}$	**3**
$\{h, l\}$	**3**
$\{i, j\}$	**3**
$\{j, k\}$	3
$\{e, f\}$	4
$\{e, i\}$	4
$\{g, k\}$	4
$\{d, h\}$	5

Add the edges $\{a, e\}$, $\{b, c\}$, $\{g, h\}$, $\{h, l\}$ and $\{i, j\}$. The edge $\{f, g\}$ was not added because it forms a circuit.

Example of Kruskal's Algorithm



Final solution.

edge	\$
$\{b, f\}$	**1**
$\{c, d\}$	**1**
$\{k, l\}$	**1**
$\{a, b\}$	**2**
$\{c, g\}$	**2**
$\{f, j\}$	**2**
$\{a, e\}$	**3**
$\{b, c\}$	**3**
$\{f, g\}$	3

edge	\$
$\{g, h\}$	**3**
$\{h, l\}$	**3**
$\{i, j\}$	**3**
$\{j, k\}$	3
$\{e, f\}$	4
$\{e, i\}$	4
$\{g, k\}$	4
$\{d, h\}$	5