Closures of Relations

Discrete Mathematics

Definition: Closure of a Relation

Let R be a relation on a set A. The relation R may or may not have some property **P** such as reflexivity, symmetry or transitivity.

If there is a relation S

- with property **P**,
- containing R,
- and such that S is a subset of every relation with property **P** containing R,

then S is called the **closure** of R with respect to **P**.

Definition: Reflexive Relation

Definition

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Is this relation reflexive? If no, what is the reflexive closure of this relation?

Definition: Reflexive Closure

Let R be a relation on a set A. The **reflexive closure** of R is

$$R \cup \Delta$$

where

$$\Delta = \{(a, a) \mid a \in A\}$$

is called the **diagonal relation** on A.

Definition: Symmetric Relation

Definition

A relation R on a set A is called **symmetric** if $(a, b) \in R$ implies that $(b, a) \in R$ for all $a, b \in A$.

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Is this relation symmetric? If no, what is the symmetric closure of R?

Definition: Symmetric Closure

Let R be a relation on a set A. The **symmetric closure** of R is

$$R \cup R^{-1}$$

where

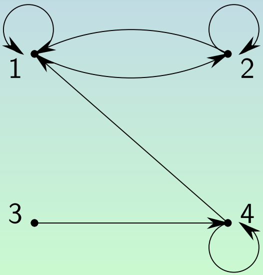
$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

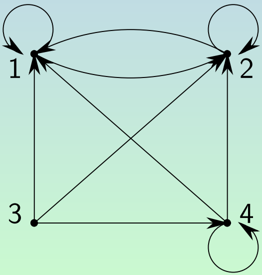
is **inverse relation** of R.

Definition

A relation R on a set A is called **transitive** if, whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. Is this relation transitive? If not, what is the transitive closure of R?

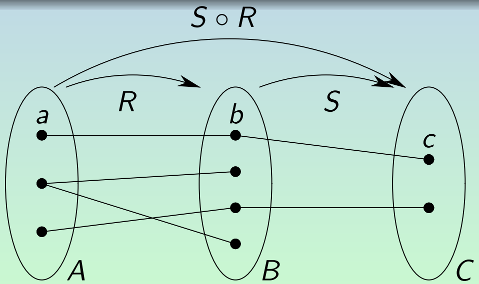




Definitions: Composite of Relations

Definition

Let R be a relation from a set A to a set B, and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.



Definitions: Path and Length

Definition

A **path** from a to b in a directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n) in G, where n is a non negative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex of the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has **length** n. We view the empty set of edges as a path from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a **circuit** or **cycle**.

Definition

There is a **path** from a to b in a relation R if there is a sequence of elements $a, x_1, x_2, \ldots, x_{n-1}, b$ with $(a, x_1) \in R$, $(x_1, x_2) \in R$, ..., $(x_{n-1}, b) \in R$. This path is of **length** n.

Definition: Powers of a Relation

Definition

Let R be a relation on the set A. The **powers** R^n, $n = 1, 2, \ldots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

Theorem

Let R be a relation on the set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Definition: Join Matrix

Definition

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero-one matrices. Then, the **join** of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \vee \mathbf{B}$, is the $m \times n$ zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$.

Example. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Definition: Boolean Product

Definition

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. Then, the **Boolean product** of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Remark 1: $\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$.
Remark 2: $\mathbf{M}_{R \circ R} = \mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_R^{[2]}$.

Paths and Connectivity

Definition

Let R be a relation on the set A. The **connectivity relation** R^* consists of pairs (a, b) such that there is a path of length at least one from a to b in R.

Transitive Closure and Connectivity

Theorem

The **transitive closure** *of a relation* R *equals the connectivity relation* R^*.

Theorem

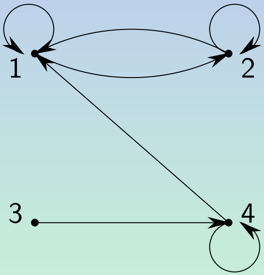
Let $\mathbf{M}_R$ *be the zero-one matrix of the relation* R *on a set with* n *elements. Then the zero-one matrix of the transitive closure* R^* *is*

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

Procedure for Computing the Transitive Closure

procedure *transitive closure*($\mathbf{M}_R$: zero-one $n \times n$ matrix)
$\{\mathbf{P}$ will store the powers of $\mathbf{M}_R\}$
$\mathbf{P} := \mathbf{M}_R$
$\{\mathbf{J}$ will store the join of the powers of $\mathbf{M}_R\}$
$\mathbf{J} := \mathbf{M}_R$
for $i := 2$ **to** n
begin
    $\mathbf{P} := \mathbf{P} \odot \mathbf{M}_R$
    $\mathbf{J} := \mathbf{J} \vee \mathbf{P}$
end
$\{\mathbf{J}$ is the zero-one matrix for $R^*\}$

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$. What is the transitive closure of R?



$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Example of Transitive Closure, Step 2 of 4

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_R^{[4]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Example of Transitive Closure, Step 3 of 4

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \mathbf{M}_R^{[4]}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Example of Transitive Closure, Step 4 of 4

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{R^*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$