Bayes Theorem

If $E_1, E_2, ..., E_n$ are mutually disjoint events with $p(E_i) \neq 0$; $i = 1, 2, ..., n$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $p(A) > 0$, we have

$$p(E_i|A) = \frac{p(E_i)\, p(A|E_i)}{\sum_{i=1}^{n} p(E_i)\, p(A|E_i)} \ , \ i = 1, 2, ..., n .$$

Example

In a bolt factory machines A, B and C manufacturer respectively 25%, 35% and 40% of the total. Of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at randomly from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

Solution

Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event of its being defective. Then we have, $p(E_1) = 0.25$, $p(E_2) = 0.35$ and $p(E_3) = 0.40$.

The probability of drawing a defective bolt manufactured by machine A is $p(E|E_1) = 0.05$. Similarly, we have $p(E|E_2) = 0.04$ and $p(E|E_3) = 0.02$.

Hence, the probability that a defective bolt selected at random is manufactured by machine A is given by

$$p(E_1|E) = \frac{p(E_1)\, p(E|E_1)}{\sum_{i=1}^{3} p(E_i)\, p(E|E_i)} = \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.362$$

Similarly, $$p(E_2|E) = \frac{p(E_2)\, p(E|E_2)}{\sum_{i=1}^{3} p(E_i)\, p(E|E_i)} = \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.406$$

and $$p(E_3|E) = \frac{p(E_3)\, p(E|E_3)}{\sum_{i=1}^{3} p(E_i)\, p(E|E_i)} = \frac{0.40 \times 0.02}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.232$$